Gröbner bases on projective bimodules and the Hochschild cohomology *

Part I. Rewriting on vector spaces

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In this paper we consider an algebra F based on a suitable well-ordered semigroup over a commutative ring K. We develop the theory of Gröbner bases on the algebra F as well as Gröbner bases on projective F-(bi)modules. We generalize the methods developed in [3] and [4] to construct projective bimodule resolutions of algebras and (bi)modules. It gives an effective way to compute the Hochschild cohomology of algebras and modules.

To discuss the three types of Gröbner bases above in a uniform way, we consider rewriting systems on free K-modules generated by a well-ordered set in Section 1. The results are applied to algebras based on well-ordered reflexive semigroups in Section 2, projective left modules in Section 3 and projective bimodules in Section 4.

1   Rewriting on K-spaces

Let K be a commutative ring with 1 and let $(X, \succ)$ be a well ordered set. Let $K \cdot X$ be the free K-module generated by X. An element f of $K \cdot X$ is uniquely written as a finite sum

$$f = \sum k_i x_i \tag{2.1}$$

with $k_i \in K \backslash \{0\}$ and $x_i \in X$, where x_i are different. For this f, if $x_j \succ x_i$ for all $j \neq i$, $k_i x_i$ is the *leading term* of f and is denoted by $\mathrm{lt}(f)$. Set $\mathrm{rt}(f) = f - \mathrm{lt}(f)$. We extend the order $\succ$ on X to a partial order $\succ$ on $K \cdot X$ denoted by the same symbol $\succ$ as follows: First, $f \succ 0$ for any $f \neq 0$. Let f and g be nonzero elements in $K \cdot X$ with the leading terms $k \cdot x$ and $\ell \cdot y$ with $k, \ell \in K$ and $x, y \in X$ respectively. If $x \succ y$ then $f \succ g$. If $x = y$, then $f \succ g$ if and only if $\mathrm{rt}(f) \succ \mathrm{rt}(g)$. Since $\succ$ is a well-order on X, $\succ$ is well-founded on $K \cdot X$, that is, there is no infinite sequence

$$f_1 \succ f_2 \succ \cdots \succ f_n \succ \cdots$$

in $K \cdot X$.

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*This is a preliminary report and the details will appear elsewhere.

Let $\mathcal{R}$ be a set of mappings $r : X \to 2^{(K \cdot X)}$ such that $r(x)$ is a finite subset of $K \cdot X$ and $x \succ t$ for any $x \in X$ and $t \in r(x)$. The couple $(X, \mathcal{R})$ is called a *set with rewriting structure* and an element $r \in \mathcal{R}$ is called a *rewriting rule* on $K \cdot X$. Let $r \in R$, $x \in X$ and $t \in r(x)$. We say that r is applied to x to get t and we write as $x \to t \in \mathcal{A}(r)$. More generally, for an element f of $K \cdot X$ written as (2.1) with $x_1 = x$, we have an element

$$g = k_1 \cdot t + \sum_{i \neq 1} k_i x_i = f - k_1(x - t)$$

of $K \cdot X$, which is called the element obtained from f by the application $x \to t$. In this situation we write

$$f \to_r g.$$

A *rewriting system* on $K \cdot X$ is a subset of $\mathcal{R}$. For $f, g \in K \cdot X$ if $f \to_r g$ for some $r \in R$, we write as

$$f \to_R g.$$

The relation $\to_R$ is called a *one-step reduction* by R. Let $\to_R^*$ and $\leftrightarrow_R^*$ denote the reflexive transitive closure and the reflexive symmetric transitive closure of $\to_R$, respectively.

Proposition 1.1. *Let R be a rewriting system on $K \cdot X$. For any $f, g, f', g' \in X$ and $k, \ell \in K$, if $f \leftrightarrow_R^* f'$ and $g \leftrightarrow_R^* g'$, then*

$$kf + \ell g \;\leftrightarrow_R^*\; kf' + \ell g'.$$

Set

$$L_0(R) = \left\{ f \in K \cdot X \mid f \leftrightarrow_R^* 0 \right\}.$$

Corollary 1.2. *$L_0(R)$ is a K-submodule of $K \cdot X$ and $\leftrightarrow_R^*$ is equal to the congruence modulo $L_0(R)$.*

Since $L_0(R)$ is a K-submodule of $K \cdot X$, we have the quotient module $M(R) = K \cdot X / L_0(R)$. Let $\eta_R : K \cdot X \to M(R)$ be the canonical surjection.

An element $x \in X$ is *R-irreducible*, if $r(x) = \emptyset$ for every $r \in R$, and the set of R-irreducible elements in X is denoted by $\mathrm{Irr}(R)$. An element f of $K \cdot X$ is *R-irreducible*, if no rule from R is applicable to f, that is, every element x_i in (2.1) is irreducible. An element which is not R-irreducible is *R-reducible*.

