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On Golden inequalities

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Abstract

We consider an inequality with equality condition where one side is greater (*resp.* less) than or equal to a multiple of the other side and an equality holds if and only if one value is a multiple of the other variable. When the two multiples constitute a Golden ratio, the inequality is called Golden. This paper presents six Golden inequalities both among one-variable functions and among two-variable functions. We show a cross-duality between four pairs of Golden inequalities for one-variable functions. Similar results for two-variable functions are stated. Further a graphic representation for Golden inequalities is shown.

1 Introduction

Both historically and practically, it is well known that the Golden raio has been taking an interesting part in many fields in science, technology, art, architecture, biology and so on [3, 13]. In mathematical science, the Golden ratio has recently been incorporated into optimization field [8–10, 12]. This paper is motivated by trying to introduce and discuss the Golden ratio in the fields of inequalities [1, 2, 4–7, 11]. In fact, there exists a mutual relationship between optimizaion and inequality [1, 2, 4, 5, 7, 10].

In this paper we consider a class of Golden inequalities between two given functions. A pair of two real values is called Golden if it constitutes the Golden ratio. We consider an inequality of the following form. One side consists of one function only. The other side consists of a *multiple* of the other function. One side is greater (*resp.* less) than or equal to the other side. Further the sign of equality holds if and only if one value is a *multiple* of the other value. If the pair of two multiples is Golden, the inequality is called Golden.

We present six Golden inequalities for one-variable quadratic functions. We show a cross-duality between four pairs of Golden inequalities for one-variable functions. Similar results for two-variable quadratic functions are stated. Further a graphic representation for Golden inequalities is shown.

2 The Golden Ratio

We take a basic standard real number

$$\phi = \frac{1+\sqrt{5}}{2} \approx 1.61803$$

The number ϕ is called the *Golden number*. It is defined as the positive solution to quadratic equation

$$x^2 - x - 1 = 0.$$

A Fibonacci sequence $\{a_n\}$ is defined by second-order linear difference equation

$$a_{n+2} - a_{n+1} - a_n = 0.$$

Then we have a famous relation.

Lemma 2.1

$$\begin{aligned} \phi^n &= a_n\phi + a_{n-1} \\ (\phi-1)^n &= a_{-n}\phi + a_{-n-1} \end{aligned} \qquad n = \cdots, -2, -1, 0, 1, 2, \cdots$$

where $\{a_n\}$ is the Fibonacci sequence with $a_0 = 0,\ a_1 = 1$.

The Fibonacci sequence is tabulated in Table 1:

n	−12	−11	−10	−9	−8	−7	−6	−5	−4	−3	−2	−1	0	1
a_n	−144	89	−55	34	−21	13	−8	5	−3	2	−1	1	0	1

n	0	1	2	3	4	5	6	7	8	9	10	11	12
a_n	0	1	1	2	3	5	8	13	21	34	55	89	144

Table 1 Fibonacci sequence $\{a_n\}$

On the other hand, the Fibonacci sequence has the analytic form

Lemma 2.2

$$a_n = \frac{1}{2\phi - 1}\{\phi^n - (1-\phi)^n\} \qquad n = \cdots, -2, -1, 0, 1, 2, \cdots$$

Lemma 2.3

(i) $$\frac{\phi}{1} = \frac{1+\phi}{\phi} = \frac{1+2\phi}{1+\phi} = \frac{2+3\phi}{1+2\phi} = \frac{3+5\phi}{2+3\phi} = \cdots = \frac{a_n + a_{n+1}\phi}{a_{n-1} + a_n\phi} \approx 1.61803$$

(ii) $$\frac{a_n + a_{n+1}\phi}{a_{n-1} + a_n\phi} = \cdots = \frac{2\phi - 3}{5 - 3\phi} = \frac{2-\phi}{2\phi - 3} = \frac{\phi - 1}{2 - \phi} = \frac{1}{\phi - 1} = \frac{\phi}{1}$$

(iii) $$\frac{0.236}{0.146} \approx \frac{0.382}{0.236} \approx \frac{0.618}{0.382} \approx \frac{1}{0.618} \approx \frac{1.618}{1} \approx \frac{2.618}{1.618} \approx \frac{4.236}{2.618} \approx \cdots \approx 1.618$$

where $\{a_n\}$ is the Fibonacci sequence.

