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An Intermediate Solution for Transferable Utility Games*

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Abstract. We try to define the Shapley value alternatively by using averages of excesses. We make a quantity (h-excess) after we combine usual excesses with these averages of excesses. Then we define an intermediate solution by using h-excesses. This solution exists for every game and consists of one point.

Keywords: cooperative game, solution, Shapley value, nucleolus.

1. Introduction and Preliminaries

In a society, suppose that a project team must decide a distribution of profits among the members of the team. Affected by social, economic and ethical situations, on one occasion it may adopt one criterion for distribution, and on another occasion, it may adopt another. In this society, on average, with some probability, each criterion is adopted. Translating this situation into statements in cooperative games (transferable utility games, or TU games), on one occasion, one solution is adopted and on another occasion, another solution is adopted.

In this note, we consider two solutions in TU games, that is, the Shapley value and the (pre-)nucleolus. We imagine a society where with some probability (say, $1-\alpha$), the Shapley value is adopted as a distribution of profits and with the remaining probability (α), the prenucleolus is adopted. A way for defining an intermediate solution is simply to make a convex combination of the Shapley value with the prenucleolus. However, we do define in another way.

In TU games, the excess of each coalition to some payoff vector is defined as an expression of dissatisfaction of this coalition to that vector. The (pre-)nucleolus is defined by using excesses for all coalitions. In this note, first we try to define the Shapley value alternatively by using averages of excesses (Theorem 1). We make a quantity (h-excess) after we combine usual excesses with these averages of excesses. Then we define an intermediate solution by using h-excesses. This solution exists for every game and consists of one point (Theorem 2).

The purpose of this note is to give a bridge between two single-valued solutions, that is, the Shapley value and the (pre-)nucleolus. In this note, we call this solution $h-$prenucleolus, for the sake of convenience, because the definition of it is similar to that of the nucleolus. Alternatively we could call this an extended Shapley value. It is a problem to define this solution alternatively by using incremental contributions $v(S)-v(S\setminus\{i\}), S\subseteq N, i\in S$. Also

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we may define another intermediate solution (say, an extended Shapley value) by combining the Shapley value with the (pre-)nucleolus, via incremental contributions.

A (cooperative) *game* (with side payments) is a pair (N, v), where $N = \{1, \ldots, n\}$ is a finite set of *players* and v is a real-valued function defined on 2^N with $v(\emptyset) = 0$, called the *characteristic-function* of the game. An element of 2^N is called a *coalition*. We let $\Sigma^0 \equiv 2^N \setminus \{\emptyset, N\}$. For any set Z, $|Z|$ denotes the cardinality of Z. For a coalition S, R^S is the $|S|$-dimensional product space $R^{|S|}$ with coordinates indexed by players in S. The i-th component of $x \in R^S$ is denoted by x_i. For $S \subseteq T \subseteq N$ and $x \in R^T$, $x|S$ means the projection of x to R^S. For $x \in R^N$, we let $x(S) = \sum_{i \in S} x_i$ (if $S \neq \emptyset$), and $x(\emptyset) = 0$. A *pre-imputation* for a game v is a vector $x \in R^N$ that satisfies

$$x(N) = v(N). \tag{1}$$

$X^*(v)$ is the set of all pre-imputations. A function ϕ which associates a set $\phi(v) \subseteq X^*(v)$ to every game v is called a *solution*. When $\phi(v)$ consists of one element, we call the unique element itself the solution (point). The Shapley value is a solution consisting of one element $\varphi(v)$ defined by

$$\varphi_i(v) = \sum_{S: i \in S} \gamma_n(S)\{v(S) - v(S \setminus \{i\})\},$$

where

$$\gamma_n(S) = \frac{(n - |S|)!(|S| - 1)!}{n!}.$$

For a game v, a pre-imputation $x \in X^*(v)$, and $S \in \Sigma^0$, the quantity $e(S, x) = e(S, x, v) \equiv v(S) - x(S)$ is called the *excess* of S at x, which is considered as an expression of dissatisfaction of S to x. Let $X \subseteq R^N$, and let $H = (h_i)_{i \in D}$ be a finite sequence of real-valued functions defined on X. Let $d = |D|$. For $x \in X$, let $\theta(x) = (\theta^1(x), \ldots, \theta^d(x))$ be the vector in R^d whose components are the numbers $(h_i(x))_{i \in D}$ arranged in non-increasing order, that is,

$$\theta : X \to R^d, \theta^t(x) = \max_{T \subseteq D, |T| = t} \min_{i \in T} h_i(x), \forall t = 1, \ldots, d.$$

Let $\geq_{lex}$ denote the lexicographical ordering of R^d; that is, $x \geq_{lex} y$, where $x, y \in R^d$, if either $x = y$ or there is $1 \leq t \leq d$ such that $x^j = y^j$ for $1 \leq j < t$ and $x^t > y^t$.

