

Some Structures of Maximal Prefix Codes Generated by Petri Nets *

Yoshiyuki Kunimochi

Department of Computer Science, Shizuoka Institute of Science and Technology
Toyosawa 2200-2, Fukuroi-shi, Shizuoka 437-8555, Japan

Abstract

A Petri net is a mathematical model which is applied to descriptions of parallel processing systems. We define a prefix code, called a Petri net code, based on a Petri net under a certain terminal condition. In this paper we consider the two families of Petri net codes. One of them is defined by the maximality of prefix codes, called the family of maximal Petri net codes. The other is the family of codes generated by restricted Petri nets, called the family of input-ordinal Petri net code. It is easily seen that the later is a subfamily of the former. But it is still open whether the later includes the former. So we show that the inclusion is true in case that the component of a Petri net is simple.

1 Introduction

In the section 1.1, we introduce the notation of languages and codes in this paper. In the next section, we define a Petri net code and explain their properties.

1.1 languages and Codes

Let X be a nonempty finite set called an alphabet, X^* be the free monoid generated by X under the concatenation. An element of X^* is called a word. The identity of X^* is called the empty word, denoted by 1. We denote $X^* \setminus \{1\}$ by X^+ , the concatenation of two words x and y by xy , and the length of a word $w \in X^*$ by $|w|$ (especially $|1| = 0$).

If for two words $w, u \in X^*$ there exists some word $v \in X^*$ (resp. $v \in X^+$) with $w = uv$, then u is called a prefix (resp. a proper prefix) of w , we represent $u \leq_p w$ (resp. $u <_p w$). A language over X is a subset of X^* . The concatenation of two languages L_1 and L_2 is defined by $L_1L_2 = \{w_1w_2 | w_1 \in L_1, w_2 \in L_2\}$. A nonempty language L is a code if for any two integers $n, m \geq 1$ and $u_1, u_2, \dots, u_n, v_1, v_2, \dots, v_m \in L$,

$$u_1u_2 \cdots u_n = v_1v_2 \cdots v_m$$

implies

$$n = m \text{ and } u_i = v_i \text{ for } i = 1, \dots, n.$$

A code L is a prefix code if $u, uv \in L$ implies $v = 1$. A code $C \subset X^+$ is maximal (resp. maximal prefix) in X if C is not included by any code (resp. prefix code) over X .

Remark A maximal and prefix code is clearly a maximal prefix code because it is not included by any code by the maximality. But a maximal prefix code is a prefix code, but is not necessarily a maximal code.

1.2 Petri net codes

DEFINITION 1.1 (Petri net) A Petri net PN is a quadruple (P, X, W, μ_0) satisfying the following conditions.

- (1) P and X are finite sets with $P \cap X = \emptyset$ and $P \cup X \neq \emptyset$.
- (2) W is a weighting function from $(P \times X) \cup (X \times P)$ to the set N of all the nonnegative integers.
- (3) μ_0 is a function from P to N , called an initial marking. ■

* This is an abstract and the paper will appear elsewhere.

A marking is positive if it is a function from P to $N \setminus \{0\}$. And PN is input-ordinal if $W(p, a) \leq 1$ for any $(p, a) \in P \times X$. In the above Petri net PN , we may call $(p, a) \in P \times X$ an arc when $W(p, a) > 0$ holds, and then $W(p, a)$ is called the weight of the arc (p, a) . The similar definition holds about $(a, p) \in X \times P$.

When $W(p, a) \leq \mu(p)$ holds for any place $p \in P$, the transition $a \in X$ is enable under the Petri net PN . Then the new marking μ' is defined as follows:

$$\mu'(p) = \mu(p) - W(p, a) + W(a, p) \text{ for } \forall p \in P.$$

The transition function δ_{PN} of PN is defined by $\delta_{PN}(\mu, a) = \mu'$. $\delta_{PN}(\mu, a)$ is undefined if $a \in X$ is not enable under PN . This function is extended from $P \times X \mapsto N$ to $P \times X^* \mapsto N$ as follows: $\delta_{PN}(\mu, 1) = \mu$ and $\delta_{PN}(\mu, ua) = \delta_{PN}(\delta_{PN}(\mu, u), a)$. We may denote δ_{PN} by δ if no confusion is possible.