3 One-variable Functions

Let us consider an inequality between two one-variable functions $f,\ g : R^1 \to R^1$ with an equality condition. We assume that inequality

$$f(u) \ \le\ (\ge)\ \alpha g(u) \qquad \text{on}\ \ R^1 \tag{1}$$

holds. The sign of equality holds if and only if $u = \beta$, where α and β are real constants.

Definition 3.1 *We say that the pair* $(\alpha,\ \beta)$ *constitutes the Golden ratio if*

$$\left|\frac{\beta}{\alpha}\right| = \phi \quad or \quad \left|\frac{\alpha}{\beta}\right| = \phi.$$

Definition 3.2 *When the pair constitutes the Golden ratio, the inequality (1) is called Golden.*

For instance we see that inequality

$$1+u^2 \ \le\ (1+\phi)\{1+(u-1)^2\} \qquad \text{on}\ \ R^1 \tag{2}$$

holds. The sign of equality holds if and only if $u = \phi$. Further $(\phi, 1+\phi)$ constitutes the Golden ratio. Thus (2) is a Golden inequality.

First we consider six Golden inequalities between one-variable quadratic functions. The inequalities (3) and (4) are pairs of Golden inequalities. The inequalities (5) and (6) are Golden. Thus we have six Golden inequalities in the following.

Lemma 3.1 (i) *It holds that*

$$(2-\phi)\{1+(u-1)^2\} \le 1+u^2 \le (1+\phi)\{1+(u-1)^2\} \qquad \text{on}\ \ R^1. \tag{3}$$

The sign of left equality holds if and only if $u = 1-\phi$ *and the sign of right equality holds if and only if* $u = \phi$ *([8–10]).*

(ii) *It holds that*

$$(1-\phi)\{1+(v-1)^2\} \le -1+v^2 \le \phi\{1+(v-1)^2\} \qquad \text{on}\ \ R^1. \tag{4}$$

The sign of left equality holds if and only if $v = 2-\phi$ *and the sign of right equality holds if and only if* $v = 1+\phi$ *(Figure 2).*

(iii) *The middle-right inequality (resp. left-middle) in (3) is equivalent to the left-middle (resp. middle-right) inequality in (4).*

Lemma 3.2 (i) *It holds that*

$$(-1+\phi)(1-u^2) \le u^2+(u-1)^2 \qquad \text{on}\ \ R^1 \tag{5}$$

The sign of equality holds if and only if $u = 2-\phi$ *(Figure 3).*
It holds that

$$-u^2-(u-1)^2 \le \phi(1-u^2) \qquad \text{on}\ \ R^1 \tag{6}$$

The sign of equality holds if and only if $u = 1+\phi$ *(Figure 4).*

(ii) *The inequality (5) is equivalent to the middle-right inequality in (4). The inequality (6) is equivalent to the left-middle inequality in (4).*