Definition. The nucleolus of H with respect to X is defined by

$$\mathcal{N}(H, X) = \{x \in X | \theta(y) \geq_{lex} \theta(x), \forall y \in X\}.$$

Definition. Let (N, v) be a game, let $X \subseteq R^N$, and let

$$H = (e(S, \bullet, v))_{S \in 2^N}.$$

$\mathcal{N}(H, X)$ is the nucleolus of (N, v) with respect to X and denoted by $\mathcal{N}(N, v, X)$. When $X = I(N, v)$, it is called the nucleolus of (N, v) and denoted by $\mathcal{N}(N, v)$. When $X = X^*(N, v)$, it is called the prenucleolus of (N, v) and denoted by $\mathcal{PN}(N, v)$.

2. The Shapley Value

In this section we express the Shapley value as a solution of a linear programming problem.

Theorem 1. Let v be a game. The Shapley value is the unique solution of the following linear programming problem:

$$(P) \qquad \begin{aligned} &\text{Minimize } K \\ &\text{s.t.} \\ &\sum_{S:i\in S, S\neq N} \gamma_{n-1}(S)e(S,x) \leq K, \quad \forall i \in N, \\ &x(N) = v(N). \end{aligned}$$

The minimum value of this problem (P) is

$$K^* = \sum_{S\neq N} \frac{|S|}{n-|S|}\gamma_n(S)v(S) - \frac{v(N)}{n}\sum_{s=1}^{n-1}\frac{s}{n-s}.$$

Lemma 1A. A pre-imputation $x \in X^*(v)$ is the Shapley value if and only if

$$\sum_{S:i\in S} \gamma_n(S)\{e(S,x) - e(S\setminus\{i\},x)\} = 0, \forall i \in N. \tag{2}$$

Proof: If $x \in X^*(v)$ is the Shapley value, by definition we have

$$x_i = \sum_{S:i\in S} \gamma_n(S)\{v(S) - v(S\setminus\{i\})\}, \forall i \in N. \tag{3}$$

For every $T \subseteq N$, $v(T) = e(T,x) + x(T)$. From this and (3), we have

$$x_i = \sum_{S:i\in S} \gamma_n(S)\{e(S,x) - e(S\setminus\{i\},x) + x_i\}, \forall i \in N. \tag{4}$$

Noting $\sum_{S:i\in S}\gamma_n(S) = 1, \forall i \in N$, we have the relation (2).

Conversely if $x \in X^*(v)$ satisfies (2), it satisfies (4), which implies x is the Shapley value. □

Lemma 1B. For a game v, let $x \in X^*(v)$. x is the Shapley value, that is, $x = \varphi(v)$ if and only if

$$\sum_{S:i\in S, j\notin S} \gamma_{n-1}(S)e(S,x) = \sum_{S:j\in S, i\notin S} \gamma_{n-1}(S)e(S,x), \forall i,j \in N, i \neq j. \tag{5}$$

Proof: Let $x \in X^*(v)$ be the Shapley value. By Lemma 1A,

$$\sum_{S:i\in S} \gamma_n(S)e(S,x) = \sum_{S:i\in S} \gamma_n(S)e(S\setminus\{i\},x). \tag{6}$$

The left hand side of (6) becomes

$$\sum_{S:i\in S} \gamma_n(S)e(S,x) = \sum_{S:i\in S, j\notin S} \gamma_n(S)e(S,x) + \sum_{S:i,j\in S} \gamma_n(S)e(S,x). \tag{7}$$

The right hand side of (6) becomes

$$
\begin{aligned}
\sum_{S:i\in S}\gamma_n(S)e(S\setminus\{i\},x) &= \sum_{S:i\notin S}\gamma_n(S\cup\{i\})e(S,x)\\
&= \sum_{S:j\in S,i\notin S}\gamma_n(S\cup\{i\})e(S,x)+\sum_{S:j\notin S,i\notin S}\gamma_n(S\cup\{i\})e(S,x).
\end{aligned}
\tag{8}
$$

