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Weak high order stochastic Runge-Kutta methods

九州工業大学・情報工学部 小守 良雄 (Yoshio Komori)
Department of Systems Innovation and Informatics,
Kyushu Institute of Technology

Abstract

A new explicit stochastic Runge-Kutta (SRK) scheme of weak order 2 is proposed for non-commutative Stratonovich-type stochastic differential equations (SDEs), which is derivative-free, which attains order 4 for ordinary differential equations and which uses $2m-1$ random variables for one step in the m-dimensional Wiener process case. It is compared with other derivative-free and weak second order schemes in a numerical experiment. In addition, the weak third order conditions are given as a preliminary to seeking higher weak order SRK schemes for multi-dimensional SDEs with one-dimensional Wiener process.

1 Introduction

As the importance of stochastic differential equations (SDEs) increases, numerical methods for SDEs get studied more by many researchers. Especially, many numerical methods in the weak sense have been recently proposed for multi-dimensional SDEs with multiplicative noise in the multi-dimensional Wiener process case, whereas counterparts in the strong sense have been enormously developed in the last 10 years [3].

Among such weak methods, we are concerned with derivative-free methods. Let us introduce results concerning such methods, which attain weak order 2 at least. Kloeden and Platen [6, 10] have proposed derivative-free schemes by replacing necessary derivatives with finite differences. Tocino and Vigo-Aguiar [16] have also proposed one of them as an example in their Runge-Kutta family. Rößler [11, 12] has proposed other derivative-free schemes by assuming a commutativity condition [1, 14], which means

$$\boldsymbol{g}_j^{(1)}(\boldsymbol{y})\boldsymbol{g}_l(\boldsymbol{y}) = \boldsymbol{g}_l^{(1)}(\boldsymbol{y})\boldsymbol{g}_j(\boldsymbol{y}) \qquad (\forall \boldsymbol{y} \in \boldsymbol{R}^d,\ 1 \le j, l \le m,\ j \ne l)$$

in (1). Here, $\boldsymbol{g}_j^{(1)}$ or $\boldsymbol{g}_l^{(1)}$ denotes the derivative of $\boldsymbol{g}_j$ or $\boldsymbol{g}_l$, respectively. On the other hand, Talay and Tubaro [15] have proposed the extrapolation method for SDEs. This method also makes it possible to obtain an approximate solution without using any derivative.

Komori [7] has also proposed a new stochastic Runge-Kutta (SRK) family and developed Butcher's rooted tree analysis [4, 5] (which is for ordinary differential equations (ODEs)) to derive weak order conditions for the new family transparently. Then, utilizing the analysis, he [8] has proposed a new explicit SRK scheme of weak order 2, which is derivative-free and which attains order 4 for ODEs, under the commutativity condition.

In [7, 11, 13, 16], it has been shown that each SRK family includes the scheme proposed by Platen, its counterpart or its derivations when the commutativity condition is not satisfied. It, however, still remains to find a solution of the order conditions of an SRK family in order to obtain another new scheme. Therefore, we aim at solving the order conditions of our SRK family and deriving a new explicit SRK scheme of weak order 2 for non-commutative SDEs. The new scheme will become a piece of evidence that our SRK family is sufficiently general to provide other new schemes.

The present paper is organized as follows. In the next section we will give a brief introduction of our SRK family as well as the expression of its order conditions with rooted trees. In Section 3 we will find a solution of them after giving simplifying assumptions, and give a numerical experiment in the non-commutative case. In Section 4 we will give the summary. The weak third order conditions will be shown in the appendix.

2 SRK family

In this section we introduce an SRK family for SDEs with a multi-dimensional Wiener process. To derive weak order conditions for the family, we utilize rooted tree analysis.

2.1 Weak order

First of all, we introduce the definition of weak (global) order. Let τ_n be an equidistant grid point nh $(n = 0, 1, \ldots, M)$ with step size $h \stackrel{\text{def}}{=} T_{end}/M < 1$ (M is a natural number) and $\boldsymbol{y}_n$ a discrete approximation to the solution $\boldsymbol{y}(\tau_n)$ of the d-dimensional stochastic integral equation

$$\boldsymbol{y}(t) = \boldsymbol{x}_0 + \int_0^t \boldsymbol{g}_0(\boldsymbol{y}(s))\mathrm{d}s + \sum_{j=1}^{m} \int_0^t \boldsymbol{g}_j(\boldsymbol{y}(s)) \circ \mathrm{d}W_j(s), \quad 0 \le t \le T_{end}, \tag{1}$$

where $W_j(s)$ is a scalar Wiener process and $\circ$ means the Stratonovich formulation. The initial approximate random variable $\boldsymbol{y}_0$ is supposed to have the same probability law with all moments finite as that of $\boldsymbol{x}_0$. In addition, let $C_P^L(\boldsymbol{R}^d, \boldsymbol{R})$ be the totality of L times continuously differentiable $\boldsymbol{R}$-valued functions on $\boldsymbol{R}^d$, all of whose partial derivatives of order less than or equal to L have polynomial growth. Then, the definition of weak order is given as follows [2].