3.1 A pair of Golden inequalities between $1+(1-v)^2$ and $-1+v^2$

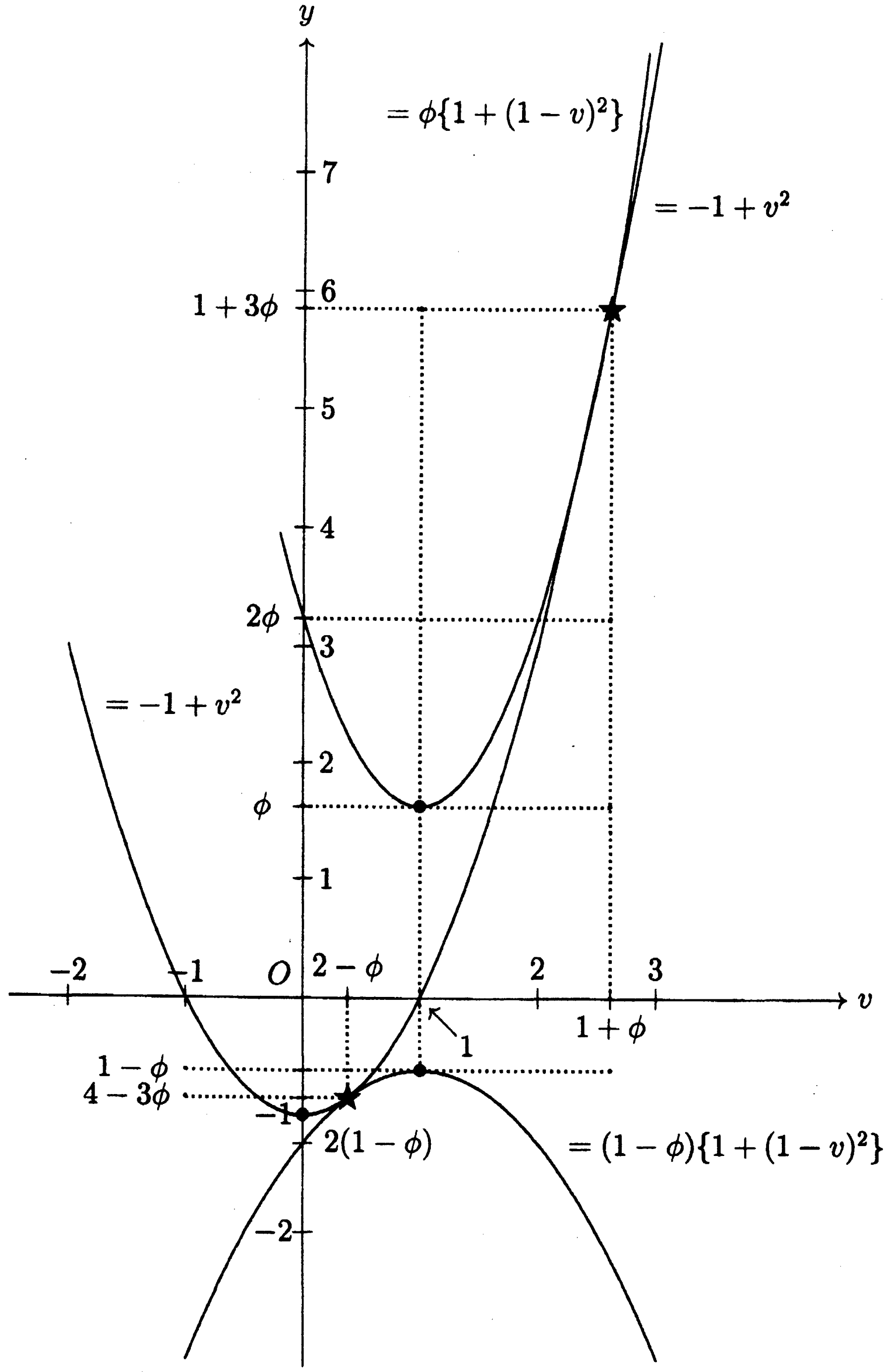

Figure 2 A pair of Golden Inequalities
$(1-\phi)\{1+(1-v)^2\} \leq -1+v^2 \leq \phi\{1+(1-v)^2\}$.
The left and right equalities attain at $\hat{v} = 2-\phi$ and $v^* = 1+\phi$, respectively.