From (6) – (8),

$$
\sum_{S:i\in S,j\notin S}\gamma_n(S)e(S,x)+\sum_{S:i,j\in S}\gamma_n(S)e(S,x)=\sum_{S:j\in S,i\notin S}\gamma_n(S\cup\{i\})e(S,x)+\sum_{S:j\notin S,i\notin S}\gamma_n(S\cup\{i\})e(S,x). \tag{9}
$$

Replacing i with j and j with i, we have

$$
\sum_{S:j\in S,i\notin S}\gamma_n(S)e(S,x)+\sum_{S:i,j\in S}\gamma_n(S)e(S,x)=\sum_{S:i\in S,j\notin S}\gamma_n(S\cup\{j\})e(S,x)+\sum_{S:i\notin S,j\notin S}\gamma_n(S\cup\{j\})e(S,x). \tag{10}
$$

Noting $\gamma_n(S\cup\{i\})=\gamma_n(S\cup\{j\})$, from (9) and (10),

$$
\sum_{S:i\in S,j\notin S}\gamma_n(S)e(S,x)-\sum_{S:j\in S,i\notin S}\gamma_n(S)e(S,x)=\sum_{S:j\in S,i\notin S}\gamma_n(S\cup\{i\})e(S,x)-\sum_{S:i\in S,j\notin S}\gamma_n(S\cup\{j\})e(S,x). \tag{11}
$$

Noting $\gamma_n(S)+\gamma_n(S\cup\{i\})=\gamma_{n-1}(S)$, we have the relation (5).

Conversely if the relation (5) holds, then the relation (11) holds, which implies

$$
\begin{aligned}
\sum_{S:i\in S}\gamma_n(S)e(S,x)-\sum_{S:j\in S}\gamma_n(S)e(S,x)
&=\sum_{S:i\notin S}\gamma_n(S\cup\{i\})e(S,x)-\sum_{S:j\notin S}\gamma_n(S\cup\{j\})e(S,x)\\
&=\sum_{S:i\in S}\gamma_n(S)e(S\setminus\{i\},x)-\sum_{S:j\in S}\gamma_n(S)e(S\setminus\{j\},x).
\end{aligned}
$$

Hence

$$
\sum_{S:i\in S}\gamma_n(S)\{e(S,x)-e(S\setminus\{i\},x)\}=\sum_{S:j\in S}\gamma_n(S)\{e(S,x)-e(S\setminus\{j\},x)\}.
$$

That is,

$$
\sum_{S:i\in S}\gamma_n(S)\{v(S)-v(S\setminus\{i\})-x_i\}=\sum_{S:j\in S}\gamma_n(S)\{v(S)-v(S\setminus\{j\})-x_j\}.
$$

This implies

$$
\varphi_i(v)-x_i=\varphi_j(v)-x_j\equiv\alpha,\forall i,j,i\neq j.
$$

Adding for all $i\in N$, we have

$$
n\alpha=\sum_{i\in N}\{\phi_i(v)-x_i\}=v(N)-v(N)=0.
$$

Hence $\alpha=0$, which means $x=\varphi(v)$. □

Lemma 1C. For a game v,let $x\in X^*(v)$. x is the Shapley value, that is, $x=\varphi(v)$ if and only if

$$
\sum_{S:i\in S,S\neq N}\gamma_{n-1}(S)e(S,x)=\sum_{S:j\in S,S\neq N}\gamma_{n-1}(S)e(S,x),\forall i,j,i\neq j. \tag{12}
$$