If $f \to_R g$, then we can see from the compatibility of $\succ$ that $f \succ g$. Hence the relation $\to_R$ is *noetherian*, that is, there is no infinite sequence

$$f_1 \to_R f_2 \to_R \cdots \to_R f_n \to_R \cdots .$$

Therefore we have

Proposition 1.3. *The one-step reduction $\to_R$ is noetherian, and for any $f \in K \cdot X$ there is an R-irreducible element $g \in K \cdot X$ such that $f \to_R^* g$.*

For $f, g \in K \cdot X$ if there is $h \in K \cdot X$ such that $f \to_R^* h$ and $g \to_R^* h$, we say $f \downarrow_R g$ holds. A system R is *confluent*, if $f \downarrow_R g$ holds for any $f, g, h \in K \cdot X$ such that $h \to_R^* f$ and $h \to_R^* g$. A noetherian and confluent system is called *complete* (see [1]), but a confluent system is complete in this paper because any system we consider is noetherian.

We state the fundamental results on complete systems.

Theorem 1.4. *Let R be a complete rewriting system on $K \cdot X$. Then, for any $f \in K \cdot X$, there is a unique R-irreducible element $\hat{f} \in K \cdot X$ such that $f \to_R^* \hat{f}$. For $f, g \in K \cdot X$ we have*

$$\hat{f} = \hat{g} \Leftrightarrow f \downarrow_R g \Leftrightarrow f \leftrightarrow_R^* g \Leftrightarrow f \equiv g \pmod{L_0(R)}.$$

In particular,

$$\hat{f} = 0 \Leftrightarrow f \to_R^* 0 \Leftrightarrow f \in L_0(R).$$

The element $\hat{f}$ in Theorem 1.4 is called the *normal form* of f.

Corollary 1.5. *If R is a complete rewriting system, then the surjection η_X is bijective on $\mathrm{Irr}(R)$, and the K-module $M(R) = K \cdot X / L_0(R)$ is a free K-module with base $\eta_R(\mathrm{Irr}(X))$. Any element of $M(R)$ is uniquely represented by the normal form in $K \cdot X$ with respect to R.*

Lemma 1.6. *For a rewriting system R on $K \cdot X$, $f - g \to_R^* 0$ implies $f \downarrow_R g$ for any $f, g \in K \cdot X$.*

Proposition 1.7. *For a rewriting system R on $K \cdot X$, the following conditions are equivalent.*

(1) R is complete.
(2) For any $r, r' \in R$, $x \in X$, $t \in r(x)$ and $t' \in r'(x)$, $t \downarrow_R t'$ holds.
(3) $f \to_R^* 0$ for all $f \in L_0(R)$.
(4) Any nonzero element in $L_0(R)$ is R-reducible.

For a rewriting rule r, we set

$$\mathrm{Dom}(r) = \{x \in X \mid r(x) \neq \emptyset\}.$$

A rule $r \in \mathcal{R}$ is *contained in* $\leftrightarrow_R^*$ if $x \leftrightarrow_R^* t$ for any $x \to t \in \mathcal{A}(r)$. For $r, r' \in \mathcal{R}$, r' *scoops* r, if for any $x \to t \in \mathcal{A}(r)$ there is $x \to t' \in \mathcal{A}(r')$ such that $t \succ t'$. A system R is *reduced* if

(i) For $r, r' \in R$, $\mathrm{Dom}(r) \subset \mathrm{Dom}(r')$ implies $r = r'$, and
(ii) No rule $r \in R$ is scooped by any rule $r' \in \mathcal{R}$ contained in $\leftrightarrow_R^*$.

Two systems R and R' on $K \cdot X$ are *equivalent* if they induce the same quotient, that is, $\leftrightarrow_R^* = \leftrightarrow_{R'}^*$.

Proposition 1.8. *For any complete rewriting system R there is a reduced complete system R' equivalent to R. If R is finite, so is R'.*

2 Rewriting on K-algebras

Let $S = B \cup \{0\}$ be a semigroup with zero element 0. S is called *reflexive* if for all $a \in B$ there are elements $e, f \in B$ such that $a = eaf$. Of course, if S has the identity element, it is reflexive. S is *well-ordered*, if B has a well-order $\succ$ which is *compatible* in the following sense:

   (i) $a \succ b, ca \neq 0, cb \neq 0 \Rightarrow ca \succ cb$, and
   (ii) $a \succ b, ac \neq 0, bc \neq 0 \Rightarrow ac \succ bc$,

for any $a, b, c, d \in B$.

