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FINITE SEMIGROUPS AND DECIDABILITY OF AMALGAMATION BASES FOR SEMIGROUPS*

KUNITAKA SHOJI
DEPARTMENT OF MATHEMATICS, SHIMANE UNIVERSITY
MATSUE, SHIMANE, 690-8504 JAPAN

In this paper, we prove that the decision problem of whether or not a finite semigroup S is an amalgamation base for all semigroups is decidable.

1 Introduction and preliminaries

In [3], M. Spair investigated problems of amalgams in the class of finite semigroups and showed that it is undecidable whether or not an amalgam of finite semigroups is embedded in a semigroup or a finite semigroup. However it is not known whether or not the problem of decidability of amalgamation bases. In this paper we prove decidablity of whether or not a finite semigroup is an amalgamation base for all semigroups.

Let S be a semigroup. Let M be a nonempty set with a unitary and associative operation of S : $S^1 \times M \longrightarrow M((s,w) \longmapsto sw)$, where S^1 is the monoid obtained from S by adjoining a new identity 1. Then M is called a *left S-set*. Dually, *right* S-set is defined. If M is a left S-set and a right S-set satisfying that $(sm)t = s(mt)$ for all $s, t \in S$ and $m \in M$, then M is called an S-biset.

A relation ρ of a left S-set M [resp. right S-set] is called S*-congruence* if $(m, m') \in \rho$ and $s \in S$ implies $(sm, sm') \in \rho$ [resp. $(ms, sm's) \in \rho$]. Let M, N be a right S-sets [left S-set]. Then a map $\phi : M \longrightarrow N$ is called an S-map if $\phi(sm) = s\phi(m)$ for any $m \in M$ and $s \in S$ [resp. $\phi(sm) = s\phi(m)$] for any $m \in M$ and $s \in S$.

Result 1 ([5, Proposition 1.5]). *Let S be a semigroup and $A, B, C, D \in S$-Ens [Ens-S, S-Ens-S] such that $A \supset C$, $B \supset D$. Let α be a bijective S-map [(S,S)-map] : $C \longrightarrow D$. Then there exist $W \in S$-Ens [Ens-S, S-Ens-S] and injective S-maps [(S,S)-maps] $\beta : A \longrightarrow W$, $\lambda : B \longrightarrow W$ such that $\alpha\lambda = \beta$ on C, $W = A\beta \cup B\lambda$, $A\beta \cap B\lambda = C\beta$.*

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*This is an absrtact and the paper will appear elsewhere.

In this case, we say that the left S-set [-biset]W is the left S-set gluing C and D by ξ and write $W = C\#_\xi D$, where $\xi = \beta^{-1}\alpha : A \longrightarrow B$.

If A, B are generated by x and y respectively and $\xi(x) = y$, then we write $C\ \#_{x=y}D$ instead of $C\#_\xi D$.

Let Y a left S-set and $s_i, t_i \in S$, $y_i \in Y$ with $s_iy_i = t_iy_{i+1}$ for all $1 \le i \le n-1$. Then for any $1 \le i \le n-1$, we define left congruences $\rho(s_i)$, $\rho(t_i)$ on S^1 as follows :
$s\rho(s_i)$ [resp. $\rho(t_i)$ if and only if $ssiy_i = tt_iy_{i+1}$ in Y for all $s, t \in S$. Let $\overline{Y_i} = S/1_S \cup \rho(s_i) \cup \rho(t_i)$ and $\overline{y_i} = (1_S \cup \rho(s_i) \cup \rho(t_i))1$. $S^1/1 \cup \rho(s)\#_{s\rho(s)=t\rho(t)}S^1/1 \cup \rho(t)$. Then we can obtain a left S-set $\overline{Y}$

$$\overline{Y} = \overline{Y}_1\#_{s_1\overline{y}_1=t_1\overline{y}_2}\overline{Y}_2\cdots\#\overline{Y}_{i-1}\#_{s_i\overline{y}_i=t_i\overline{y}_{i+1}}\overline{Y}_{i+1}\#\cdots\#_{s_{n-1}\overline{y}_{n-1}=t_{n-1}\overline{y}_n}\overline{Y}_n.$$

Then we have the set A' of equations $s_i\overline{y}_i = t_{i+1}\overline{y}_{i+1}$ in $\overline{X}$ $(1 \le i \le n-1)$.
We call $\overline{Y}$ *relatively free* the relatively free left S-set associated to Y with respect to A.

2 The decision problem of amalgamation bases for all semigroups

A semigroup S is called an *amalgamation base* in the class of all semigroups (simply called a *semigroup amalgamation base*) if for any semigroups T_i $(i \in I)$ containing S as a subsemigroup the semigroup amalgam $[T_i\ (i \in I)\ ; S]$ is embedded into a semigroup.