Definition 2.1 *Suppose that a discrete approximation $\boldsymbol{y}_M$ is given by a scheme. Then, we say that the scheme is of weak (global) order q if for each $G \in C_P^{2(q+1)}(\boldsymbol{R}^d, \boldsymbol{R})$, $C > 0$ (independent of h) and $\delta > 0$ exist such that*

$$|E[G(\boldsymbol{y}(\tau_M)] - E[G(\boldsymbol{y}_M)]| \le Ch^q, \quad h \in (0, \delta).$$

In order to obtain an approximate solution $\boldsymbol{y}_{n+1}$ of the solution $\boldsymbol{y}(t_{n+1})$ when $\boldsymbol{y}_n$ is given, we consider the SRK family given by

$$\begin{aligned} \boldsymbol{y}_{n+1} &= \boldsymbol{y}_n + \sum_{i=1}^{s} \sum_{j_a, j_b=0}^{m} c_i^{(j_a,j_b)} \boldsymbol{Y}_i^{(j_a,j_b)}, \\ \boldsymbol{Y}_{i_a}^{(j_a,j_b)} &= \tilde{\eta}_{i_a}^{(j_a,j_b)} \Bigg\{ \boldsymbol{g}_{j_b}(\boldsymbol{y}_n + \sum_{i_b=1}^{s} \sum_{j_c,j_d=0}^{m} \alpha_{i_a i_b}^{(j_a,j_b,j_c,j_d)} \boldsymbol{Y}_{i_b}^{(j_c,j_d)}) \\ &\qquad + \boldsymbol{g}_{j_b}^{(1)}(\boldsymbol{y}_n) \sum_{i_b=1}^{s} \sum_{j_c,j_d=0}^{m} \tilde{\gamma}_{i_a i_b}^{(j_a,j_b,j_c,j_d)} \boldsymbol{Y}_{i_b}^{(j_c,j_d)} \Bigg\} \end{aligned} \tag{2}$$

$(1 \le i_a \le s,\ 0 \le j_a, j_b \le m)$, where the constants $c_i^{(j_a,j_b)}$, $\alpha_{i_a i_b}^{(j_a,j_b,j_c,j_d)}$ and $\tilde{\gamma}_{i_a i_b}^{(j_a,j_b,j_c,j_d)}$ are defined by the Butcher tableau and where each $\tilde{\eta}_{i_a}^{(j_a,j_b)}$ is a random variable independent of $\boldsymbol{y}_n$ and satisfies

$$E\left[\left(\tilde{\eta}_{i_a}^{(j_a,j_b)}\right)^{2k}\right] = \begin{cases} K_1 h^{2k} & (j_b = 0), \\ K_2 h^k & (j_b \ne 0) \end{cases}$$

for constants K_1, K_2 and $k = 1, 2, \ldots$. Note that this formulation includes stochastic Rosenbrock-Wanner methods [9].

2.2 Weak order conditions by multi-colored rooted trees

In this subsection we express weak order conditions by multi-colored rooted trees (MRTs). As preliminaries, we introduce several notations and definitions.

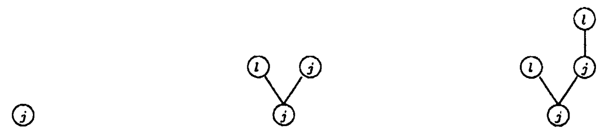
First, we introduce the multi-colored rooted tree (MRT) and a function on its set.

Definition 2.2 (MRT) *An MRT with a root ⓙ (colored with a label j from 0 to m) is a tree recursively defined in the following manner:*

1) $\tau^{(j)}$ is the primitive tree having only a vertex ⓙ.

2) If $t_1, \ldots, t_k$ are MRTs, then $[t_1, \ldots, t_k]^{(j)}$ is also an MRT with the root ⓙ.

The totality of MRTs is denoted by T.