Example 2.1. Let Γ be a quiver. Let B be the set of all paths in Γ. Then, $S = B \cup \{0\}$ is a reflexive semigroup with zero with the following operation $\circ$: For two paths p and q, $p \circ q$ is the path obtained by concatenating them at the end point v of p, if v coincides with the initial vertex of q, and $p \circ q = 0$ otherwise. We can define a compatible well-order $\succ$ on B as follows for example. Let $p, q \in B$. If $|p| > |q|$, then $p \succ q$, where $|p|$ and $|q|$ are the lengths of p and q respectively. If $|p| = |q|$, then $p \succ q$ if and only if p is greater than q in lexicographic order with respect to a linear order given beforehand on the vertices and the edges of Γ.

Example 2.2. Let $n \geq 2$ and let $N = \{1 \succ a \succ a^2 \succ \cdots \succ a^{n-1}, a^n = 0\}$. Then, N is a well-ordered reflexive semigroup with 0.

In the rest of this section $S = B \cup \{0\}$ is a well-ordered reflexive semigroup with 0. Let $E(B)$ denote the set of idempotents in B. The following lemmas are given in [2].

Lemma 2.3. *For any $a \in B$ there is a unique pair $(e, e') \in E(B) \times E(B)$ such that $a = eae'$.*

Lemma 2.4. *For $e, e' \in E(B)$, if $e \neq e'$, then $ee' = 0$.*

For a unique $(e, e') \in E(B) \times E(B)$ in Lemma 2.3, e and e' are called the *source* and the *terminal* of a and denoted by $\sigma(a)$ and $\tau(a)$, respectively. For $e, e' \in E(B)$, set

$$_eB = \{a \in B \mid \sigma(a) = e\} = e \cdot B \setminus \{0\},$$

$$B_{e'} = \{a \in B \mid \tau(a) = e'\} = B \cdot e' \setminus \{0\}$$

and

$$_eB_{e'} = \{a \in B \mid \sigma(a) = e, \tau(a) = e'\} = e \cdot B \cdot e' \setminus \{0\}.$$

The semigroup S is *normally ordered* if $a \succ b$ and $cad = 0$ imply $cbd = 0$ for any $e, e' \in E(B)$, $a, b \in {}_eB_{e'}$, $c \in B_e$ $d \in {}_{e'}B$. S is *coherent* if for any $a, b \in B$, $\tau(a) = \sigma(b)$ implies $ab \neq 0$. If S is coherent, it is normal. The semigroup S in Example 2.1 is coherent, but the semigroup N in Example 2.2 is not coherent though it is normal.

Let K be a commutative ring and $F = K \cdot B$ be the free K-module generated by B. Then, F has an algebra structure with the product induced from the semigroup operation of S. An element f of F is uniquely written as a finite sum

$$f = \sum_{i=1}^{n} k_i x_i \tag{2.1}$$

with $k_i \in K \setminus \{0\}$ and x_i are different elements in B. The element f is *uniform* if $\sigma(x_i) = \sigma(x_j)$ and $\tau(x_i) = \tau(x_j)$ for all i, j, and for this uniform f we define the source $\sigma(f) = \sigma(x_i)$ and the terminal $\tau(f) = \tau(x_i)$. Two uniform elements f and g are *parallel* and written as $f \| g$, if $\sigma(f) = \sigma(g)$ and $\tau(f) = \tau(g)$.

For $e, e' \in E(B)$, eF, Fe' and eFe' are the subalgebras of F spanned by $_eB$, $B_{e'}$ and $_eB_{e'}$ over K, respectively. We have

$$F = \bigoplus_{e \in E(B)} eF = \bigoplus_{e' \in E(B)} Fe' = \bigoplus_{e, e' \in E(B)} eFe'.$$

The well-order on B is extended to a well-founded partial order $\succ$ on F, and we can define the leading term $\mathrm{lt}(f)$ of $f \in F$ and the rest $\mathrm{rt}(f) = f - \mathrm{lt}(f)$ as we did in Section 1.

A rewriting rule on F is a pair $r = (u, v)$ with $u \in B$ and $v \in F$ such that $u \succ v$ and $u \| v$. If $x = x_1 u x_2$ in B, the rule r is applied to $x \in \Sigma^*$ to get $x_1 v x_2$. The rule r is written $u \to v$. Since $\succ$ is compatible, we have $x \succ x_1 v x_2$. Let $\mathcal{R}$ be the set of all rewriting rules on F, then the couple $(B, \mathcal{R})$ is a set with rewriting structure in the sense of Section 1.