This is a characterization of semigroup amalgamation bases.

Result 2 [4, Theorem 2.2]. *A semigroup S is a semigroup amalgamation base if and only if for each $X \in$ Ens-S, $Y \in S$-Ens and $N \in S$-Ens-S with $N \supset S^1$, the map : $X \otimes Y \longrightarrow X \otimes N \otimes Y$ $(x \otimes y \longrightarrow x \otimes 1 \otimes y)$ is injective.*

We recall Bulman-Flemming and MxDowell's characterization of equality in tenser product.

Result 3 [1, Lemma 1.2]. *Let $X \in$ Ens-S, $Y \in S$-Ens. Then $x \otimes y = x' \otimes y'$ in $X \otimes Y$ if and only if there exist $s_1, \cdots, s_n$, $t_1, \cdots, t_n \in S^1$, $x_1, \cdots, x_n \in X$ and $y_2, \cdots, y_n \in Y$*

such that

$$\begin{array}{rclcrcl} x &=& x_1s_1, && s_1y &=& t_1y_2 \\ x_1t_1 &=& x_2s_2, && s_2y_2 &=& t_2y_3 \\ &\vdots& && &\vdots& \\ x_{n-1}t_{n-1} &=& x_ns_n, && s_ny_n &=& t_ny' \\ x_nt_n &=& x' && && \end{array} \tag{1}$$

Then we call the system of equations (1) *a scheme of length n over X and Y joining* (x, y) *to* (x', y').

The main theorem . *The decision problem whether or not a finite semigroup is an amalgamation base for all semigroups is decidable.*

2.1 Schemes and Automata

We know that a semigroup S is is an amalgamation base for all semigroups if and only if so is the semigroup S^1 obtained from S by adjoining an identity element. So we assume that S is a monoid.

In this section we construct an automata associated to an equation on tensor product of a certain right S-set and certain a left S-set in order to complete the proof of Theorem 1.

Let S be a finite monoid.

I. Let $RC(S)$ be the set of all right congruences of S.

Let $\{(\xi, a) \mid \xi \in RC(S), a \in S\}$ be the sets of initial vertices and terminal vertices and $\{(\xi, a, b) \mid \xi \in RC(S), a, b \in S\}$ be the sets of vertices.

Edges are of the form $(\xi, a) \xrightarrow{\theta} (\xi', a', b')$, where θ is an S-isomorphism : $(\xi a)S \to (\xi' a')S$ with $\theta(\xi a) = \xi' a'$ or of the form $(\xi, a, b) \xrightarrow{\theta} (\xi', a', b')$, where θ is an S-isomorphism : $(\xi b)S \to (\xi' a')S$ with $\theta(\xi b) = \xi' a'$.

II. Let $LC(S)$ be the set of all left congruences of S.

Let $\{(\phi, u) \mid \phi \in LC(S), u \in S\}$ be the sets of initial vertices and terminal vertices and $\{(\phi, u, v) \mid \phi \in LC(S), u, v \in S\}$ be the sets of vertices.

Edges are of the form $(\phi, u) \xrightarrow{\theta} (\phi', u', v')$, where θ is an S-isomorphism : $(\phi u)S \to (\phi' u')S$ with $\theta(\xi u) = \xi' u'$ or of the form $(\phi, u, v) \xrightarrow{\theta} (\phi', u', v')$, where θ is an S-isomorphism : $S(\phi v) \to S(\phi' u')$ with $\theta(\xi v) = \xi' u'$.

III. $\{(\xi, a, b, m, \phi, u, v) \mid \xi \in RC(S), a, b \in S, m \in E({}_SS_S), \phi \in LC(S), u, v \in S\}$

To obtain all sechemes joining $(\xi_0, *, 1, 1, \rho_0, *, u)$ to $(\xi_0', 1, *, 1, \phi_0' u', *)$ over right S-sets, the bi-S-sets $E({}_SS_S)$ and left S-sets, we make a non-dterministic automaton $\mathcal{A}(\xi_0, 1, \phi_0, u,$

$E({}_SS_S), \xi_0', a', \rho_0', u')$ as follows :

Vertices are of the form $(\xi, a, b, m, \phi, u, v)$ where $\xi \in RC(S), a, b, u, v \in S, m \in E({}_SS_S), \phi \in LC(S)$,

$(\xi_0, *, 1, 1, \phi_0, *, v_0)$ is the initial vertix, $(\xi_0', 1, *, 1, \phi_0', u_0, *)$ is the terminal vertix, where $\xi_0, \xi_0' \in RC(S)$, $v \in S$ and $\phi_0, \phi_0' \in LC(S)$.