Proof: The relation (12) is obtained by adding

$$\sum_{i,j\in S, S\neq N} \gamma_{n-1}(S)e(S,x)$$

to both sides of the relation (5). □

Proof of the theorem:

$$\begin{aligned}
\sum_{S:i\in S,S\neq N} \gamma_{n-1}(S)x(S) &= x_i \sum_{S:i\in S,S\neq N} \gamma_{n-1}(S) + \sum_{j\neq i} x_j \sum_{S:i,j\in S,S\neq N} \gamma_{n-1}(S) \\
&= x_i \sum_{s=1}^{n-1} \binom{n-1}{s-1} \gamma_{n-1}(S) + \sum_{j\neq i} x_j \sum_{s=2}^{n-1} \binom{n-2}{s-2} \gamma_{n-1}(S) \\
&= x_i \sum_{s=1}^{n-1} \frac{1}{n-s} + x(N\setminus\{i\}) \sum_{s=2}^{n-1} \frac{s-1}{(n-1)(n-s)} \\
&= x_i \{ \sum_{s=1}^{n-1} \frac{1}{n-s} - \sum_{s=2}^{n-1} \frac{s-1}{(n-1)(n-s)} \} + v(N) \sum_{s=2}^{n-1} \frac{s-1}{(n-1)(n-s)} \\
&= x_i + v(N) \sum_{s=2}^{n-1} \frac{s-1}{(n-1)(n-s)}.
\end{aligned}$$

Hence

$$\begin{aligned}
\sum_{i\in N} \sum_{S:i\in S,S\neq N} \gamma_{n-1}(S)e(S,x) &= \sum_{i\in N} \sum_{S:i\in S,S\neq N} \gamma_{n-1}(S)v(S) - \sum_{i\in N} \{ x_i + v(N) \sum_{s=2}^{n-1} \frac{s-1}{(n-1)(n-s)} \} \\
&= \sum_{S\neq N} |S| \gamma_{n-1}(S)v(S) - v(N) - nv(N) \sum_{s=2}^{n-1} \frac{s-1}{(n-1)(n-s)} \qquad (13) \\
&= \sum_{S\neq N} \frac{n|S|}{n-|S|} \gamma_n(S)v(S) - v(N) \sum_{s=1}^{n-1} \frac{s}{n-s} \\
&= nK^*.
\end{aligned}$$

From this and the inequality constraint of the problem (P),

$$nK \geq \sum_{i\in N} \sum_{S:i\in S,S\neq N} \gamma_{n-1}(S)e(S,x) = nK^*.$$

Hence $K \geq K^*$. If x is the Shapley value, by Lemma 1C, we can let

$$L \equiv \sum_{S:i\in S,S\neq N} \gamma_{n-1}(S)e(S,x)$$

for all $i \in N$. From this and (13) we have $nL = nK^*$, or $L = K^*$, which means that the Shapley value is an optimal solution for the problem (P). Next suppose $y \in X^*(v)$ is an optimal solution. Then

$$\sum_{S:i\in S,S\neq N} \gamma_{n-1}(S)e(S,y) \leq K^*, \forall i \in N.$$

Adding these together, we have $nK^* \leq nK^*$ by (13). Hence

$$\sum_{S:i\in S,S\neq N} \gamma_{n-1}(S)e(S,y) = K^*, \forall i \in N.$$

From this and Lemma 1C, y is the Shapley value. □

Example 1. When $n = 3$, the constraints in (P) become

$$\begin{aligned} x_1 + K &\geq \frac{v(1) + v(12) + v(13) - v(N)}{2}, \\ x_2 + K &\geq \frac{v(2) + v(12) + v(23) - v(N)}{2}, \\ x_3 + K &\geq \frac{v(3) + v(23) + v(13) - v(N)}{2}, \\ x_1 + x_2 + x_3 &= v(N). \end{aligned}$$

Adding the first three inequalities and using the efficiency, we get

$$K \geq K^* \equiv \frac{v(1) + v(2) + v(3) - 5v(N)}{6} + \frac{v(12) + v(13) + v(23)}{3}.$$

When $K = K^*$, we see $x_i \geq \phi_i(v)$ for all $i \in N$. From this and the last equality constraint, we have $x_i = \phi_i(v)$ for all $i \in N$.

Remark. (Sobolev 1975, Peleg/Sudholter 2003,pp.218-224) For a game v and $x \in X^*(v)$, define a game $(N \setminus \{n\}, w)$ by

$$w(T) = \begin{cases} x(N \setminus \{n\}), & \text{if } T = N \setminus \{n\}; \\ \frac{n-1-|T|}{n-1} v(T) + \frac{|T|}{n-1}\{v(T \cup \{n\}) - x_n\}, & \text{if } T \neq N \setminus \{n\}, T \neq \emptyset; \\ 0, & \text{if } T = \emptyset. \end{cases} \tag{14}$$

For a game v and $x \in X^*(v)$, let $(N \setminus \{n\}, w)$ be a game defined by (14). If $x_i = \varphi_i(w)$ for $i \neq n$, then $x_i = \varphi_i(v)$ for all $i \in N$, that is, x is the Shapley value.