A rule $r = u \to v$ is *normal* if $x \cdot u \cdot y = 0$ implies $x \cdot v \cdot y = 0$ for any $x, y \in B$. If S is normally ordered, any rule is normal. A rewriting system R on F is a (not necessarily finite) set of rewriting rules on F. R is *normal* if every rule in R is normal. If f has a nonzero term $k \cdot x$ and $x = x_1 u x_2$ with $x_1, x_2 \in B$ and $u \to v \in R$, then

$$f \to_R k x_1 (v - u) x_2 + f.$$

Set

$$I_0(R) = \{f \in F \mid f \leftrightarrow_R^* 0\}$$

and

$$G_R = \{u - v \mid u \to v \in R\}.$$

By Corollary 1.2, $I_0(R)$ is a K-submodule of F, but, in general, $I_0(R)$ is not an ideal of F and $\leftrightarrow_R^*$ is not the congruence modulo an ideal. To fill this gap, define

$$Z(R) = \{xvy \mid x, y \in B, u \to v \in R, xuy = 0\}.$$

Let $I(R)$ denote the (two-sided) ideal generated by G_R.

Proposition 2.5. *Let R be a rewriting system on F. If $Z(R) \subset I_0(R)$, then $I_0(R) = I(R)$, and $\leftrightarrow_R^*$ coincides with the congruence modulo $I(R)$.*

When R is normal, $Z(R) = \{0\}$, and the condition in Proposition 2.5 is satisfied. When $I_0(R) = I(R)$, we have the quotient algebra

$$A = F/ \leftrightarrow^*_R = F/I(R).$$

The set $\mathrm{Irr}(R)$ of R-irreducible elements in B is given by

$$\mathrm{Irr}(R) = B \setminus B \cdot \mathrm{Left}(R) \cdot B,$$

where $\mathrm{Left}(R) = \{u \mid u \to v \in R\}$, and $f \in F$ is irreducible if and only if f is a K-linear combination of irreducible elements.

Let I be an ideal of F and let $A = F/I$ be the quotient algebra. For $e, e' \in E(B)$, eA, Ae' and eAe' are the set of elements of A coming from elements of eF, Fe' and eFe', and are isomorphic to $eF/(I \cap eF)$, $Fe'/(I \cap Fe')$ and $eFe'/(I \cap eFe')$ as K-modules, respectively. We have

$$A = \bigoplus_{e \in E(B)} eA = \bigoplus_{e' \in E(B)} Ae' = \bigoplus_{e,e' \in E(B)} eAe'.$$

A set G of monic uniform elements of F is called a *Gröbner basis*, if the system

$$R_G = \{\mathrm{lt}(g) \to -\mathrm{rt}(g) \mid g \in G\}$$

associated with G is a complete rewriting system on F and $Z(R_G) \subset I_0(R_G)$. If G is a Gröbner basis, then $I_0(R_G)$ is equal to the ideal $I(G)$ of F generated by G by Proposition 2.5, so G is called a Gröbner basis of the ideal $I(G)$.

Proposition 2.6. *A set of monic uniform elements of an ideal I of F is a Gröbner basis of I if and only if $f \to^*_{R_G} 0$ for all $f \in I$.*

We confuse a Gröbner basis G with the associated rewriting system R_G. We write $g = u - v \in G$, implicitly assuming that $u = \mathrm{lt}(g)$ and $v = -\mathrm{rt}(g)$, and we just write $\to_G$ for the relation $\to_{R_G}$. We say $f \in F$ is *G-irreducible* if it is R_G-irreducible, and $\mathrm{Left}(G)$ and $\mathrm{Irr}(G)$ denote $\mathrm{Left}(R_G)$ and $\mathrm{Irr}(R_G)$ respectively.

In this situation Theorem 1.4 becomes

Theorem 2.7. *Let G be a Gröbner basis of an ideal I of F. Let $A = F/I$ be the quotient algebra of F by I and let $\rho : F \to A$ be the canonical surjection. Then, ρ is injective on $\mathrm{Irr}(G)$ and $\rho(\mathrm{Irr}(G))$ forms a free K-base of $A = F/I$. Any f has the unique normal form $\hat{f}$, and we have*

$$\hat{f} = \hat{g} \iff f \downarrow g \iff f \leftrightarrow^*_G g \iff f - g \to^*_G 0 \iff \rho(f) = \rho(g)$$

for any $f, g \in F$. In particular, we have

$$I = \{f \in F \mid \hat{f} = 0\} = \{f \in F \mid f \to^*_G 0\}.$$

It is easy to see that a complete rewriting system R is *reduced* in the sense in Section 1, if for any $r = u \to v \in R$, u and v are both $(R \setminus \{r\})$ irreducible. By Proposition 1.7 we see that for a complete rewriting system R on F there is a reduced complete system R' equivalent to R. Moreover, this R' is unique.

3 Rewriting on projective left modules

In this section, G is a reduced Gröbner basis of an ideal I of the algebra $F = KB$ based on a well-ordered reflexive semigroup $B \cup \{0\}$ over a commutative ring K, $A = F/I$ is the quotient algebra and $\rho : F \to A$ is the natural surjection.