$\tau^{(j)}$ $\qquad$ $[\tau^{(l)},\tau^{(j)}]^{(j)}$ $\qquad$ $[\tau^{(l)},[\tau^{(l)}]^{(j)}]^{(j)}$

Figure 1: Examples of MRTs

Definition 2.3 (Elementary weight $\Phi(t)$ on T) *An elementary weight of $t \in T$ is given recursively as follows:*

$$\Phi(\tau^{(j)};s)=\int_{\tau_n}^{s}\circ dW_j(s_1),\qquad \Phi(t;s)=\int_{\tau_n}^{s}\prod_{i=1}^{k}\Phi(t_i;s_1)\circ dW_j(s_1)\quad \text{for } t=[t_1,\dots,t_k]^{(j)},$$

where $\circ dW_0(s_1)\stackrel{\text{def}}{=}ds_1$.

For ease of notation we will denote $\Phi(t;\tau_{n+1})$ by $\Phi(t)$.

Next, we introduce several matrices related to the formula parameters of (2), the multi-colored rooted tree with labels (MRTL) and a function on its set. Let us adopt nominal symbols $\tilde{\eta}_{s+1}^{(j_a,j_b)}$, $\alpha_{s+1,i_b}^{(j_a,j_b,j_c,j_d)}$ and $\tilde{\gamma}_{s+1,i_b}^{(j_a,j_b,j_c,j_d)}$ and define $\alpha_{s+1,i_b}^{(0,0,j_c,j_d)}\stackrel{\text{def}}{=}c_{i_b}^{(j_c,j_d)}$ for $i_b\geq 1$ and

$$A^{(j,j')}\stackrel{\text{def}}{=}\begin{bmatrix}
\alpha_{11}^{(0,j,0,j')} & \cdots & \alpha_{11}^{(m,j,0,j')} & \cdots & \alpha_{s+1,1}^{(0,j,0,j')} & \cdots & \alpha_{s+1,1}^{(m,j,0,j')}\\
\vdots & & \vdots & & \vdots & & \vdots\\
\alpha_{11}^{(0,j,m,j')} & \cdots & \alpha_{11}^{(m,j,m,j')} & \cdots & \alpha_{s+1,1}^{(0,j,m,j')} & \cdots & \alpha_{s+1,1}^{(m,j,m,j')}\\
\vdots & & \vdots & & \vdots & & \vdots\\
\alpha_{1s}^{(0,j,0,j')} & \cdots & \alpha_{1s}^{(m,j,0,j')} & \cdots & \alpha_{s+1,s}^{(0,j,0,j')} & \cdots & \alpha_{s+1,s}^{(m,j,0,j')}\\
\vdots & & \vdots & & \vdots & & \vdots\\
\alpha_{1s}^{(0,j,m,j')} & \cdots & \alpha_{1s}^{(m,j,m,j')} & \cdots & \alpha_{s+1,s}^{(0,j,m,j')} & \cdots & \alpha_{s+1,s}^{(m,j,m,j')}\\
\mathbf{0} & \cdots & \mathbf{0} & \cdots & \mathbf{0} & \cdots & \mathbf{0}
\end{bmatrix}$$

for $\alpha_{i_a i_b}^{(j_a,j,j_c,j')}$, where $\mathbf{0}$ stands for an $m+1$-dimensional column vector of 0's. Similarly, define the matrix $\tilde{\Gamma}^{(j,j')}$ for $\tilde{\gamma}_{i_a i_b}^{(j_a,j,j_c,,j')}$ and set $\tilde{A}^{(j,,j')}\stackrel{\text{def}}{=}A^{(j,,j')}+\tilde{\Gamma}^{(j,,j')}$. In addition, define the $(m+1)(s+1)\times(m+1)(s+1)$ diagonal matrix $D^{(j)}$ by

$$D^{(j)}\stackrel{\text{def}}{=}\operatorname{diag}(\tilde{\eta}_1^{(0,j)},\dots,\tilde{\eta}_1^{(m,j)},\dots,\tilde{\eta}_{s+1}^{(0,j)},\dots,\tilde{\eta}_{s+1}^{(m,j)}).$$

In the sequel, let us use a label $X^{(j)}\in\{A^{(j)},\tilde{A}^{(j)}\}$ for labels $A^{(j)},\tilde{A}^{(j)}$ as well as a matrix $X^{(j,j')}\in\{A^{(j,j')},\tilde{A}^{(j,j')}\}$.

