数理解析研究所講究録
1505 巻 2006 年 101-105

Rayleigh-Bénard 問題の大域分岐構造に対する精度保証付き数値計算

九州大学情報基盤センター
渡部 善隆 (Yoshitaka Watanabe)
Computing and Communications Center, Kyushu University

1 The Rayleigh-Bénard Problems

Consider a plane horizontal layer (see Fig.1) of an incompressible viscous fluid heated from below. At the lower boundary: $z=0$ the layer of fluid is maintained at temperature $T+\delta T$ and the temperature of the upper boundary $(z=h)$ is T.

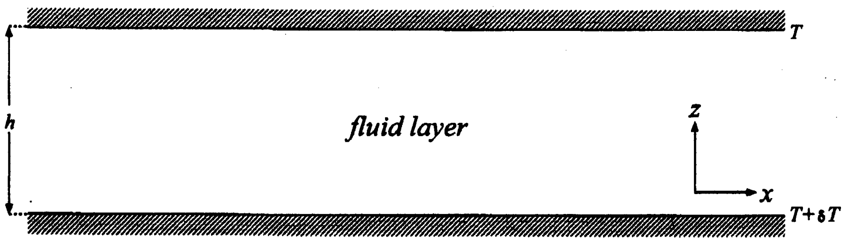

Fig.1. Fluid layer model

As well known, under the vanishing assumption in y-direction, the two-dimensional (x-z) heat convection model can be described as the following Oberbeck-Boussinesq approximations [1]:

$$\left\{\begin{array}{rcl} u_t+uu_x+wu_z & = & -p_x/\rho_0+\nu\Delta u, \\ w_t+uw_x+ww_z & = & -(p_z+g\rho)/\rho_0+\nu\Delta w, \\ u_x+w_z & = & 0, \\ \theta_t+u\theta_x+w\theta_z & = & \kappa\Delta\theta. \end{array}\right. \tag{1}$$

Here, u, w: velocity in x and z, respectively, p: pressure, θ: temperature, ρ: fluid density, ρ_0: density at temperature $T+\delta T$, ν: kinematic viscosity, g: gravitational acceleration, κ: coefficient of thermal diffusivity, $*_\xi:=\partial/\partial\xi(\xi=x,z,t)$, $\Delta:=\partial^2/\partial x^2+\partial^2/\partial z^2$. And ρ is assumed to be represented by $\rho-\rho_0=-\rho_0\alpha(\theta-T-\delta T)$, where α is the coefficient of thermal expansion.

The Oberbeck-Boussinesq equations (1) have the following stationary solution:

$$u^* = 0, \quad w^* = 0, \quad \theta^* = T + \delta T - \frac{\delta T}{h} z, \quad p^* = p_0 - g\rho_0 (z + \frac{\alpha \delta T}{2h} z^2),$$

where p_0 is a constant. By setting

$$\hat{u} := u, \qquad \hat{w} := w, \qquad \hat{\theta} := \theta^* - \theta, \qquad \hat{p} := p^* - p,$$

we obtain the transformed equations:

$$\begin{cases} \hat{u}_t + \hat{u}\hat{u}_x + \hat{w}\hat{u}_z &= \hat{p}_x/\rho_0 + \nu\Delta\hat{u}, \\ \hat{w}_t + \hat{u}\hat{w}_x + \hat{w}\hat{w}_z &= \hat{p}_z/\rho_0 - g\alpha\hat{\theta} + \nu\Delta\hat{w}, \\ \hat{u}_x + \hat{w}_z &= 0, \\ \hat{\theta}_t + \delta T\hat{w}/h + \hat{u}\hat{\theta}_x + \hat{w}\hat{\theta}_z &= \kappa\Delta\hat{\theta}. \end{cases} \tag{2}$$

By further transforming to dimensionless variables:

$$t \to \kappa t, \quad u \to \hat{u}/\kappa, \quad w \to \hat{w}/\kappa, \quad \theta \to \hat{\theta}h/\delta T, \quad p \to \hat{p}/(\rho_0\kappa^2)$$

of (2), we have the dimensionless equations:

$$\begin{cases} u_t + uu_x + wu_z &= p_x + \mathcal{P}\Delta u, \\ w_t + uw_x + ww_z &= p_z - \mathcal{P}\mathcal{R}\,\theta + \mathcal{P}\Delta w, \\ u_x + w_z &= 0, \\ \theta_t + w + u\theta_x + w\theta_z &= \Delta\theta. \end{cases} \tag{3}$$

Here $\mathcal{R} := (\delta T\alpha g)/(\kappa\nu h)$ is the Rayleigh number and $\mathcal{P} := \nu/\kappa$ is the Prandtl number.