A *left edged set* is a (possibly infinite) set X of elements ξ such that the source $\sigma(\xi) \in E(B)$ is assigned. For a nonempty left edged set X we set

$$F \cdot X = \bigoplus_{\xi \in X} F\sigma(\xi).$$

and

$$A \cdot X = \bigoplus_{\xi \in X} A\sigma(\xi).$$

Clearly, $F \cdot X$ is a left F-module and $A \cdot X$ is a left A-module. Moreover,

Proposition 3.1. $F \cdot X$ *is a projective F-module and $A \cdot X$ is a projective A-module.*

We call $F \cdot F$ and $A \cdot X$ the projective left F-module and the projective left A-module generated by X, respectively. $F \cdot X$ is the free K-module generated by the set $B \cdot X = \bigcup_{\xi \in X} B_{\sigma(\xi)}$ (disjoint union) with left F-action. An element $x \cdot \xi \in B_{\sigma(\xi)}$ with $\xi \in X$ and $x \in B_{\sigma(\xi)}$ is written as $x[\xi]$. Then, an element f of $F \cdot X$ is expressed as

$$f = \sum k_i x_i[\xi_i] \tag{3.1}$$

with $k_i \in K \setminus \{0\}$, $x_i \in B_{\sigma(\xi_i)}$ and $\xi_i \in X$. If (x_i, ξ_i) are different for i in (3.1), this expression is unique.

Let $\succ$ be a well-order on the set $B \cdot X$. We assume that it is left compatible, that is, for any $f = x[\xi] \in B_{\sigma(\xi)}$ and $f' = x'[\xi'] \in B_{\sigma(\xi')}$ and for any $a, b \in B$, $f \succ f'$, $af \neq 0$ and $af' \neq 0$ imply $af \succ af'$, and $a \succ b$ in B, $af \neq 0$ and $bf \neq 0$ imply $af \succ bf$. The order $\succ$ on $B \cdot X$ can be extended to a partial order $\succ$ on $F \cdot X$, and we can define the *leading term* $\mathrm{lt}(f)$ of $f \in F \cdot X$ and the rest $\mathrm{rt}(f) = f - \mathrm{lt}(f)$ as before. An element f written as (3.1) is *(left) uniform* if $\sigma(x_i) = \sigma(x_j) = e$ for all i, j. For this uniform f we define $\sigma(f) = e$.

A *rewriting rule* on $F \cdot X$ is a pair (s, t) with $s \in B \cdot X$ and $t \in F \cdot X$ such that $s \succ t$ and $s - t$ is uniform. Let $s = u[\xi]$ with $\xi \in X$ and $u \in B$. If $f \in F \cdot X$ has a term $k \cdot x[\xi]$ such that $x = x'u, x' \in B$, then we have a reduction $f \to_r f - k \cdot x'(u[\xi] - t)$ by an application of the rule $r = s \to t$. Let $\mathcal{T}$ is the set of all rewriting rules on $F \cdot X$.

A rule $r = u \to v \in \mathcal{R}$ on F is applied also to an element f of $F \cdot X$, if f has a term $k \cdot x[\xi]$ such that $x = x'ux''$ with $x', x'' \in B$. In this situation r is applied to f to get $g = f - kx'(u - v)x''[\xi]$ and we write $f \to_r g$. Thus, we have a couple $(B \cdot X, \mathcal{R} \cup \mathcal{T})$, which is a set of rewriting structure we discuss in this section.

Recall that a reduced Gröbner basis G on F is given and fixed. We write $f \to_G g$ if $f \to_r g$ for some $r \in R_G$. The relation $\to_G$ on $F \cdot X$ is complete,

because $\to_G$ is complete on F. So, any $f \in F \cdot X$ has the unique normal form $\hat{f}$ with respect to $\to_G$. An element f expressed as (3.1) is G-irreducible, if and only if every x_i is G-irreducible. Thus, we have

$$\hat{f} = \sum k_i \hat{x}_i [\xi_i].$$

Let T be a rewriting system on $F \cdot X$, that is, T is a subset of $\mathcal{T}$. Let $\to_{T,G} = \to_T \cup \to_G$ be the union of one-step reductions by T and G. Because $f \to_{T,G} g$ implies $f \succ g$ by the compatibility of $\succ$, $\to_{T,G}$ is a noetherian relation on $F \cdot X$. Let $\to_{T,G}^*$ and $\leftrightarrow_{T,G}^*$ be the reflexive transitive closure and the reflexive symmetric transitive closure of $\to_{T,G}$, respectively. Let

$$H = H_T = \{s - t \mid s \to t \in T\},$$

and let $L^\ell(T, G)$ be the submodule of $F \cdot X$ generated by $H \cup G \cdot B \cdot X$. Set