3.2 One Golden inequality between $1-u^2$ and $u^2+(1-u)^2$

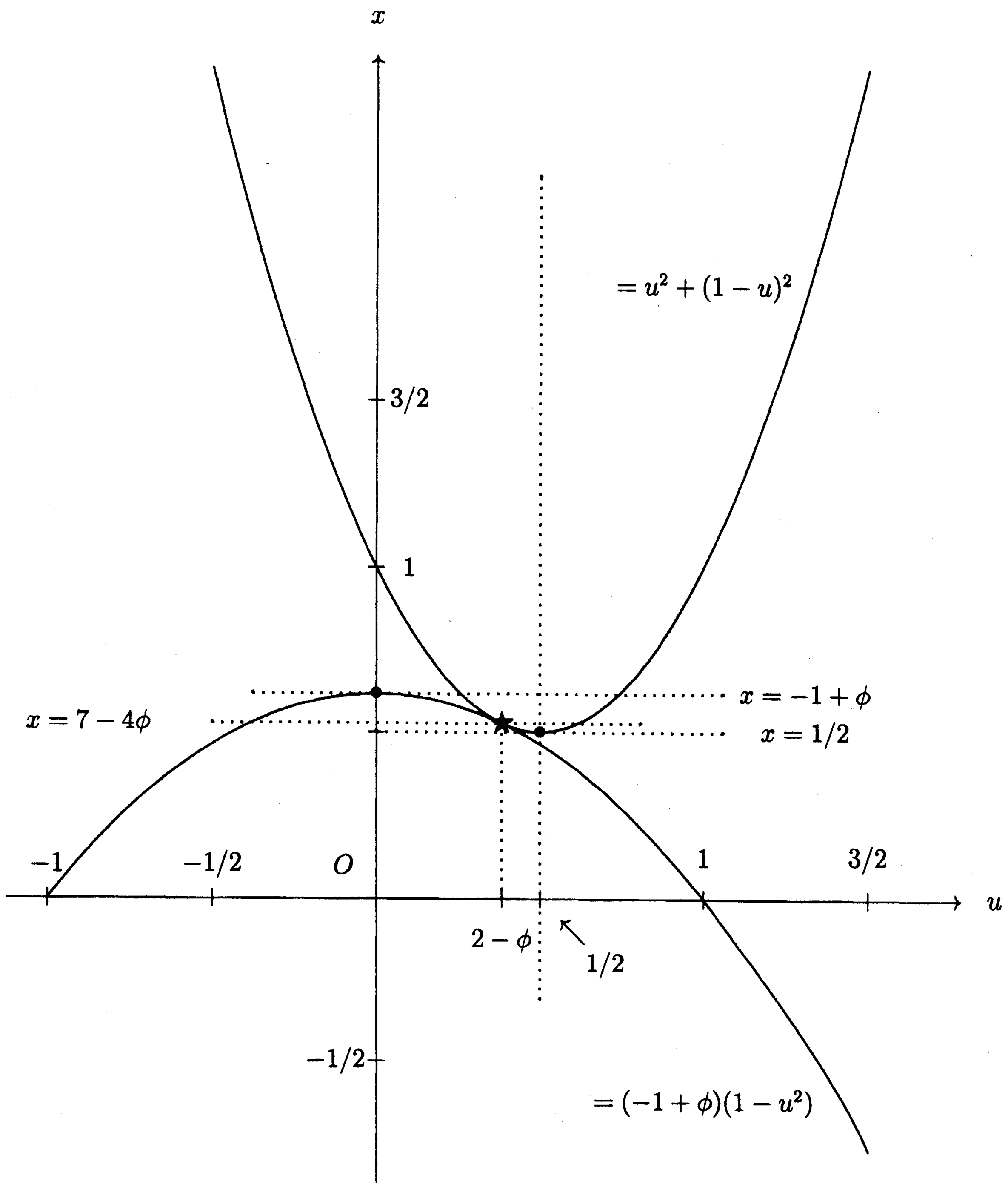

Figure 3 Golden Inequality $(-1+\phi)(1-u^2) \leq u^2 + (1-u)^2$.
The equality attains at $u^* = 2 - \phi$.