Example 2. Let $n = 4$. For a game v and $x \in X^*(v)$, define a game $(\{1, 2, 3\}, w)$ by

$$\begin{aligned} w(123) &= x(123), \\ w(ij) &= \frac{1}{3}v(ij) + \frac{2}{3}\{v(ij4) - x_4\}, \forall i, j \in N \setminus \{4\}, i \neq j, \\ w(i) &= \frac{2}{3}v(i) + \frac{1}{3}\{v(i4) - x_4\}, \forall i \neq 4. \end{aligned}$$

If $x_i = \varphi_i(w)$ for $i \neq 4$, then $x_i = \varphi_i(v)$ for all i.

3. An Intermediate Solution

In this section we define a solution and examines its properties. For $0 \leq \alpha \leq 1$, $x \in R^N$ and $S \subsetneq N$, define

$$h(S, x) \equiv \alpha e(S, x) + (1 - \alpha) \max_{i \in S}\{ \sum_{T: i \in T \neq N} \gamma_{n-1}(T) e(T, x)\}.$$

We consider a variation of the prenucleolus for (N, v) where the excess is replaced by $h(\bullet, \bullet)$. We call this variation the h-prenucleolus.

Interpretation. $h(S,x)$ is a weighted sum of the usual excess and the maximum of averages of excesses for all members in the coalition. If $\alpha = 1$, the solution is the prenucleolus. If $\alpha = 0$, the solution is the Shapley value. The variation is an intermediate solution between the prenucleolus and the Shapley value.

Theorem 2. For a fixed $0 \leq \alpha \leq 1$, the h-prenucleolus is nonempty and consists of one point. If $\alpha = 0$ it is the Shapley value, while if $\alpha = 1$, it is the prenucleolus.

Lemma 2A. For every $T \subseteq N$, $h(T,x)$ is continuous and convex with respect to $x \in X^*(v)$.
Proof: Let

$$f_i(x) = \sum_{T:i\in T\neq N} \gamma_{n-1}(T)e(T,x).$$

For $0 \leq r \leq 1$,

$$e(T, rx + (1-r)y) = re(T,x) + (1-r)e(T,y).$$

So, we have $f_i(rx + (1-r)y) = rf_i(x) + (1-r)f_i(y)$. From this and definition,

$$\begin{aligned} rh(T,x) + (1-r)h(T,y) &= \alpha re(T,x) + \alpha(1-r)e(T,y) \\ &\quad + (1-\alpha)r \max_{i\in T}\{f_i(x)\} + (1-\alpha)r \max_{i\in T}\{f_i(y)\} \\ &\geq \alpha e(T, rx + (1-r)y) + (1-\alpha)\max_{i\in T}\{rf_i(x) + (1-r)f_i(y)\} \\ &= \alpha e(T, rx + (1-r)y) + (1-\alpha)\max_{i\in T}\{f_i(rx + (1-r)y)\} \\ &= h(T, rx + (1-r)y). \end{aligned}$$

Hence $h(T,x)$ is convex. It is continuous by definition. □

The next two lemmas are from [Peleg/Sudholter 2003, esp. pp.108-112].

Lemma 2B. If $X \neq \emptyset$, compact and if $h_i, i \in D$ are continuous, then $\mathcal{N}(H,X) \neq \emptyset$.

Lemma 2C. Assume that X is convex and all $h_i, i \in D$ are convex. Then $\mathcal{N}(H,X)$ is convex. Furthermore, if $x, y \in \mathcal{N}(H,X)$, then $h_i(x) = h_i(y)$ for all $i \in D$.