$$L_0^\ell(T, G) = \{f \in F \cdot X \mid f \leftrightarrow_{T,G}^* 0\}$$

and

$$Z^\ell(T) = \{xt \mid x \in B, s \to t \in T, xs = 0\}.$$

A rule $(s, t) \in \mathcal{T}$ is *normal* if $x \cdot s = 0$ implies $x \cdot t = 0$, and T is *normal* if every rule in it is normal. If $Z^\ell(T) \subset L_0(T, G)$, in particular, if T is normal, then $L_0^\ell(T, G)$ coincides with $L^\ell(T, G)$ and the relation $\leftrightarrow_{T,G}^*$ is equal to the left F-module congruence on $F \cdot X$ modulo $L^\ell(T, G)$;

$$f \leftrightarrow_{T,G}^* g \iff f \equiv g \pmod{L^\ell(T, G)}.$$

The quotient $M = M(T, G) = F \cdot X / \leftrightarrow_{T,G}^* = F \cdot X / L^\ell(T, G)$ is a left F-module, and actually, it is a left A-module in a natural way. Let $\eta_M : F \cdot X \to M$ be the natural surjection.

Considering the case $T = \emptyset$, the module $M(\emptyset, G)$ is isomorphic to $A \cdot X$, and we have a natural surjection $\rho_X = \eta_{A \cdot X} : F \cdot X \to A \cdot X$; $\rho_X(x[\xi]) = \rho(x)[\xi]$ for $x \in B_{\sigma(\xi)}$ and $\xi \in X$. For the quotient $M = M(T, G)$ above, we have a surjection $\overline{\eta}_M : A \cdot X \to M$ such that $\eta_M = \overline{\eta}_M \circ \rho_X$. Hence, $\mathrm{Ker}(\overline{\eta}_M) = \rho_X(L^\ell(T, G))$, which is denoted by $L_A^\ell(H)$, is the A-submodule of $A \cdot X$ generated by $\rho_X(H)$ and we have

$$M \cong A \cdot X / L_A^\ell(H).$$

If the system $T \cup R_G$ is complete (resp. reduced) on $F \cdot X$ in the sense of Section 2, we say T is *complete* (resp. *reduced*) modulo G. An element $f \in F \cdot X$ is (T, G)-*irreducible*, if no rule from $T \cup R_G$ is applied to f, otherwise f is (T, G)-*reducible*.

Let L be a left F-submodule of $F \cdot X$. A set H of monic (i.e. the coefficient of the leading term is 1) and left uniform elements in $F \cdot X$ is a *Gröbner basis* (modulo G) of L, if the associated system

$$T_H = \{\mathrm{lt}(f) \to -\mathrm{rt}(f) \mid f \in H\}$$

is a complete rewriting system on $F \cdot X$ modulo G and $L = L_0(T, G)$. It is also called a Gröbner basis for the A-submodule $\rho_X(L)$ of $A \cdot X$. We write $\to_{H,G}$ and $\to^*_{H,G}$ for $\to_{T_H,G}$ and $\to^*_{T_H,G}$ respectively. A $(\to_H, G)$-(ir)reducible element is called (H, G)-(ir)reducible. Similar to Theorem 2.7, we have

Theorem 3.2. *Let H be a Gröbner basis on $F \cdot X$ of a left F-submodule L of $F \cdot X$. Then, for any $f \in F \cdot X$, there is a unique (H, G)-irreducible element (the normal form of f) $\tilde{f} \in F \cdot X$ such that $f \to^*_{H,G} \tilde{f}$, and for any $f, g \in F \cdot X$,*

$$\tilde{f} = \tilde{g} \Leftrightarrow f \leftrightarrow^*_{T,G} g \Leftrightarrow f - g \to^*_{T,G} 0 \Leftrightarrow f \equiv g \pmod{L}.$$

If H is a Gröbner basis on $F \cdot X$ of L, then the quotient $F \cdot X/L$ is a left A-module, which is said to be defined by a pair (G, H) of Gröbner bases.

As easily seen, a complete rewriting system T is reduced modulo G if for any $s \to t \in T$, s and t are $(H \setminus \{s - t\}, G)$-irreducible. As before, if a left F-submodule of $F \cdot X$ has a Gröbner basis H modulo G, it has a unique reduced Gröbner basis H' modulo G on $F \cdot X$.

4 Rewriting on projective bimodules

In this section we treat bimodules over the algebra A. An A-bimodule is considered to be a left module over the enveloping algebra $A^e = A \otimes_K A^o$, where A^o is the opposite algebra of A, and we may apply the results in Section 3. However, if A is a quotient of an algebra F based on a well-ordered reflexive semigroup defined by a Gröbner basis G, then A^e is a quotient of an algebra based a larger semigroup and a Gröbner basis for A must be much larger than G. So, here we treat A-bimodules as they are.