2 Fixed-point formulation of problem

We describe the problem concerned as a fixed-point equation of a compact map on the appropriate function space. Since we only consider the the *steady-state solutions*, u_t, w_t and θ_t vanish in (3). And also assume that all fluid motion is confined to the rectangular region $\Omega := \{0 < x < 2\pi/a,\ 0 < z < \pi\}$ for a given wave number $a > 0$.

Let us impose periodic boundary condition (period $2\pi/a$) in the horizontal direction, stress-free boundary conditions ($u_z = w = 0$) for the velocity field and Dirichlet boundary conditions ($\theta = 0$) for the temperature field on the surfaces $z = 0, \pi$, respectively. Furthermore, we assume the following evenness and oddness conditions:

$$u(x,z) = -u(-x,z), \quad w(x,z) = w(-x,z), \quad \theta(x,z) = \theta(-x,z).$$

We use the stream function Ψ satisfying $u = -\Psi_z$, $w = \Psi_x$ so that $u_x + w_z = 0$. By some simple calculations in (3) with setting $\Theta := \sqrt{\mathcal{P}\mathcal{R}}\theta$, we obtain

$$\begin{cases} \mathcal{P}\Delta^2\Psi &= \sqrt{\mathcal{P}\mathcal{R}}\,\Theta_x - \Psi_z\Delta\Psi_x + \Psi_x\Delta\Psi_z, \\ -\Delta\Theta &= -\sqrt{\mathcal{P}\mathcal{R}}\,\Psi_x + \Psi_z\Theta_x - \Psi_x\Theta_z. \end{cases} \tag{4}$$

From the boundary conditions, the functions Ψ and Θ can be assumed to have the following representations:

$$\Psi = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}\sin(amx)\sin(nz), \quad \Theta = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} B_{mn}\cos(amx)\sin(nz). \tag{5}$$

We now define the following function spaces for integers $k \geq 0$:

$$X^k := \left\{\Psi = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}\sin(amx)\sin(nz) \mid A_{mn} \in \mathbf{R}, \quad \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}((am)^{2k} + n^{2k})A_{mn}^2 < \infty\right\},$$

$$Y^k := \left\{\Theta = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} B_{mn}\cos(amx)\sin(nz) \mid B_{mn} \in \mathbf{R}, \quad \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}((am)^{2k} + n^{2k})B_{mn}^2 < \infty\right\}.$$

In order to get the enclosure of the exact solutions for the problem (4), we need some appropriate finite dimensional subspaces. For $M_1, N_1, M_2 \geq 1$ and $N_2 \geq 0$, we set $N := (M_1, N_1, M_2, N_2)$ and define the finite dimensional approximate subspaces by

$$S_N^{(1)} = \left\{\sum_{m=1}^{M_1}\sum_{n=1}^{N_1} \hat{A}_{mn}\sin(amx)\sin(nz) \mid \hat{A}_{mn} \in \mathbf{R}\right\},$$

$$S_N^{(2)} = \left\{\sum_{m=0}^{M_2}\sum_{n=1}^{N_2} \hat{B}_{mn}\cos(amx)\sin(nz) \mid \hat{B}_{mn} \in \mathbf{R}\right\},$$

$$S_N = S_N^{(1)} \times S_N^{(2)}.$$

Let denote an approximate solution of (4) by $\hat{u}_N := (\hat{\Psi}_N, \hat{\Theta}_N) \in S_N$. We now set

$$\begin{cases} f_1(\Psi,\Theta) & := & \sqrt{\mathcal{PR}}\,\Theta_x - \Psi_z\Delta\Psi_x + \Psi_x\Delta\Psi_z, \\ f_2(\Psi,\Theta) & := & -\sqrt{\mathcal{PR}}\,\Psi_x + \Psi_z\Theta_x - \Psi_x\Theta_z, \end{cases}$$

where $\Psi = \hat{\Psi}_N + w^{(1)}$, $\Theta = \hat{\Theta}_N + w^{(2)}$. Then (4) is rewritten as the problem with respect to $(w^{(1)}, w^{(2)}) \in X^4 \times Y^2$ satisfying

$$\begin{cases} \mathcal{P}\Delta^2 w^{(1)} & = & f_1(\hat{\Psi}_N + w^{(1)}, \hat{\Theta}_N + w^{(2)}) - \mathcal{P}\Delta^2\hat{\Psi}_N, \\ -\Delta w^{(2)} & = & f_2(\hat{\Psi}_N + w^{(1)}, \hat{\Theta}_N + w^{(2)}) + \Delta\hat{\Theta}_N, \end{cases} \tag{6}$$