DEFINITION 1.2 Let $PN = (P, X, W, \mu_0)$ be a Petri net, μ_0 be a positive marking. Then we define the language $C(P, X, W, \mu_0)$ as follows:

$$C(P, X, W, \mu_0) = \{w \in X^* \mid w = uv, v \in X^+, \delta(\mu, w) \text{ is not positive, } \delta(\mu, u) \text{ is positive}\}.$$

If $C = C(P, X, W, \mu_0) \neq \emptyset$ then C is a prefix code. Because both $u, uv \in C$ and $v \neq 1$ yield a contradiction since $\delta(\mu, u)$ is positive. We call such a code a Petri net code. The family of all the Petri net codes is denoted by **CPN**. Moreover a Petri net code is said to be a maximal Petri net code if it is maximal prefix. The family of all the maximal Petri net codes is denoted by **mCPN**. A Petri net code is said to be a input-ordinal Petri net code if it is generated by some input-ordinal Petri net. The family of all the input-ordinal Petri net codes is denoted by **NmCPN**.

Since an input-ordinal Petri net code is clearly a maximal Petri net code, we get the inclusion relation $\text{NmCPN} \subset \text{mCPN}$. In this paper, we consider the following problem.

[Problem] $\text{mCPN} \subset \text{NmCPN}$?

Since it is too difficult to solve this problem in general Petri nets, in the next section we prove that the problem is solved affirmatively in a restricted Petri net.

2 Fundamental Properties

Here we state some fundamental properties used in the next section.

DEFINITION 2.1 Let $PN = (P, X, W, \mu_0)$ be a Petri net and μ_0 be a positive marking. For $w \in X^*$ and $a \in X$, we define the set $K_w(a)$ of places as follows.

$$K_w(a) = \{p \in P \mid \delta(\mu_0, w) = \mu, \quad W(a, p) = 0, W(p, a) > 0 \text{ and } s = \frac{\mu(p)}{w(p, a)} \text{ is an integer,}$$

$$\forall q \in P \setminus \{p\} \quad (W(a, q) = 0 \text{ and } W(p, a) > 0 \text{ implies } s \leq \frac{\mu(q)}{w(q, a)})\}.$$

An element of $K_w(a)$ is called a critical place (after reading the word w) Especially $K_1(a)$ is denoted by $K(a)$, where 1 is the empty word. K_w is a mapping from X to 2^P , called the critical place mapping of the Petri net (P, X, W, μ_0) . ■

A critical place p of a transition a means that p is a place where the number of tokens first becomes zero when a fires one after another (see Figure 1).

THEOREM 2.1 (Fundamental Theorem) Let K be a critical place mapping of a Petri net (P, X, W, μ_0) , $C(P, X, W, \mu_0)$ be a maximal Petri net code. Let $p \in P$ and $B = \{a \in X \mid W(p, a) > 0\}$. Then, For any $a, b \in B$,

- (1) $p \in K(a)$ implies $W(p, a) \geq W(p, b)$,
- (2) $p \in K(a) \cap K(b)$ implies $W(p, a) = W(p, b)$.

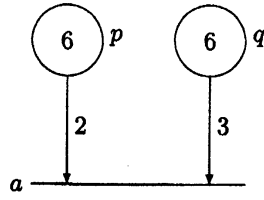


Figure 1: Example of critical places, $K(a) = \{q\}$.

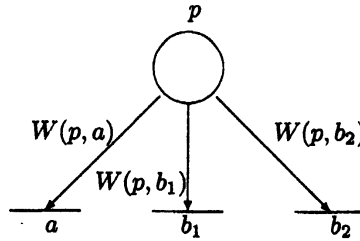


Figure 2: $p \in K(a) \Rightarrow W(p, a) \geq W(p, b_1), W(p, b_2)$.