An *edged set* is a set X of an element ξ such that the source $\sigma(\xi) \in E(B)$ and the terminal $\tau(\xi) \in E(B)$ of ξ are assigned. For a nonempty edged set X we consider the F-bimodule

$$F \cdot X \cdot F = \bigoplus_{\xi \in X} F\sigma(\xi) \times \tau(\xi)F$$

and the A-bimodule

$$A \cdot X \cdot A = \bigoplus_{\xi \in X} A\sigma(\xi) \times \tau(\xi)A.$$

The bimodule $F \cdot X \cdot F$ is the free K-module generated by

$$B \cdot X \cdot B = \bigcup_{\xi \in X} B_{\sigma(\xi)} \times_{\tau(\xi)} B \quad \text{(disjoint union)}$$

with two-sided F-action. An element (x, y) in $B_{\sigma(\xi)} \times_{\tau(\xi)} B$ with $x \in B_{\sigma(\xi)}$ and $y \in \tau(\xi)B$ is written as $x[\xi]y$. In particular, if $x = \sigma(\xi)$ (resp. $y = \tau(\xi)$), this element is simply written $[\xi]y$ (resp. $x[\xi]$). An element f of $F \cdot X \cdot F$ is uniquely written as

$$f = \sum k_i x_i[\xi_i]y_i, \tag{4.1}$$

with $k_i \in K \setminus \{0\}$, $x_i \in B_{\sigma(\xi_i)}$, $y_i \in {}_{\tau(\xi_i)}B$ and $\xi_i \in X$, where (x_i, ξ_i, y_i) are different.

Proposition 4.1. *$F \cdot X \cdot F$ is a projective F-bimodule and $A \cdot X \cdot A$ is a projective A-bimodule.*

Let $\succ$ be a well-order on the set $B \cdot X \cdot B$. We assume that it is compatible, that is, for any $f = x[\xi]y \in B_{\sigma(\xi)} \times {}_{\tau(\xi)}B$, $f' = x'[\xi']y' \in B_{\sigma(\xi')} \times {}_{\tau(\xi')}B$ and for any $a, b \in B$, $f \succ f'$, $afb \neq 0$ and $af'b \neq 0$ imply $afb \succ af'b$, and $a \succ a'$ in B, $af \neq 0$ and $bf \neq 0$ imply $af \succ a'f$, and $b \succ b'$ in B, $fb \neq 0$ and $fb' \neq 0$ imply $fb \succ fb'$. This order $\succ$ can be extended to a partial order $\succ$ on $F \cdot X \cdot F$ and the *leading term* $\mathrm{lt}(f)$ of $f \in F \cdot X \cdot F$ and the rest $\mathrm{rt}(f)$ are defined.

The element f in (5.1) is monic if the coefficient k_i of the leading term $k_i x_i[\xi_i]y_i$ is 1. If moreover $x_i = \sigma(\xi_i)$ (resp. $y_i = \tau(\xi_i)$), f is called *left* (resp. *right*) *very monic*. f is *uniform* if $\sigma(x_i) = \sigma(x_j) = e$ and $\tau(y_i) = \tau(y_j) = e'$ for all i, j. For this uniform f we define $\sigma(f) = e$ and $\tau(f) = e'$.

A *rewriting rule* on $F \cdot X \cdot F$ is a pair (s, t) with $s \in B \cdot X \cdot B$ and $t \in F \cdot X \cdot F$ such that $s \succ t$ and $s - t$ is uniform. If $f \in F \cdot X \cdot F$ has a term $k \cdot x[\xi]y$, $x = x'u$, $y = vy'$ and $s = u[\xi]v$, then $f \to_r f - k \cdot x'(u[\xi]v - t)y'$ by an application of the rule $r = s \to t$. Let $\mathcal{T}$ be the set of rewriting rules on $F \cdot X \cdot F$. A rule $r = u \to v \in \mathcal{R}$ on F is applied also to an element $f \in F \cdot X \cdot F$ with a term $k \cdot x[\xi]y$ such that x or y are G-reducible, that is, $x = x'ux''$ or $y = y'uy''$. In the former case, $f \to_r f - k \cdot x'(u - v)x''[\xi]y$, and in the latter, $f \to_r f - k \cdot x[\xi]y'(u - v)y''$. Again, $(B \cdot X \cdot B, \mathcal{R} \cup \mathcal{T})$ forms a set with rewriting structure. A *normal* rule and a *normal* rewriting system are defined in a similar way to Section 3.