Edges are of the form

(1) $(\xi, a, b, m, \phi, u, v) \xrightarrow{\theta} (\xi', a', b', m', \phi, u, v)$, where $\xi, \xi' \in RC(S), a, a', b, b', u, v \in S, m, m' \in E({}_SS_S)$, $\phi, \phi \in LC(S)$, θ is an S-isomorphism : $(\xi b)S \to (\xi' a')S$ with $\theta(\xi b') = \xi' a'$ and there exists an element $z \in E({}_SS_S)$ with $m = bz$ and $m' = a'z$.

(2) $(\xi, a, b, m, \rho, u, v) \xrightarrow{\theta} (\xi, a, b, m', \rho', u', v')$, where $\xi, \xi' \in RC(S), a, b, u, u', v, v' \in S, m, m' \in E({}_SS_S)$, $\rho, \rho' \in LC(S)$, θ is an S-isomorphism : $S(\phi v) \to (\phi' u')S$ with $\theta(\phi v) = \phi' u'$ and there exists an element $w \in E({}_SS_S)$ with $m = wv$ and $m' = wu'$.

(3) $(\xi_0, *, 1, 1, \rho_0, *, v_0) \xrightarrow{\theta} (\xi, a, b, a, \rho_0, *, v_0)$, where $\xi, \in RC(S), v \in S, m \in E({}_SS_S), \phi, \phi \in LC(S)$, θ is an S-isomorphism : $(\xi_0 1)S \to (\xi a)S$ with $\theta(\xi_0 1) = \xi' a$, where and there exists an element $w \in E({}_SS_S)$ with $a = bw$. These edges are labelled by θ.

(4) $(\xi, a, b, m, \rho_0, *, v_0) \xrightarrow{\theta} (\xi', a', b', m', \rho_0, *, v_0)$, where $\xi, \xi' \in RC(S), a, b, a', b' \in S, m, m' \in E({}_SS_S)$, θ is an S-isomorphism : $(\xi b)S \to (\xi' a')S$ with $\theta(\xi b) = \xi' a'$, where and there exists an element $z \in E({}_SS_S)$ with $m = bz, m' = a'z$. These edges are labelled by θ.

(5) Edges $(\xi, a, b, b, \rho_0', u_0, *) \xrightarrow{\theta} (\xi', 1, *, 1, \rho_0', u_0, *)$ are with no label, where $\xi, \xi' \in RC(S)$, $a, b, u \in S$, $m \in E({}_SS_S)$ and there exist elements $z \in E({}_SS_S)$ with $b = aw$ and the edge is labelled by θ.

(6) $(\xi, a, b, m, \rho_0', u_0', *) \xrightarrow{\theta} (\xi', a', b', m', \rho_0', *, u_0', *)$, where $\xi, \xi' \in RC(S), a, b, a', b' \in S$, $m, m' \in E({}_SS_S)$, θ is an S-isomorphism : $(\xi b)S \to (\xi' a')S$ with $\theta(\xi b) = \xi' a'$, where and there exists an element $z \in E({}_SS_S)$ with $m = bz, m' = a'z$. These edges are labelled by θ.

We make an automaton $\mathcal{B}(\xi_0, 1, \phi_0, v, \xi_0', a', \rho_0', u')$ as follows :

Vertices are of the form (ξ, a, b, ϕ, u, v) where $\xi \in RC(S), a, b, u, v \in S, m \in E({}_SS_S), \phi \in LC(S)$,

$(\xi_0, *, 1, \phi_0, *, v)$ is the initial vertix, $(\xi_0', 1, *, \phi_0', u, *)$ is the terminal vertix, where $\xi, \xi' \in RC(S)$, $v \in S$ and $\phi_0, \phi_0' \in LC(S)$.

Edges are of the form : (1) $(\xi, a, b, \phi, u, v) \xrightarrow{\theta} (\xi', a', b', \phi, u, v)$, where $\xi, \xi' \in RC(S), a, a', b, b'$, $u, v \in S, \phi, \phi \in LC(S)$, θ is an S-isomorphism : $(\xi b)S \to (\xi' a')S$ with $\theta(\xi b') = \xi' a'$.

(2) $(\xi, a, b, \rho, u, v) \xrightarrow{\theta} (\xi, a, b, m', \rho', u', v')$, where $\xi, \xi' \in RC(S), a, b, u, u', v, v' \in S, m, m' \in$

$E({}_SS_S)$, $\rho, \rho' \in LC(S)$, θ is an S-isomorphism : $S(\phi v) \to (\phi' u')S$ with $\theta(\phi v) = \phi' u'$.

(3) $(\xi_0, *, 1, \rho_0, *, v_0) \xrightarrow{\theta} (\xi, a, b, a, \rho_0, *, v_0)$, where $\xi, \in RC(S), v \in S, \phi, \phi \in LC(S)$, θ is an S-isomorphism : $(\xi_0 1)S \to (\xi a)S$ with $\theta(\xi_0 1) = \xi' a$ and these edges are labelled by θ.