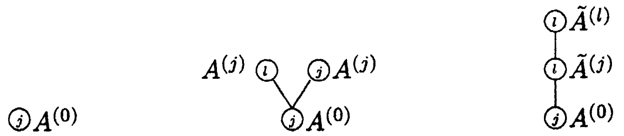
Definition 2.4 (MRTL) *An MRTL denoted by $t_{X^{(j)}}$ is one attached by labels according to the following rules:*

1) *The label of the root is $X^{(j)}$.*

2) *The label of the other vertices is decided by the number of branches and the color of the parent vertex:*

   - *the label is $\tilde{A}^{(j)}$ if the parent vertex has a single branch and it is colored with j,*
   - *the label is $A^{(j)}$ if the parent vertex has more than one branch and it is colored with j.*

The totality of MRTL's whose roots are labeled with $X^{(j)}$, is denoted by $T_{X^{(j)}}$. For example, some MRTL's are listed in Fig. 2.



$\tau^{(j)}_{A^{(0)}}$ $\quad [\tau^{(l)}_{A^{(j)}}, \tau^{(j)}_{A^{(j)}}]^{(j)}_{A^{(0)}}$ $\quad \left[[\tau^{(l)}_{\tilde{A}^{(l)}}]^{(l)}_{\tilde{A}^{(j)}}\right]^{(j)}_{A^{(0)}}$

Figure 2: Examples of trees in $\mathcal{T}_{A^{(0)}}$

Definition 2.5 (Elementary numerical weight $\bar{\Phi}(t)$ on $\mathcal{T}_{X^{(j)}}$) *An elementary numerical weight of $t \in \mathcal{T}_{X^{(j)}}$ is given recursively as follows:*

$$\bar{\Phi}(\tau^{(j')}_{X^{(j)}}) = \mathbf{1}D^{(j')}X^{(j,j')}, \qquad \bar{\Phi}(t) = (\prod_{i=1}^{k} \bar{\Phi}(t_i))D^{(j')}X^{(j,j')} \quad \text{for } t = [t_1, \ldots, t_k]^{(j')}_{X^{(j)}}$$

$(0 \le j, j' \le m)$, where $\tau^{(j')}_{X^{(j)}}$ and $[t_1, \ldots, t_k]^{(j')}_{X^{(j)}}$ express MRTL's whose roots are labeled by $X^{(j)}$. In addition, $\mathbf{1}$ stands for an $(m+1)(s+1)$-dimensional row. vector of 1's, and $\prod_{i=1}^{k} \bar{\Phi}(t_i)$ means the elementwise product of row vectors $\bar{\Phi}(t_i)$.

Now, we can give weak order conditions. Let $\rho(t)$ be the number of vertices of $t \in T$ and $r(t)$ the number of vertices of t with the color 0, and suppose that any component of $\boldsymbol{g}_j$ belongs to $C_P^{2(q+1)}(\boldsymbol{R}^d, \boldsymbol{R})$ $(0 \le j \le m)$ and the regularity of the time discrete approximation is satisfied [6, 7]. In addition, if the following are satisfied, the time discrete approximation $\boldsymbol{y}_M$ converges to the $\boldsymbol{y}(\tau_M)$ with weak (global) order q as $h \to 0$:

$$E\left[\prod_{j=1}^{L} \bar{\Phi}_{(m+1)s+1}(t_j)\right] = E\left[\prod_{j=1}^{L} \Phi(\hat{t}_j)\right] \tag{3}$$

for any $t_1, \ldots, t_L \in \mathcal{T}_{A^{(0)}}$ $(1 \le L \le 2q)$ satisfying $\sum_{j=1}^{L}(\rho(\hat{t}_j) + r(\hat{t}_j)) \le 2q$ and

$$E\left[\bar{\Phi}_{(m+1)s+1}(t)\right] = 0 \tag{4}$$

for any $t \in \mathcal{T}_{A^{(0)}}$ satisfying $\rho(\hat{t}) + r(\hat{t}) = 2q+1$ [7], where $\bar{\Phi}_{(m+1)s+1}(t_j)$ denotes the $((m+1)s+1)$st element of $\bar{\Phi}(t_j)$ and $\hat{t}_j$ denotes an MRT obtained by removing all labels from t_j.

Remark 2.1 *[7] The expectations of both sides of (3) or of the left-hand side of (4) can be obtained directly from diagrams for MRTs or MRTL's. This will be much helpful to seek the order conditions when the order becomes higher. As a result, we should note that even weak third order conditions in the appendix can be obtain transparently.*

3 Solution of order conditions

In the previous section we have shown the order conditions with MRTL's. In this section we will find a solution of them for weak order 2 in the non-commutative case.