Proof of the theorem: For $x^* = (\frac{v(N)}{n}, \ldots, \frac{v(N)}{n})$, let $q \equiv max\{h(T,x^*) : T \neq \emptyset\}$. There exists K such that for every $i \in N$, $h(\{i\},x) \geq q$ if $x_i \leq K$. So we need to consider points in $Y \equiv \{y \in X^*(v) : y_i \geq K, \forall i \in N\}$. Hence by Lemma 2B, the h-pre-nucleolus is nonempty. Furthermore, Y is convex, and so by Lemma 2C and Lemma 2A, we have that Y is convex and, if x and y are in the h-pre-nucleolus, then $h(S,x) = h(S,y)$ for all $S \subseteq N$. In particular, $h(\{i\},x) = h(\{i\},y)$ for all $i \in N$. This implies $x_i = y_i$ for all $i \in N$. □

Remark. For a game v, $i \in N$ and $x \in X^*(v)$, by an elementary calculation, we see

$$\sum_{T:i\in T\neq N} \gamma_{n-1}(T)e(T,x) = \ell_i(v) - x_i,$$

where

$$\ell_i(v) = \sum_{T:i\in T\neq N} \gamma_{n-1}(T)v(T) - v(N)\sum_{s=2}^{n-1} \frac{s-1}{(n-1)(n-s)}.$$

For $0 \leq \alpha \leq 1, x \in R^N$ and $i \in S \subseteq N$, define

$$h(S,i,x,v) \equiv \alpha e(S,x) + (1-\alpha)(\ell_i(v) - x_i).$$

Definition. Let v be a game. For $x \in X^*(v)$ and β, let

$$\mathcal{D}(\beta,x,v) \equiv \{(S,i) : i \in S, h(S,i,x,v) \geq \beta\}.$$

A vector $x \in X^*(v)$ has h–property I with respect to v if for all β such that $\mathcal{D}(\beta,x,v) \neq \emptyset$: If $y \in R^N$ satisfies $y(N) = 0$ and $\alpha y(S) + (1-\alpha)y_i \geq 0$ for all $(S,i) \in \mathcal{D}(\beta,x,v)$, then $\alpha y(S) + (1-\alpha)y_i = 0$ for all $(S,i) \in \mathcal{D}(\beta,x,v)$.

Theorem 3. For a game v and $x \in X^*(v)$, x is the h–prenucleolus if and only if x has h–property I.

Proof: *Necessity.* Assume x is the h–prenucleolus. Let β satisfy $\mathcal{D}(\beta,x,v) \neq \emptyset$ and let $y \in R^N$ satisfy $y(N) = 0$ and $\alpha y(S) + (1-\alpha)y_i \geq 0$ for all $(S,i) \in \mathcal{D}(\beta,x,v)$. Define $z_\epsilon = x + \epsilon y$ for $\epsilon > 0$. Then $z_\epsilon \in X^*(v)$. Choose $\epsilon^* > 0$ such that, for all $(S,i) \in \mathcal{D}(\beta,x,v)$ and all $T \in 2^N \setminus \mathcal{D}(\beta,x,v)$,

$$h(S,i,z_{\epsilon^*},v) > h(T,i,z_{\epsilon^*},v). \tag{15}$$

For every $(S,i) \in \mathcal{D}(\beta,x,v)$,

$$\begin{aligned} h(S,i,z_{\epsilon^*},v) &= \alpha e(S,z_{\epsilon^*}) + (1-\alpha)(\ell_i(v) - z_{\epsilon^*}) \\ &= \alpha(e(S,x) - \epsilon^* y(S)) + (1-\alpha)(\ell_i(v) - x_i - \epsilon^* y_i) \\ &= h(S,i,x,v) - \epsilon^*(\alpha y(S) + (1-\alpha)y_i) \\ &\leq h(S,i,x,v). \end{aligned} \tag{16}$$

Assume, on the contrary, that there is $(S,i) \in \mathcal{D}(\beta,x,v)$ such that $\alpha y(S) + (1-\alpha)y_i > 0$. By (15) and (16), we obtain $\theta(x) >_{lex} \theta(z_{\epsilon^*})$, which is a contradiction.

Sufficiency. Let $x \in X^*(v)$ have h–property I and let z be the h–prenucleolus. Denote

$$\{h(S,i,x,v) : (S,i)\} = \{\beta_1, \ldots, \beta_p\},$$

where $\beta_1 > \cdots > \beta_p$. Define $y = z - x$. Then $y(N) = 0$. Also, since $\theta(x) >_{lex} \theta(z)$, if $(S,i) \in \mathcal{D}(\beta_1,x,v)$, then $h(S,i,x,v) = \beta_1 \geq h(S,i,z,v)$. Hence