THEOREM 2.2 (Deletion of useless places) Let $PN = (P, X, W, \mu_0)$ be a Petri net and μ_0 be a positive marking. Let $C = C(P, X, W, \mu_0)$ be a maximal Petri net code. Let $p \in P$ be a place such that $\delta(\mu_0, w)(p) \neq 0$ for any $w \in C$. And the Petri net $PN' = (P', X', W', \mu'_0)$ is defined as follows, which is obtained by removing the place p and the arcs from p and the arcs to p .

$$\begin{aligned} P' &= P \setminus \{p\}, X' = X \\ W' &\text{ is a restriction of } W \text{ on } (P' \times X) \cup (X \times P'), \\ \mu'_0 &\text{ is a restriction of } \mu_0 \text{ on } P', \end{aligned}$$

Then,

$$C(P, X, W, \mu_0) = C(P', X', W', \mu'_0). \blacksquare$$

Generally set $P_0 = \{q \in P \mid \exists w \in C, \delta(\mu_0, w)(q) = 0\}$. Applying the above theorem repeatedly, the theorem holds even if we replace P' in the theorem by P_0 . The maximality in the theorem is needed as the following example shows.

EXAMPLE 2.1 Let $P = \{p, q\}$, $X = \{a, b\}$, $W(p, a) = W(p, b) = 1$, $W(q, b) = 2$, $\mu_0(p) = \mu_0(q) = 1$. The other arcs weigh 0. Then $C = C(P, X, W, \mu_0) = \{a\}$ is not maximal. For any $w \in C$, $\delta(\mu_0, w)(q) \neq 0$, where δ is the transition function of (P, X, W, μ_0) . However, since $P' = P \setminus \{q\} = \{p\}$, $X' = \{a, b\}$, $W'(p, a) = W'(p, b) = 1$, $\mu'_0(p) = 1$, the other arcs weigh 0, $C' = C(P', X', W', \mu'_0) = \{a, b\}$. This means that $C = C'$ does not necessarily hold.

THEOREM 2.3 (Reduction rule of two-way arcs) Let $PN = (P, X, W, \mu_0)$ be a Petri net and μ_0 be a positive marking. Let $C = C(P, X, W, \mu_0)$ be a maximal Petri net code. Let $p \in P$, $a \in X$ with $W(p, a) > 0$ and $W(a, p) > 0$. Then the Petri net $PN' = (P, X, W', \mu_0)$ is defined as follows, which is obtained by replacing the weights of the two arcs (p, a) and (a, p) .

$$\begin{aligned} W(p, a) > W(a, p) &\Rightarrow W'(p, a) = W(p, a) - W(a, p), W'(a, p) = 0 \\ W(p, a) = W(a, p) &\Rightarrow W'(p, a) = W'(a, p) = 0 \\ W(p, a) < W(a, p) &\Rightarrow W'(a, p) = W(a, p) - W(p, a), W'(p, a) = 0 \\ q \neq p \text{ or } b \neq a &\Rightarrow W'(b, q) = W(b, q), W'(q, b) = W(q, b) \end{aligned}$$

Then

$$C(P, X, W, \mu_0) = C(P, X, W', \mu_0).$$

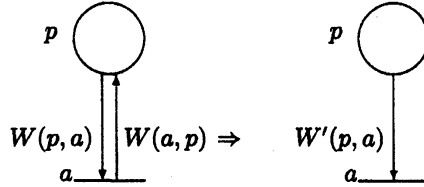


Figure 3: Replacing the weights of arcs (if $W(p, a) > W(a, p)$).

