Markov version of Bruss' odds-theorem

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1. Introduction

Let $I_1, I_2, \ldots, I_n$ be a sequence of independent indicator functions with

$$
I_k = \begin{cases} 1, & \text{if the } kth \text{ event is a success} \\ 0, & \text{if the } kth \text{ event is a failure.} \end{cases}
$$

We observe $I_1, I_2, \ldots$ sequentially and may stop at any time. We want to find a stopping rule which maximizes the probability of stopping on the last success. Bruss[1] considered this problem and gave the so called odds-theorem, which can be stated as follows:

(**Bruss' odds-theorem**)   Let $p_j = P(I_j = 1)$ and define $q_j = 1 - p_j$ and $r_j = p_j/q_j$. Then an optimal rule stops on the first index k with $I_k = 1$ and $k \geq s$, where

$$
s = \sup\{1 \leq k \leq n : \sum_{j=k}^{n} r_j \geq 1\}.
$$

The optimal reward, probability of stopping on the last success, is given by $(\prod_{j=s}^{n} q_j)(\sum_{j=s}^{n} r_j)$. Interprete $s = 1$ if $\sum_{j=1}^{n} r_j < 1$.

In this note, the odds-theorem is generalized in such a way that the sequence $I_1, I_2, \ldots, I_n$ are Markov dependent. That is, we assume here that $I_1, I_2, \ldots, I_n$ constitute a Markov chain with transition probabilities

$$
\begin{aligned}
\alpha_j &= P\{I_{j+1} = 1 | I_j = 0\} \\
\beta_j &= P\{I_{j+1} = 0 | I_j = 1\}
\end{aligned}
$$

for $1 \leq j < n$. We assume $\alpha_j = 0, \beta_j = 1$ for $j \geq n$ for convenience and use the notations $\bar{\alpha}_j = 1 - \alpha_j, \bar{\beta}_j = 1 - \beta_j$. The main result of this note can be stated as follows:

Theorem 1 (Markov version of the odds-theorem)   Assume that
(a) α_j is non-increasing in j.
(b) β_j is non-decreasing concave in j.

Then an optimal rule stops on the first index k with $I_k = 1$ and $k \geq s$, where

$$s = \sup \left\{ 1 \leq k \leq n : \frac{\bar{\beta}_k}{\beta_k} \frac{\beta_{k+1}}{\bar{\alpha}_{k+1}} + \sum_{j=k+1}^{n-1} \frac{\alpha_j}{\bar{\alpha}_j} \frac{\beta_{j+1}}{\bar{\alpha}_{j+1}} \geq 1 \right\}.$$

In this note, the vacuous sum is assumed to be 0 and the vacuous product to be 1.

2. Derivation

We are said to be in state k if $I_k = 1$ at time k. If we stop in state k, the reward is

$$\begin{aligned} P_k &= \beta_k \bar{\alpha}_{k+1} \cdots \bar{\alpha}_{n-1} \\ &= \frac{\beta_k}{\bar{\alpha}_k} \prod_{j=k}^{n-1} \bar{\alpha}_j. \end{aligned}$$

If we pass over the kth success and stop as soon as a new success occurs, then since the probability that the $jth (j > k)$ is the first new success is

$$q_{k,j} = \begin{cases} \bar{\beta}_k, & \text{if } j = k+1 \\ \beta_k \alpha_{j-1} \prod_{i=k+1}^{j-2} \bar{\alpha}_i, & \text{if } k+2 \leq j \leq n, \end{cases}$$

the expected reward is given by

$$\begin{aligned} Q_k &= \sum_{j=k+1}^{n} q_{k,j} P_j \\ &= P_k \left[\frac{\bar{\beta}_k}{\beta_k} \frac{\beta_{k+1}}{\bar{\alpha}_{k+1}} + \sum_{j=k+1}^{n-1} \frac{\alpha_j}{\bar{\alpha}_j} \frac{\beta_{j+1}}{\bar{\alpha}_{j+1}} \right]. \end{aligned}$$

Now let

$$\begin{aligned} B &= \{ 1 \leq k \leq n : P_k \geq Q_k \} \\ &= \left\{ 1 \leq k \leq n : 1 \geq \frac{\bar{\beta}_k}{\beta_k} \frac{\beta_{k+1}}{\bar{\alpha}_{k+1}} + \sum_{j=k+1}^{n-1} \frac{\alpha_j}{\bar{\alpha}_j} \frac{\beta_{j+1}}{\bar{\alpha}_{j+1}} \right\}. \end{aligned}$$

Hence B represents the set of states for which stopping immediately is at least as good as continuing and then stopping with the first success. The rule that stops the first time the process enters a state in B is called OLA(one-stage look-ahead) rule. It is known that the OLA rule is optimal if B is

closed in a sense that once the process enters B, then it stays in B(see Ross[2]). Since $\sum_{j=k+1}^{n-1} \frac{\alpha_j}{\bar{\alpha}_j} \frac{\beta_{j+1}}{\bar{\alpha}_{j+1}}$ is non-increasing in k, to show that B is closed, it suffices to show that

$$R_k \equiv \frac{\bar{\beta}_k}{\bar{\beta}_k} \frac{\beta_{k+1}}{\bar{\alpha}_{k+1}}$$

is non-increasing in k.

Lemma 1   A set of ocnditions for R_k to be non-increasing in k are

(i) α_j is non-increasing in j.

(ii) β_j is non-decreasing in j.

(ii) β_j/β_{j+1} is non-increasing in j.

Proof   Obviously we have, from (i), (ii) and (iii),

$$\frac{1}{\bar{\alpha}_j} \geq \frac{1}{\bar{\alpha}_{j+1}}$$

$$\bar{\beta}_{j-1} \geq \bar{\beta}_j$$

$$\frac{\beta_j}{\beta_{j-1}} \geq \frac{\beta_{j+1}}{\beta_j}$$

respectively. Thus multiplying term by term we have

$$R_{j-1} \geq R_j$$

which completes the proof.

To prove Theorem 1, it is sufficient to show that the condition (b) satisfies the condition (iii) of Lemma 1. From (b) and the well known inequality we have

$$\beta_j \geq \frac{\beta_{j-1} + \beta_{j+1}}{2} \geq \sqrt{\beta_{j-1}\beta_{j+1}},$$

implying that

$$\beta_j^2 \geq \beta_{j-1}\beta_{j+1}$$

or equivalently

$$\frac{\beta_j}{\beta_{j-1}} \geq \frac{\beta_{j+1}}{\beta_j}$$

which guarantees the condition (iii) and completes the proof.

It is easy to see that, if for each j,

$$\alpha_j + \beta_j = 1$$

then $I_1, I_2, \ldots, I_n$ are independent and Theorem 1 reduces to Bruss' odds-theorem.

参考文献

[1] Bruss, F. T. (2000), Sum the odds to one and stop, Ann. Prob., 28, 1384-1391.

[2] Ross, S. M. (1983), Introduction to Stochastic Dynamic Programming, Academic Press, Orland.