$$h(S.i,x,v) - h(S,i,z,v) = \alpha y(S) + (1-\alpha)y_i \geq 0.$$

Therefore, by assumption, $\alpha y(S) + (1-\alpha)y_i = 0$ for all $(S,i) \in \mathcal{D}(\beta_1,x,v)$. Assume now that $\alpha y(S) + (1-\alpha)y_i = 0$ for all $(S,i) \in \mathcal{D}(\beta_t,x,v)$ for some $1 \leq t < p$. Then, since $\theta(x) >_{lex} \theta(z)$,

$$h(S,i,x,v) = \beta_{t+1} \geq h(S,i,z,v), \forall (S,i) \in \mathcal{D}(\beta_{t+1},x,v) \setminus \mathcal{D}(\beta_t,x,v).$$

Hence $\alpha y(S) + (1-\alpha)y_i \geq 0$ for all $(S,i) \in \mathcal{D}(\beta_{t+1},x,v)$. Again, by assumption, $\alpha y(S) + (1-\alpha)y_i = 0$ for all $(S,i) \in \mathcal{D}(\beta_{t+1},x,v)$. We conclude that $\alpha y(S) + (1-\alpha)y_i = 0$ for all (S,i). Hence, $y = 0$ and $x = z$. □

Example 3. Let $n = 3$, and

$$v(1) = v(2) = v(3) = 0, v(12) = 3, v(13) = 5, v(23) = 7, v(N) = 10.$$

The prenucleolus for this game is

$$(\frac{3}{2}, \frac{13}{4}, \frac{21}{4}).$$

The Shapley value for this game is

$$(\frac{7}{3}, \frac{10}{3}, \frac{13}{3}).$$

$$h(S,x) = \begin{cases} -x_1 + \alpha - 1, & \text{if } S = \{1\}; \\ -x_2, & \text{if } S = \{2\}; \\ -x_3 - \alpha + -1, & \text{if } S = \{3\}; \\ \alpha(3 - x_1 - x_2) + (1-\alpha)\, max\{-1 - x_1, -x_2\}, & \text{if } S = \{1,2\}; \\ \alpha(5 - x_1 - x_3) + (1-\alpha)\, max\{-x_2, 1 - x_3\}, & \text{if } S = \{1,3\}; \\ \alpha(7 - x_2 - x_3) + (1-\alpha)\, max\{-x_2, 1 - x_3\}, & \text{if } S = \{1,2\}. \end{cases}$$

Both of the Shapley value and the prenucleolus satisfy

$$x_3 \geq x_2 + 1, \text{ and } x_2 \geq x_1 + 1, x \in X^*(v). \tag{17}$$

We will find the solution for the region in $X^*(v)$ which is defined by (17). First we see

$$h(\{2\},x) = -x_2 \leq -x_1 - 1 \leq h(\{1\},x),$$

$$h(\{3\},x) = -x_3 + 1 - \alpha \leq -x_2 - 1 + 1 - \alpha \leq h(\{2\},x) \leq h(\{1\},x),$$

$$h(\{1,2\},x) = h(\{1\},x) + \alpha(3 - x_2),$$

$$h(\{1,3\},x) = h(\{1\},x) + \alpha(5 - x_3),$$

$$h(\{2,3\},x) = \alpha(7 - x_3) - x_2 \leq \alpha(7 - x_3) + h(\{1\},x).$$

Case 1: $x_2 \leq 3$.
This implies $x_1 \leq 2$ and $x_1 + x_3 \geq 7$ and so $x_3 \geq 5$. Hence $h(\{1,2\},x) \geq h(\{1\},x) \geq h(\{1,3\},x)$. So we need to compare $h(\{1,2\},x)$ with $h(\{2,3\},x)$.

$$h(\{1,2\},x) \geq h(\{2,3\},x) \iff (1-\alpha)x_2 + \alpha x_3 - x_1 \geq 3\alpha + 1.$$