EXAMPLE 2.2 Let X be an alphabet and k be a positive integer. Suppose that subsets X_1 and X_2 of X satisfy $X = X_1 \cup X_2$ and $X_1 \cap X_2 = \emptyset$. Then, the following language C is an input-ordinal Petri net code.

$$C = \left(\bigcup_{0 \leq i < k} X_2^i X_1 \right) \cup X_2^k$$

(proof) If the input-ordinal Petri net (P, X, W, μ_0) is defined as follows (see Figure 4), then $C = C(P, X, W, \mu_0)$ holds.

$$\begin{aligned} P &= \{p, q\}, X = X_1 \cup X_2, \\ \mu_0(p) &= 1, \mu_0(q) = k, \\ W(p, a) &= 1, W(a, p) = 0 \quad \text{if } a \in X_1 \\ W(q, b) &= 1, W(b, q) = 0 \quad \text{if } b \in X_2 \end{aligned}$$

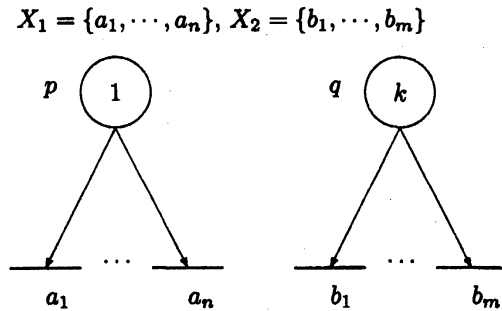


Figure 4: the Petri net generating C of Example 2.2.

Especially, in the above example setting $X_1 = \emptyset$ and $X_2 = X$, $C = X^k = \{w \in X^* \mid |w| = k\}$. X^k is called a (full) uniform code over X . Therefore a uniform code becomes an input-ordinal Petri net code.

3 Maximal Petri net codes and input-ordinal Petri net code

Here we solve the problem whether $m\text{CPN} \subset n\text{mCPN}$ holds or not under some conditions. At first we consider the case the number $|P|$ of places equals 1 and the case the number $|X|$ of transitions equals 1.

3.1 In case of $|P| = 1$ or $|X| = 1$

THEOREM 3.1 *Let $PN = (P, X, W, \mu_0)$ be a Petri net and μ_0 be a positive marking. Assume that $|X| = 1$ or $|P| = 1$. If $C = C(P, X, W, \mu_0)$ is a maximal Petri net code, then C is an input-ordinal Petri net code. ■*

Assume that $|P| = 1$, that is $P = \{p\}$ in this theorem. Setting $X_1 = \{a \in X | W(p, a) > 0, W(a, p) = 0\}$ and $X_2 = X - X_1$, Then

$$C(P, X, W, \mu_0) = (X_1^{n-1} \diamond (\bigcup_{a_i \in X_2} a_i X_1^{n_i})^\circ) X_1,$$

where $n_i = W(a_i, p)/n$, \diamond is the shuffle product over two languages $L, K \subset X^*$ defined by $L \diamond K = \bigcup_{x \in L, y \in K} x \diamond y$, $x \diamond y = \{x_1 y_1 x_2 y_2 \cdots x_n y_n | x = x_1 x_2 \cdots x_n, y = y_1 y_2 \cdots y_n, x_i, y_i \in X^* \text{ for } 1 \leq i \leq n\}$ for $x, y \in X^*$ and L° is the shuffle closure of a language L , defined by $L^\circ = \bigcup_{i \geq 0} L^{\circ i}$, $L^{\circ 0} = \{1\}$, $L^{\circ(i+1)} = L^{\circ i} \diamond L$.

In case that a Petri net has only a place or only a transition, we have proven $NmCPN = mCPN$. In the next section, we consider the case that a Petri net has two places.

3.2 In case of $|P| = 2$

In a Petri net $PN = (P, X, W, \mu_0)$, for a transition $a \in X$, Set $I(a) = \{p \in P | W(p, a) > 0\}$ and $O(a) = \{p \in P | W(a, p) > 0\}$. If $I(a) \neq \emptyset$ and $O(a) = \emptyset$, a is called a consuming transition. If $I(a) \neq \emptyset$ and $O(a) \neq \emptyset$ a is called a transporting transition. If $I(a) = \emptyset$ and $O(a) \neq \emptyset$, a is called a supplying transition. If $I(a) = O(a) = \emptyset$, a is called an isolated transition.