3.1 Simplifying assumption

As seen in (3) and (4), the conditions for weak order are generally given in the form of expectations. By replacing expectations with monomials for trees which have only a few vertices, however, we can reduce the number of the order conditions to solve. In relation to $\tau^{(j)}_{A^{(0)}}$, $\tau^{(0)}_{A^{(0)}}$, $[\tau^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{A^{(0)}}$, $[\tau^{(j)}_{\tilde{A}^{(0)}}]^{(0)}_{A^{(0)}}$, $[\tau^{(0)}_{\tilde{A}^{(j)}}]^{(j)}_{A^{(0)}}$, $[\tau^{(l)}_{\tilde{A}^{(j)}}]^{(j)}_{A^{(0)}}$ and $[\tau^{(j)}_{\tilde{A}^{(l)}}]^{(l)}_{A^{(0)}}$ $(j < l)$, let us assume that the following equations hold (simplifying assumptions):

$$\sum c_{i_1}^{(j_1',j)} \tilde{\eta}_{i_1}^{(j_1',j)} = \Delta W_j, \tag{5}$$

$$\sum c_{i_1}^{(j_1',0)} \tilde{\eta}_{i_1}^{(j_1',0)} = h, \qquad \sum c_{i_1}^{(j_1',j)} \tilde{\eta}_{i_1}^{(j_1',j)} \tilde{\alpha}_{i_1 i_2}^{(j_1',j,j_2',j)} \tilde{\eta}_{i_2}^{(j_2',j)} = \frac{(\Delta W_j)^2}{2},$$

$$\sum c_{i_1}^{(j_1',0)} \tilde{\eta}_{i_1}^{(j_1',0)} \tilde{\alpha}_{i_1 i_2}^{(j_1',0,j_2',j)} \tilde{\eta}_{i_2}^{(j_2',j)} = \frac{h\Delta W_j}{2}, \qquad \sum c_{i_1}^{(j_1',j)} \tilde{\eta}_{i_1}^{(j_1',j)} \tilde{\alpha}_{i_1 i_2}^{(j_1',j,j_2',0)} \tilde{\eta}_{i_2}^{(j_2',0)} = \frac{h\Delta W_j}{2},$$

$$\sum c_{i_1}^{(j_1',j)} \tilde{\eta}_{i_1}^{(j_1',j)} \tilde{\alpha}_{i_1 i_2}^{(j_1',j,j_2',l)} \tilde{\eta}_{i_2}^{(j_2',l)} = \frac{\Delta W_j(\Delta W_l + \Delta \tilde{W}_l)}{2} \qquad (j < l), \tag{6}$$

$$\sum c_{i_1}^{(j_1',l)} \tilde{\eta}_{i_1}^{(j_1',l)} \tilde{\alpha}_{i_1 i_2}^{(j_1',l,j_2',j)} \tilde{\eta}_{i_2}^{(j_2',j)} = \frac{\Delta W_j(\Delta W_l - \Delta \tilde{W}_l)}{2} \qquad (j < l), \tag{7}$$

where ΔW_j's $(j = 1, \ldots, m)$ and $\Delta\tilde{W}_l$'s $(l = 2, \ldots, m)$ are mutually independent random variables satisfying

$$E\left[(\Delta W_j)^k\right] = \begin{cases} 0 & (k = 1,3,5), \\ (k-1)h^{k/2} & (k = 2,4), \\ O(h^3) & (k \geq 6), \end{cases} \qquad E\left[(\Delta \tilde{W}_l)^k\right] = \begin{cases} 0 & (k = 1,3), \\ h & (k = 2), \\ O(h^2) & (k \geq 4), \end{cases} \tag{8}$$

and $\tilde{\alpha}_{i_k i_{k+1}}^{(j_k',j,j_{k+1}',l)} \stackrel{\text{def}}{=} \alpha_{i_k i_{k+1}}^{(j_k',j,j_{k+1}',l)} + \tilde{\gamma}_{i_k i_{k+1}}^{(j_k',j,j_{k+1}',l)}$. Note that the expressions in the right-hand side of (6) and (7) come from the approximation

$$\Phi\left(\begin{smallmatrix} \textcircled{\scriptsize l} \\ | \\ \textcircled{\scriptsize j} \end{smallmatrix}\right) \approx \begin{cases} \dfrac{\Delta W_j(\Delta W_l + \Delta \tilde{W}_l)}{2} & (j < l), \\ \dfrac{\Delta W_l(\Delta W_j - \Delta \tilde{W}_j)}{2} & (j > l). \end{cases}$$