3.3 The other Golden inequality between $1 - u^2$ and $u^2 + (1 - u)^2$

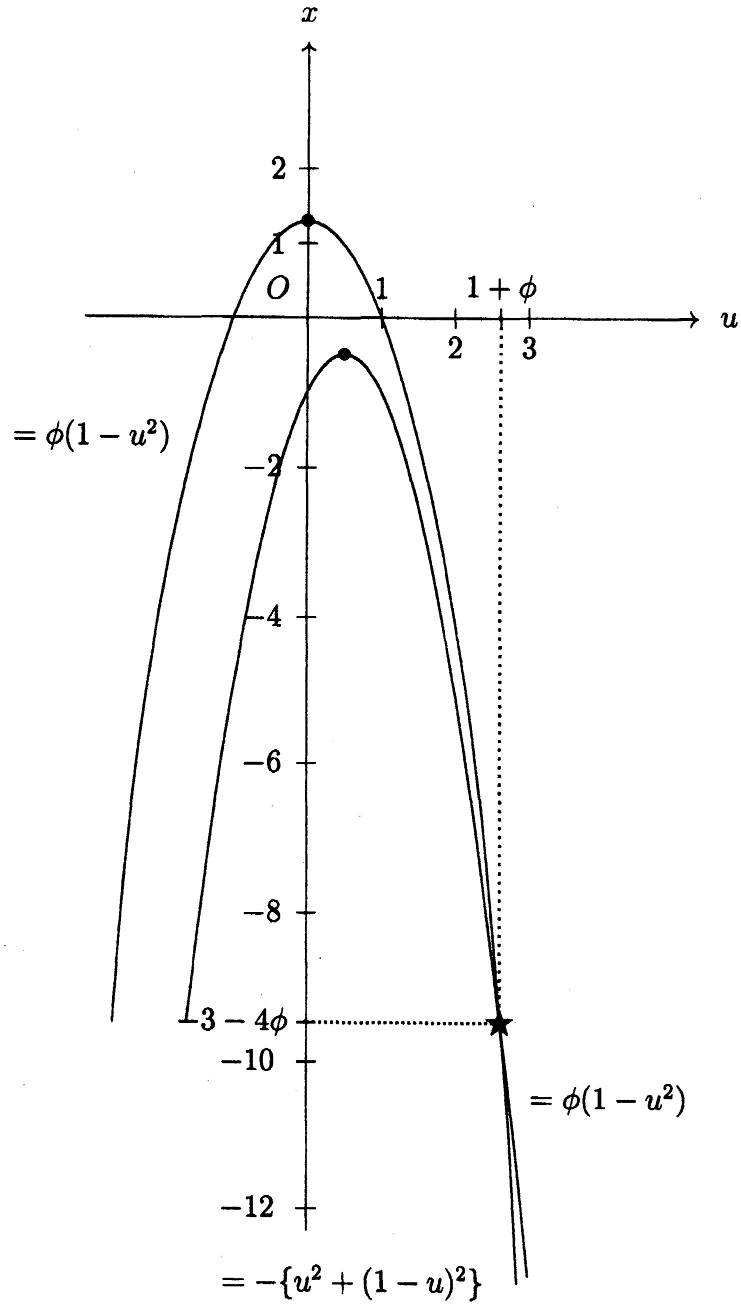

Figure 4 Golden Inequality $-\{u^2 + (1-u)^2\} \leq \phi(1-u^2)$.
The equality attains at $\hat{u} = 1 + \phi$.

4 Two-variable Functions

Let us take two two-variable functions $f,\ g : R^2 \to R^1$. We assume that inequality

$$f(x,y) \ \le \ (\ge) \ \alpha g(x,y) \qquad \text{on } R^2 \tag{7}$$

holds and that the sign of equality holds if and only if $y = \beta x$.

Definition 4.1 *When the pair* $(\alpha,\ \beta)$ *constitutes the Golden ratio, the inequality (7) is called Golden.*

For instance, the inquality

$$x^2 + y^2 \ \ge \ (2-\phi)\{x^2 + (y-x)^2\} \qquad \text{on } R^2 \tag{8}$$

holds. The sign of equality holds if and only if $y = (1-\phi)x$. Thus inequality (8) is also Golden.

4.1 Cauchy-Schwarz

Let us take $f(x,y) = (ax+by)^2,\ \ g(x,y) = x^2+y^2$, where $a\ (\neq 0)$, b are real constants. Then it holds that

$$(ax+by)^2 \ \le \ (a^2+b^2)(x^2+y^2) \qquad \text{on } R^2. \tag{9}$$

The sign of equality holds if and only if $ay = bx$.
When $\alpha = \dfrac{b}{a}$ and $\beta = a^2+b^2$ constitute the Golden ratio, the Cauchy-Schwart inequality becomes Golden.

4.2 Golden inequalities

We specify six Golden inequalities between two-variable quadratic functions. Each of (10) and (11) yields a pair of Golden inequalities. Both (12) and (13) are Golden inequalities. Thus we have also six Golden inequalities in the following.

Theorem 4.1 (i) *It holds that*

$$(2-\phi)\{x^2+(y-x)^2\} \le x^2+y^2 \le (1+\phi)\{x^2+(y-x)^2\} \qquad \text{on } R^2. \tag{10}$$

The sign of left equality holds if and only if $y = (1-\phi)x$ *and the sign of right equality holds if and only if* $y = \phi x$.

(ii) *It holds that*

$$(1-\phi)\{x^2+(y-x)^2\} \le -x^2+y^2 \le \phi\{x^2+(y-x)^2\} \qquad \text{on } R^2. \tag{11}$$

The sign of left equality holds if and only if $y = (2-\phi)x$ *and the sign of right equality holds if and only if* $y = (1+\phi)x$.

(iii) *The middle-right inequality (resp. left-middle) in (10) is equivalent to the left-middle (resp. middle-right) inequality in (11).*

Theorem 4.2 (i) *It holds that*

$$(-1+\phi)(x^2-y^2) \leq y^2+(y-x)^2 \quad \text{on} \quad R^2. \tag{12}$$

The sign of left equality holds if and only if $y=(2-\phi)x$. *It holds that*

$$-y^2-(y-x)^2 \geq \phi(x^2-y^2) \quad \text{on} \quad R^2. \tag{13}$$

The sign of right equality holds if and only if $y=(1+\phi)x$.

(ii) *The inequality (12) is equivalent to the middle-right inequality in (11). The inequality (13) is equivalent to the left-middle inequality in (11).*

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