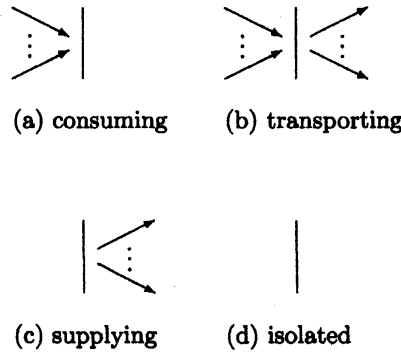


Figure 5: Classification of transitions

All through this section, we assume that a Petri net $PN = (P, X, W, \mu_0)$ with a positive marking μ_0 generating the code satisfies the following conditions.

- (1) $C(P, X, W, \mu_0)$ is a maximal Petri net code.
- (2) $|P| = 2$, that is the number of places equals 2. Set $P = \{p, q\}$.
- (3) $X \neq \emptyset$ and each element of X is a consuming transition or a transporting transition.
- (4) By theorem 2.3, for any $p \in P$ and any $a \in X$, Both the weight of (a, p) and the weight of (p, a) are not positive.

THEOREM 3.2 Let $PN = (P, X, W, \mu_0)$ be a Petri net, μ_0 be positive. Assume that $C = C(P, X, W, \mu_0)$ is a maximal Petri net code, $|P| = 2$, and each element of X is not a supplying transition, C is an input-ordinal Petri net code. ■

(proof) Since the marking is unchanged by a isolated transition's firing, we may assume that X has no isolated transition without the loss of generality.

Moreover, Set the sets X_p and X_q of transitions, respectively

$$\begin{aligned} X_p &= \{a \in X \mid p \in K(a)\}, \\ X_q &= \{a \in X \mid q \in K(a)\} \end{aligned}$$

(note that $X_p \cap X_q = K^{-1}(\{p, q\}) = \emptyset$ is not necessarily hold), where K is a critical place mapping.

By the condition (3), since $a \in X$ is a consuming or transporting transition, the number of tokens in p or q becomes zero when a is fired in succession. Namely at least one of $a \in X_p$ or $a \in X_q$ holds.

If $a \in X_p$, there exists some positive divisor n_p of $\mu_0(p)$ such that $W(p, a) = n_p$. The analogy holds if $a \in X_q$. Hence, there exist two positive integers k and l such that

$$\begin{aligned} \mu_0(p) &= kn_p, \mu_0(q) = ln_q, \\ W(p, a) &= n_p, W(a, p) = 0 \quad \text{if } a \in X_p, \\ W(q, a) &= n_q, W(a, q) = 0 \quad \text{if } a \in X_q. \end{aligned} \quad (1)$$

If $k = l = 1$, the statement of this theorem holds because the code C is a uniform code X^1 . Moreover, if no transition a exists such that $p \in K(a)$, that is $X = X_q$, the code is a uniform code $C = X^l \in \text{NmCPN}$. C is also a uniform code if $X = X_p$. So we may assume that $k = l = 1$ doesn't hold, $X_p \neq \emptyset$ and $X_q \neq \emptyset$.

If $a \in X_p$ and $b \in X_q$, the weights of arcs (p, a) and (q, b) is uniquely determined as shown the expression (1). While there are some cases: the weights of arcs (p, b) and (b, p) are not multiples of n_p , the weights of arcs (q, a) and (a, q) are not multiples of n_q . How to weigh these arcs is divided into the next five cases by the symmetry.

- (A) $\exists a \in X_p, \exists b \in X_q$ [$x = W(p, b) > 0, y = W(q, a) > 0$, and $(n_p \nmid x \text{ or } n_q \nmid y)$]
- (B) $\exists a \in X_p, \exists b \in X_q$ [$x = W(b, p) > 0, y = W(q, a) > 0$, and $(n_p \nmid x \text{ or } n_q \nmid y)$]
- (C) $\exists a \in X_p, \exists b \in X_q$ [$W(b, p) = 0, y = W(q, a) > 0$, and $n_q \nmid y$]
- (D) $\exists a \in X_p, \exists b \in X_q$ [$x = W(b, p) > 0, y = W(a, q) > 0$, and $(n_p \nmid x \text{ or } n_q \nmid y)$]
- (E) $\exists a \in X_p, \exists b \in X_q$ [$x = W(b, p) > 0, W(q, a) = 0$, and $n_p \nmid x$]

By Lemma 3.1 stated later in case of (A), by Lemma 3.2 in case of (B), by Lemma 3.3 in case of (C), by Lemma 3.5 in case of (E), we can show that C is an input-ordinal Petri net code respectively. On the other hand the case (D) is impossible because C is not a maximal Petri net code by the lemma 3.4.

