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SKEW POLYNOMIAL RINGS OVER GENERALIZED GCD DOMAINS

島根大学・総合理工学部 植田 玲 (Akira Ueda)
Department of Mathematics, Shimane University
Matsue, Shimane, 690-8504, Japan

Abstract. A ring R is said to be a **right generalized GCD** domain if any finitely generated right v-ideal is a projective generator of the category Mod-R of right R-modules. A skew polynomial ring $D[x,\sigma]$ over a commutative generalized GCD domain D is a right generalized GCD domain, where σ is an aoutomorphism with finite order.

1 Preliminaries

At first, we introduce some elementary notions and notations. We refer to [MR] and [MMU] for details about orders and v-ideals.

Throughout this note, let R be an order in a divisin ring Q, that is, any non-zero element of R has its inverse in Q, and for any element q of Q, there exist a, $b \in R$ and non-zero s, $t \in R$ such that $q = as^{-1} = t^{-1}b$.

A non-zero right R-submodule I of Q is called a **right** R**-ideal** if there exists a non-zero element a of Q such that $aI \subseteq R$. Similarly, a **left** R**-ideal** of Q is a non-zero left R-submodule J of Q with $Ja \subseteq R$ for some non-zero element a of Q.

For any subsets A and B of Q, let

$$(A : B)_l = \{q \in Q \mid qB \subseteq A\}$$

and

$$(A : B)_r = \{q \in Q \mid Bq \subseteq A\}.$$

If I is is a right R-ideal of Q, then $(R : I)_l$ is a left R-ideal. $(R : I)_r$ is a right R-ideal if J is a left R-ideal of Q.

For a right R-ideal I of Q, we set

$$I_v = (R : (R : I)_l)_r.$$

Clearly we have $I \subseteq I_v$, and I is called a **right v-ideal** if $I = I_v$. Furthermore, a right R-ideal I is said to be a **finitely generated v-ideal** if there exist finitely many elements

This is an abstract and the paper will appear elsewhere.

$a_1, \cdots, a_k$ ($\in I$) such that $I = (a_1R + \cdots + a_kR)_v$. Similarly, we set

$$_vJ = (R : (R : J)_r)_l$$

for a left R-ideal J of Q. J is called a **left v-ideal** if $J = {}_vJ$, and J is said to be a **finitely generated left v-ideal** if $J = {}_v(Ra_1 + \cdots + Ra_k)$ for some finitely many elements $a_1, \cdots, a_k$ of J.

For a right R-ideal I of Q, we put

$$O_r(I) = (I : I)_r = \{q \in Q \mid Iq \subseteq I\}.$$

$O_r(I)$ is called the **right order** of I. In fact, $O_r(I)$ is an order in Q. We define similarly the **left order** $O_l(I)$ of I:

$$O_l(I) = (I : I)_l = \{q \in Q \mid qI \subseteq I\},$$

and $O_l(I)$ is also an order in Q.

A right R-module M is called a **generator** of the category Mod-R of right R-modules if $\sum_{f \in \mathrm{Hom}_R(M,\, R)} f(M) = R$. We note that, for a right R-ideal I of Q, I is a generator of Mod-R if and only if $(R : I)_l I = R$. Furthermore, if I is a generator of Mod-R, then $O_l(I) = R$ (cf. Lemma 1.4 of [MMU]).

A right R-module M is said to be a **progenerator** of Mod-R if M is a finitely generated projective R-module and a generator. Note that a right R-ideal I of Q is projective if and only if $I(R : I)_l = O_l(I)$. If I is projective, then I is finitely generated as a right R-module and $I_v = I$ (cf. Lemma 1.5 of [MMU]).

2 Right generalized GCD domains

A commutative domain is called a GCD domain if any non-zero two elements have the greatest common divisor. In a commutative domaim D, the greatest common divisor d of elements a and b is characterized to be the element such that

$$dD = \bigcap_{eD \supseteq aD+bD} eD.$$

By Proposition 1.8 of [MMU], we have

$$\bigcap_{eD \supseteq aD+bD} eD = (aD + bD)_v.$$

Hence d is the greatest common divisor of a and b if and only if $dD = (aD + bD)_v$. Thus a domain is GCD if and only if any finitely generated v-ideal is principal.

Now, a principal ideal dD is clearly an invertible ideal, that is, $(dD)(dD)^{-1} = D$, where $(dD)^{-1} = \{q \in F \mid q(dD) \subseteq D\}$ and F is the quotient field of D. So, the notion of a GCD domain is naturally extended to that of a generalized GCD domain, that is, a commutative domain D is called a generalized GCD domain if any finitely generated v-ideal of D is invertible (cf. [FHP] Chapter VI).

By the way, the polynomial ring $D[x]$ over a generalized GCD domain D is also a generalized GCD domain (cf. Theorem 6.2.3 of [FHP]). Then, what is a skew polynomial ring over a generalized GCD domain, or what is an Ore extension over a generalized GCD domain?

From these point of view, we define a non-commutative generalized GCD domain as follows: Let R be an order in a division ring Q. If any finitely generated right v-ideal of Q is a progenerator of Mod-R, then we call R a **right generalised GCD** domain (a **right G-GCD** domain for short), that is, R is G-GCD if

1. $(R : I)_l I = R$, and

2. $I(R : I)_l = O_l(I)$.

for any finitely generated right v-ideal I of Q. We note that a right Püfer order in Q is a right G-GCD domain (cf. [MMU]).

Now we have the following characterization of right G-GCD domains.

Theorem 2.1 *Let R be an order in a division ring Q. Then the following are equivalent:*

(1) *R is a right G-GCD domain.*

(2) *For any non-zero elements a_1 and a_2 of R, the left R-ideal $Ra_1 \cap Ra_2$ is a progenerator of the category R-Mod of left R-module.*

(3) *For any left R-ideals J_1 and J_2 which are progenerator of R-Mod, $J_1 \cap J_2$ is also a progenerator of R-Mod.*

3 Skew polynomial rings over generalized GCD domains

Let D be a commutative domain and let σ be an automorphism of D. Then we can define the skew polynomial ring $D[x, \sigma]$ over D with multiplication $xa = \sigma(a)x$, where $a \in D$. Since $D[x, \sigma]$ is a prime Goldie ring, $D[x, \sigma]$ has the quotient division ring Q.

We say that an automorphism σ of D has a **finite order** if $\sigma^k = \mathrm{id}_D$ for some positive integer k, where id_D is the identity mapping of D.

Then we have the following.

Theorem 3.1 *Let D be a commutative generalized GCD domain and let σ be an automorphism of D with finite order. Then the skew polynomial ring $D[x, \sigma]$ is a right G-GCD domain.*

In particular, by Theorem 3.1, a skew polynomial ring over a commutative Prüfer domain is a right G-GCD domain. We note that the case of automorphisms with infinite order is an open promlem. Also we don't know whether an Ore extension of a G-GCD domain is right G-GCD or not.

References

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