which is so-called a residual equation. Setting $w = (w^{(1)}, w^{(2)})$ and

$$\begin{aligned} h_1(w) &= f_1(\hat{\Psi}_N + w^{(1)}, \hat{\Theta}_N + w^{(2)}) - \mathcal{P}\Delta^2\hat{\Psi}_N, \\ h_2(w) &= f_2(\hat{\Psi}_N + w^{(1)}, \hat{\Theta}_N + w^{(2)}) + \Delta\hat{\Theta}_N, \\ h(w) &= (h_1(w), h_2(w)), \end{aligned}$$

by virtue of the Sobolev embbeding theorem and the definition of f_1 and f_2, h is a bounded continuous map from $X^3 \times Y^1$ to $X^0 \times Y^0$. Moreover, it is easily shown that for all $(g_1, g_2) \in X^0 \times Y^0$, the linear problem:

$$\left\{ \begin{array}{rcl} \Delta^2 \bar{\Psi} & = & g_1, \\ -\Delta \bar{\Theta} & = & g_2 \end{array} \right. \tag{7}$$

has a unique solution $(\bar{\Psi}, \bar{\Theta}) \in X^4 \times Y^2$. We denote this mapping by $\bar{\Psi} = (\Delta^2)^{-1} g_1$ and $\bar{\Theta} = (-\Delta)^{-1} g_2$, then the operator:

$$\mathcal{K} := (\mathcal{P}^{-1}(\Delta^2)^{-1}, (-\Delta)^{-1}) : \ X^0 \times Y^0 \rightarrow X^3 \times Y^1$$

is a compact map because of the compactness of the imbedding $X^4 \hookrightarrow X^3$ and $Y^2 \hookrightarrow Y^1$ and the boundedness of $(\Delta^2)^{-1} : X^0 \rightarrow X^4$, $(-\Delta)^{-1} : Y^0 \rightarrow Y^2$. Thus, (6) is rewritten by a fixed-point equation:

$$w = Fw \tag{8}$$

for the compact operator $F := \mathcal{K} \circ h$ on $X^3 \times Y^1$. Therefore, by the Schauder fixed-point theorem, if we find a nonempty, closed, bounded and convex set $W \subset X^3 \times Y^1$, satisfying

$$FW \subset W \tag{9}$$

then there exists a solution of (8) in W. The set W in (9) is referred as *a candidate set* of solutions[2, 3].

3 Extended System

Moreover, in order to obtain the enclosure of the bifurcation point, we set

$$Z := X^3 \times Y^1, \qquad G := I - F$$

and an operator $S : Z \longrightarrow Z$ by

$$Sw = S(\Psi, \Theta) := (\Psi(x + \pi/a, z), \Theta(x + \pi/a, z))$$

satisfying $SGw = GSw$. Using this "symmetric" operator S, we have the decomposition

$$Z = Z_s \oplus Z_a,$$

where $Z_s = \{w \in Z; Sw = w\}$ and $Z_a = \{w \in Z; Sw = -w\}$. Next, considering $\mathcal{R}$ as a variable, let $\mathcal{G}$ on $Z_s \times Z_a \times \mathbb{R}$ be a map defined by

$$\mathcal{G}(w, v, \mathcal{R}) := \left(\begin{array}{l} G(w, \mathcal{R}) \\ D_w G[w, \mathcal{R}]v \\ \mathcal{L}(v) - 1. \end{array} \right) \tag{10}$$

Here $\mathcal{L}$ is an appropriate functional on Z_a. We will check the extended system $\mathcal{G}(w, v, \mathcal{R}) = 0$ has an isolate solution $(w_*, v_*, \mathcal{R}_*) \in Z_s \times Z_a \times \mathbb{R}$ and show a sufficient condition such that $\mathcal{R}_*$ is a symmetry-breaking bifurcation point [4] of $G(w, \mathcal{R}) = 0$ by computer-assisted proof.

参考文献

[1] Chandrasekhar, S.: *Hydrodynamic and Hydromagnetic Stability*, Oxford University Press, 1961.

[2] Watanabe, Y., Yamamoto, N., Nakao, M.T. and Nishida, T.: A Numerical Verification of Nontrivial Solutions for the Heat Convection Problem, *Journal of Mathematical Fluid Mechanics*, Vol.6, No.1, pp.1–20 (2004).

[3] Nakao, M.T., Watanabe, Y., Yamamoto, N. and Nishida, T.: Some Computer Assisted Proofs for Solutions of the Heat Convection Problems, *Reliable Computing*, Vol.9, No.5, pp.359-372 (2003).

[4] Kawanago, T.: A Symmetry-breaking Bifurcation Theorem and Some Related Theorems Applicable to Maps Having Unbounded Derivatives, *Japan Journal of Industrial and Applied Mathematics*, Vol. 21, pp.57-74 (2004).