If any condition from (A) to (E) does not hold, the weight of each output arc from the place p (resp. q) is n_p (resp. n_q), the weight of every input arc to p (resp. q) is a multiple of n_p (resp. n_q). Therefore, $C(P, X, W, \mu_0)$ is the same as the code which the following Petri net (P', X', W', μ'_0) generates.

$$\begin{aligned} P' &= P = \{p, q\}, X' = X \\ W'(p, a) &= W(p, a)/n_p = 1, W'(a, p) = W(a, p)/n_p \quad \text{for } \forall a \in X \\ W'(q, a) &= W(q, a)/n_q = 1, W'(a, q) = W(a, q)/n_q \quad \text{for } \forall a \in X \\ \mu'_0(p) &= \mu_0(p)/n_p = k, \mu'_0(q) = \mu_0(q)/n_q = l \end{aligned}$$

Hence, $C(P, X, W, \mu_0) \in \text{NmCPN}$. ■

We state the lemmata 3.1~3.5 in referred in the proof of the theorem 3.2. We omit thier proofs.

LEMMA 3.1 If the following condition (A) is satisfied in the Petri net $PN = (P, X, W, \mu_0)$, then the code which PN generates is a uniform code X^k , that is an input-ordinal Petri net code.

- (A) $\exists a \in X_p, \exists b \in X_q$ [$x = W(p, b) > 0, y = W(q, a) > 0$, and $(n_p \nmid x \text{ or } n_q \nmid y)$]. ■

The case (A_1) of $K(a) = \{p\}, K(b) = \{q\}$ and the case (A_3) of $K(a) = \{p, q\}, K(b) = \{p, q\}$ are impossible. In case (A_2) of $K(a) = \{p, q\}, K(b) = \{q\}$, $C = X^k$.

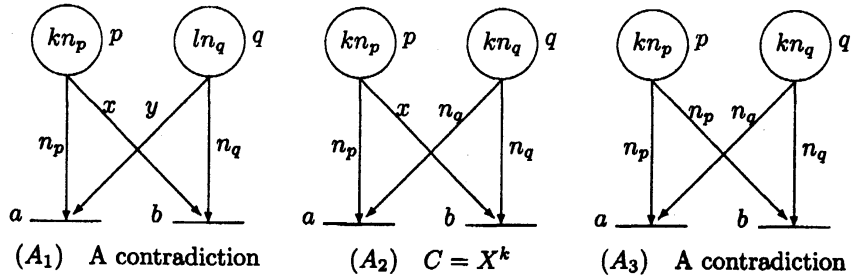


Figure 6: When the condition (A) holds.

LEMMA 3.2 *If the following condition (B) is satisfied in the Petri net $PN = (P, X, W, \mu_0)$, then the code which this Petri net generates is an input-ordinal Petri net code.*

$$(B) \exists a \in X_p, \exists b \in X_q \ [x = W(b, p) > 0, y = W(q, a) > 0, \text{ and } (n_p \not\parallel x \text{ or } n_q \not\parallel y)]. \blacksquare$$

In case (B₁) of $K(a) = \{p\}, K(b) = \{q\}$, C is an input-ordinal Petri net code, and in case (B₂) of $K(a) = \{p, q\}, K(b) = \{q\}$, $C = X^k$.