G is continued to be a reduced Gröbner basis of an ideal I of F, and $A = F/I$ is the quotient algebra. The relation $\to_G$ on $F \cdot X \cdot F$ is complete, and any $f \in F \cdot X \cdot F$ has the unique normal form $\hat{f}$ with respect to $\to_G$. An element f written as (5.1) is G-irreducible, if and only if every x_i and y_i are G-irreducible, and we have

$$\hat{f} = \sum k_i \hat{x}_i[\xi_i]\hat{y}_i.$$

An element f of the projective A-bimodule $A \cdot X \cdot A$ generated by X is written as a finite sum $f = \sum x_i[\xi_i]y_i$ with $\xi \in X$, $x_i \in A_{\sigma(\xi)}$ and $y_i \in A_{\tau(\xi)}$. We have a morphism $\rho_X : F \cdot X \cdot F \to A \cdot X \cdot A$ of K-modules defined by

$$\rho_X(x[\xi]y) = \rho(x)[\xi]\rho(y)$$

for $x \in B_{\sigma(\xi)}$, $y \in {}_{\tau(\xi)}B$ and $\xi \in X$. In fact, ρ_X is a morphism of F-bimodules.

Let T be a subset of $\mathcal{T}$, which we call a *rewriting system* on $F \cdot X \cdot F$. Let $\to_{T,G} = \to_T \cup \to_G$, then $\to_{T,G}$ is a noetherian relation on $F \cdot X \cdot F$. Let $\to_{T,G}^*$ and $\leftrightarrow_{T,G}^*$ be the reflexive transitive closure and the reflexive symmetric transitive closure of $\to_{T,G}$, respectively. If the relation $\to_{T,G}$ is complete on $F \cdot X \cdot F$, we say T is *complete modulo G*. An element $f \in F \cdot X \cdot F$ is (T, G)-*irreducible*, if no rule from $T \cup R_G$ is applied to f, otherwise, f is (T, G)-*reducible*.

Let

$$L_0(T, G) = \{f \in F \cdot X \cdot F \mid f \to_{T,G}^* 0\}.$$

and

$$Z(T) = \{xty \mid x, y \in B, s \to t \in T, xsy = 0\}.$$

A set H of monic uniform elements of $F \cdot X \cdot F$ is a *Gröbner basis* (modulo G) of a F-subbimodule L of $F \cdot X \cdot F$, if the associated system $T_H = \{ \mathrm{lt}(f) \to -\mathrm{rt}(f) \mid f \in H \}$ is a complete rewriting system on $F \cdot X \cdot F$ modulo G and $L = L_0(T, G)$. It is also called a Gröbner basis for the A-subbimodule $\rho_X(L)$ of $A \cdot X \cdot A$. We write $\to_{H,G}$ and $\to^*_{H,G}$ for $\to_{T_H,G}$ and $\to^*_{T_H,G}$ respectively. A $(\to_H, G)$-(ir)reducible element is called (H, G)-(ir)reducible.

Theorem 4.2. *If H is a Gröbner basis of L on $F \cdot X \cdot F$ modulo G, then for any $f \in F \cdot X \cdot F$, there is a unique (H, G)- irreducible element (the normal form of f) $\tilde{f} \in F \cdot X \cdot F$ such that $f \to^*_{H,G} \tilde{f}$. For any $f, g \in F \cdot X \cdot F$ we have*

$$\tilde{f} = \tilde{g} \Leftrightarrow f \leftrightarrow^*_{H,G} g \Leftrightarrow f - g \to^*_{H,G} 0 \Leftrightarrow f \equiv g \pmod{L}.$$

If H is a Gröbner basis of L modulo G, the quotient $M = M(H, G) = (F \cdot X \cdot F)/L$ is an A-bimodule and is called the A-bimodule defined by a pair (G, H) of Gröbner bases. Let $\eta : F \cdot X \cdot F \to M$ be the natural surjection. Since M is an A-bimodule, we have a surjection $\bar{\eta} : A \cdot X \cdot A \to M$ with $\eta = \bar{\eta} \circ \rho_X$. Hence, $\mathrm{Ker}(\bar{\eta}) = \rho_X(L)$, which is denoted by $L_A(H)$, is the A-subbimodule of $A \cdot X \cdot A$ generated by $\rho_X(H)$, and we have $M \cong (A \cdot X \cdot A)/L_A(H)$.

A rewriting system T on $F \cdot X \cdot F$ (and $H = H_T$) is *left* (resp. *right*) very monic if the left-hand side of each rule of T is left (resp. right) very monic, that is, every rule of T is of the form $[\xi]x \to t$ (resp. $x[\xi] \to t$) with $\xi \in X$, $x \in {}_{\tau(\xi)}\Sigma^*$ and $t \in F$. T is *unifoliate* if every rule in T is left or right very monic. T (and H) is *reduced modulo G* if for any $s \to t \in T$, s and t are $(H \setminus \{s - t\}, G)$-irreducible. We have

Proposition 4.3. *If an F-subimodule L has a Gröbner basis H on $F \cdot X \cdot F$ modulo G, it has a unique reduced Gröbner basis H' on $F \cdot X \cdot F$ modulo G. If H is left very monic (resp. unifoliate, finite), H' is left very monic (resp. unifoliate, finite).*

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