The line $h(\{1,2\},x) \geq h(\{2,3\},x)$ in $X^*(v)$ has common points

$$(\frac{5}{3}, \frac{8}{3}, \frac{17}{3}) \text{ and } (1 + \frac{1}{1+\alpha}, 3, 5 + \frac{\alpha}{1+\alpha})$$

with lines $x_2 = x_1 + 1$ and $x_2 = 3$ respectively. Since $h(\{1,2\},x)$ is a linear function and

$$h(\{1,2\}, (\frac{5}{3}, \frac{8}{3}, \frac{17}{3})) > h(\{1,2\}, (1 + \frac{1}{1+\alpha}, 3, 5 + \frac{\alpha}{1+\alpha})),$$

we conclude that the point $(1 + \frac{1}{1+\alpha}, 3, 5 + \frac{\alpha}{1+\alpha})$ minimizes $h(\{1,3\}, \bullet)$ in the segment connecting these two points. Also in the segment connecting

$$(2,3,5) \text{ and } (1 + \frac{1}{1+\alpha}, 3, 5 + \frac{\alpha}{1+\alpha}),$$

we see that $(1 + \frac{1}{1+\alpha}, 3, 5 + \frac{\alpha}{1+\alpha})$ minimizes $h(\{2,3\}, \bullet)$.

Case 2: $x_2 \geq 3, x_3 \geq 5$ and $x_3 \geq x_2 + 1$.

We have $h(\{1,2\},x) \leq h(\{1\},x)$ and $h(\{1,3\},x) \leq h(\{1\},x)$. So we need to compare $h(\{1\},x)$ with $h(\{2,3\},x)$.

$$h(\{1\},x) \geq h(\{2,3\},x) \Longleftrightarrow x_2 + \alpha x_3 - x_1 \geq 6\alpha + 1.$$

The line $x_2 + \alpha x_3 - x_1 = 6\alpha + 1$ has common points

$$(1 + \frac{1}{1+\alpha}, 3, 5 + \frac{\alpha}{1+\alpha}) \text{ and } (2 - \frac{\alpha}{2}, 3 + \frac{\alpha}{2}, 5)$$

with lines $x_2 = 3$ and $x_3 = 5$ respectively. In the segment defined by these two points, the latter minimizes $h(\{1\},\bullet)$.

Case 3: $x_3 \leq 5$ and , $x_3 \geq x_2 + 1$.

These imply $x_2 \geq 3$. We have $h(\{1,3\},x)$ with $h(\{1\},x)$ and $h(\{1\},x) \geq h(\{1,2\},x)$. So we need to compare $h(\{1,3\},x)$ with $h(\{2,3\},x)$.

$$h(\{1,3\},x) \geq h(\{2,3\},x) \Longleftrightarrow x_2 \geq x_1 + \alpha + 1.$$

The line $x_2 = x_1 + \alpha + 1$ has common points

$$(\frac{7}{3} - \frac{2\alpha}{3}, \frac{10}{3} + \frac{\alpha}{3}, \frac{13}{3} + \frac{\alpha}{3}) \text{ and } (2 - \frac{\alpha}{2}, 3 + \frac{\alpha}{2}, 5)$$

with lines $x_3 = x_2 + 1$ and $x_3 = 5$ respectively. In the segment defined by these two points, the latter minimizes $h(\{1,3\},\bullet)$ if $\alpha \geq \frac{1}{2}$ and the former minimizes if $\alpha \leq \frac{1}{2}$. When $\alpha = \frac{1}{2}$, all points in the segment minimize $h(\{1,3\},\bullet)$.

Summarizing the three cases, we see that the chosen point is

$$\begin{cases} a : (\frac{7}{3} - \frac{2\alpha}{3}, \frac{10}{3} + \frac{\alpha}{3}, \frac{13}{3} + \frac{\alpha}{3}), & \text{if } \alpha \leq \frac{1}{2}; \\ b : (2 - \frac{\alpha}{2}, 3 + \frac{\alpha}{2}, 5), & \text{if } \alpha \geq \frac{1}{2}. \end{cases}$$

As $\alpha \to 0$, the point a converges to the Shapley value, while, as $\alpha \to 1$, the point b converges to $(\frac{3}{2}, \frac{7}{2}, 5)$, which is not the prenucleolus. When $\alpha = 1$, all points in the segment connecting $(\frac{3}{2}, \frac{7}{2}, 5)$ and $(\frac{3}{2}, 3, \frac{11}{2})$ are candidates, and the midpoint of this segment minimizes the third-largest element of $h(\bullet,\bullet)$'s. This midpoint is the prenucleolus.

4. Concluding Remarks.

We defined a solution for TU games and showed that the proposed solution is nonempty and consists of one point. It is a bridge between the Shapley value and the prenucleolus. It remains to characterize it axiomatically.

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