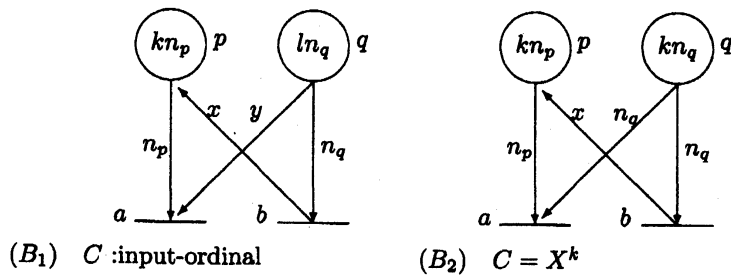


Figure 7: When the condition (B) holds.

LEMMA 3.3 *If the following condition (C) is satisfied in the Petri net $PN = (P, X, W, \mu_0)$, then the code which this Petri net generates is an input-ordinal Petri net code.*

$$(C) \exists a \in X_p, \exists b \in X_q \ [W(b, p) = 0, y = W(q, a) > 0, \text{ and } n_q \not\parallel y]. \blacksquare$$

In case (C₁) of $K(a) = \{p\}, K(b) = \{q\}$, C is an input-ordinal Petri net code in the form of EXAMPLE 2.2. The case (C₂) is impossible.

LEMMA 3.4 *If the following condition (D) is satisfied in the Petri net $PN = (P, X, W, \mu_0)$, then the code which this Petri net generates must not be a maximal Petri net code.*

$$\exists a \in X_p, \exists b \in X_q \ [x = W(b, p) > 0, y = W(a, q) > 0, \text{ and } (n_p \not\parallel x \text{ or } n_q \not\parallel y)]. \blacksquare$$

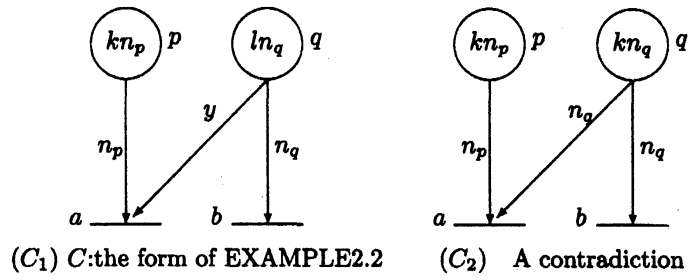


Figure 8: When the condition (C) holds.

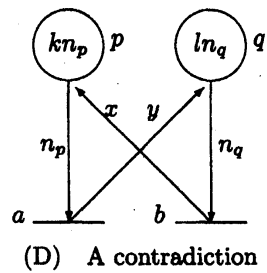
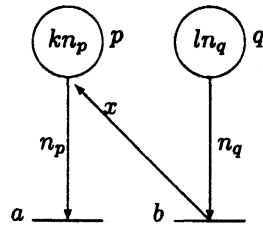


Figure 9: When the condition (D) holds.

LEMMA 3.5 *If the following condition (E) is satisfied in the Petri net $PN = (P, X, W, \mu_0)$, then the code which this Petri net generates is an input-ordinal Petri net code in the form of EXAMPLE 2.2.*

$$(E) \exists a \in X_p, \exists b \in X_q \quad [x = W(b, p) > 0, W(q, a) = 0, \text{ and } n_p \nmid x]. \blacksquare$$



(E) C : the form of EXAMPLE 2.2

Figure 10: When the condition (E) holds.

References

- [1] G. Tanaka. Prefix codes determined by Petri nets, *Algebra Colloquium* 5, pp.255–264, (1998)
- [2] M.Ito and Y.Kunimochi. Some Petri nets languages and codes, *Lecture Note in Computer Science* 2295, pp.69–80, (2002)
- [3] H. J. Shyr, *Free Monoids and Languages*, Lecture Notes (Hon Min Book Company, Taichung 1991).
- [4] J. L. Peterson, *Petri Net Theory and the Modeling of Systems*, (Prentice-Hall, 1981).
- [5] G. Rozenberg and A. Salomaa, *Handbook of Formal Languages Vol.1 WORD LANGUAGE, GRAMMAR*, (Springer, 1997).
- [6] J. Berstel and D. Perrin, *Theory of Codes*, Pure and Applied Mathematics (Academic Press, 1985).