(4) $(\xi, a, b, \rho_0, *, v_0) \xrightarrow{\theta} (\xi', a', b', \rho_0, *, v_0)$, where $\xi, \xi' \in RC(S), a, b, a', b' \in S$, θ is an S-isomorphism : $(\xi b)S \to (\xi' a')S$ with $\theta(\xi b) = \xi' a'$.
These edges are labelled by θ.

(5) Edges $(\xi, a, b, \rho_0', u_0, *) \xrightarrow{\theta} (\xi', 1, *, 1, \rho_0', u_0, *)$ are with no label, where $\xi, \xi' \in RC(S)$, $a, b, u \in S$ and the edge is labelled by θ.

(6) $(\xi, a, b, \rho_0', u_0', *) \xrightarrow{\theta} (\xi', a', b', m', \rho_0', *, u_0', *)$, where $\xi, \xi' \in RC(S), a, b, a', b' \in S$, θ is an S-isomorphism : $(\xi b)S \to (\xi' a')S$ with $\theta(\xi b) = \xi' a'$. These edges are labelled by θ.

(7) $(\xi, a, b; x, \rho, u, v; y) \xrightarrow{\theta} (\xi', a', b'; x', \rho', u, v; y)$, where $\xi, \xi' \in RC(S), a, a', b, b', u, v \in S, \rho \in LC(S)$, $x \in (\xi b)S$, $x' \in (\xi' a')S$, $y \in S(\rho u) \cup S(\rho v)$, θ is an S-isomorphism : $(\xi b)S \to (\xi' a')S$ with $\theta(\xi b') = \xi' a'$ and $\theta(x) = x'$.

(8) $(\xi, a, b; x, \rho, u, v; y) \xrightarrow{\varepsilon} (\xi, a, b; x', \rho, u, v; y')$, where $\xi, \xi' \in RC(S), a, b, u, v \in S, \rho \in LC(S)$, $x, x' \in (\xi a)S \cup (\xi b)S$, , $y, y' \in S(\rho u) \cup S(\rho v)$ and there exists $s \in S$ with $x' = xs$, $y = sy'$.

Lemma 1. *Let S be a finite monoid. Let $E({}_SS_S)$ be the injective hull of the S-biset ${}_SS_S$. Let X be a right S-set with an element x, x' and Y be a left S-set with an element y, y'. Let ξ [resp. ξ'] be right congruences on S such that there exists an isomorphism ϕ [resp. ϕ'] of S/ξ to xS with $\theta(\xi 1) = x$ [resp. S/ξ' to $x'S$ with $\theta'(\xi' 1) = x'$]. Let ρ [resp. ρ'] left congruences on S such that there exists an isomorphism γ [resp. γ'] of S/ρ to xS with $\gamma(\rho 1) = y$ [resp. S/ρ' to $x'S$ with gamma$'(\rho 1) = y'$].*
Then $x \otimes 1 \otimes y' = x' \otimes 1 \otimes y'$ in $X \otimes E({}_SS_S) \otimes Y$ if and only if there exists an element $v, u \in S$ such that there exists a successful pass from the initial vertix $(\xi, 1, 1, \rho, *, v)$ to the terminal vertix $(\xi', 1, 1, \rho', u)$ on the automaton $\mathcal{A}(\xi, \xi', E({}_SS_S, \rho, \rho', v, u)$.

Lemma 2. *Let S be a finite semigroup. Let X be a right S-set with an element x, x' and Y be a left S-set with an element y, y'. Let ξ [resp. ξ'] be right congruences on S such that there exists an isomorphism ϕ [resp. ϕ'] of S/ξ to xS with $\theta(\xi 1) = x$ [resp. S/ξ' to $x'S$ with $\theta'(\xi' 1) = x'$]. Let ρ [resp. ρ'] left congruences on S such that there exists an isomorphism γ [resp. γ'] of S/ρ to xS with $\gamma(\rho 1) = y$ [resp. S/ρ' to $x'S$ with gamma$'(\rho 1) = y'$]. Let X be the right S-set associated with $\theta_0\theta_1\theta_2\cdots\theta_n$ and Y the left S-set associated with $\gamma_0\gamma_1\gamma_2\cdots\gamma_m$.*

Then $x \otimes y' = x' \otimes y'$ in $X \otimes Y$ if and only if there exists a successful pass with the label $(\theta_0\theta_1\theta_2 \cdots \theta_n,\ \gamma_0\gamma_1\gamma_2 \cdots \gamma_m)$ from the initial vertix $(\xi, 1, 1, \rho, v)$ to the terminal vertix $(\xi', 1, 1, \rho', u)$ on the automaton $\mathcal{B}(\xi, \xi', \rho, \rho', v, u)$.

Finally, by using Lemma 1 and Lemma 2, we can prove the main theorem.

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