Whereas the first five simplifying assumptions satisfy

$$E\left[\left\{\bar{\Phi}_{(m+1)s+1}\left(\tau_{A^{(0)}}^{(j)}\right)\right\}^2 \left\{\bar{\Phi}_{(m+1)s+1}\left(\tau_{A^{(0)}}^{(l)}\right)\right\}^2\right] = h^2,$$

$$E\left[\bar{\Phi}_{(m+1)s+1}\left([\tau_{\bar{A}^{(j)}}^{(j)}]_{A^{(0)}}^{(j)}\right) \bar{\Phi}_{(m+1)s+1}\left([\tau_{\bar{A}^{(l)}}^{(l)}]_{A^{(0)}}^{(l)}\right)\right] = \frac{h^2}{4},$$

$$E\left[\bar{\Phi}_{(m+1)s+1}\left([\tau_{\bar{A}^{(j)}}^{(j)}]_{A^{(0)}}^{(j)}\right) \left\{\bar{\Phi}_{(m+1)s+1}\left(\tau_{A^{(0)}}^{(l)}\right)\right\}^2\right] = \frac{h^2}{2}$$

and the 11 order conditions of weak order 2 for $t = \tau_{A^{(0)}}^{(j)}, \tau_{A^{(0)}}^{(0)}, [\tau_{\bar{A}^{(j)}}^{(0)}]_{A^{(0)}}^{(j)}, [\tau_{\bar{A}^{(0)}}^{(j)}]_{A^{(0)}}^{(0)}$ (for details, see the appendix), (6) and (7) satisfy

$$E\left[\left\{\bar{\Phi}_{(m+1)s+1}\left(\begin{smallmatrix} \textcircled{\scriptsize l}\,\bar{A}^{(j)} \\ | \\ \textcircled{\scriptsize j}\,A^{(0)} \end{smallmatrix}\right)\right\}^2\right] = \frac{h^2}{2}, \qquad E\left[\bar{\Phi}_{(m+1)s+1}\left(\begin{smallmatrix} \textcircled{\scriptsize l}\,\bar{A}^{(j)} \\ | \\ \textcircled{\scriptsize j}\,A^{(0)} \end{smallmatrix}\right) \bar{\Phi}_{(m+1)s+1}\left(\begin{smallmatrix} \textcircled{\scriptsize j}\,\bar{A}^{(l)} \\ | \\ \textcircled{\scriptsize l}\,A^{(0)} \end{smallmatrix}\right)\right] = 0,$$

$$E\left[\bar{\Phi}_{(m+1)s+1}\left(\begin{smallmatrix} \textcircled{\scriptsize l}\,\bar{A}^{(j)} \\ | \\ \textcircled{\scriptsize j}\,A^{(0)} \end{smallmatrix}\right) \bar{\Phi}_{(m+1)s+1}\left(\textcircled{\scriptsize j}\,A^{(0)}\right) \bar{\Phi}_{(m+1)s+1}\left(\textcircled{\scriptsize l}\,A^{(0)}\right)\right] = \frac{h^2}{2}$$

for $j \neq l$. Since these cause difficulties in the construction of weak second order schemes for non-commutative SDEs, it is remarkable that the virtue of the simplifying assumptions (6) and (7) ensures that the 3 order conditions hold.

3.2 Explicit SRK methods

We consider explicit SRK methods and show how to solve the order conditions.

First of all, we set

$$\tilde{\eta}_i^{(0,0)} = h, \qquad \tilde{\eta}_i^{(j_a,j_b)} = \begin{cases} \Delta \tilde{W}_{j_b} & (j_b > j_a > 0), \\ \Delta W_{j_b} & (j_a \geq j_b > 0). \end{cases} \tag{9}$$

Here, $\tilde{\eta}_i^{(0,j_b)}$ $(j_b > 0)$ does not need to be set since it is not used below. Next, let us set $c_i^{(j_a,0)} = 0$ if $j_a \neq 0$, $\alpha_{i_a i_b}^{(j_a,j,j_c,j)} = 0$ if $j_a \neq j$ or $j_c \neq j$ when $j > 0$, $\alpha_{i_a i_b}^{(j_a,0,j_c,j)} = 0$ if $j_a \neq 0$ or $j_c \neq j$ when $j > 0$, $\alpha_{i_a i_b}^{(j_a,j,j_c,0)} = 0$ if $j_a \neq j$ or $j_c \neq 0$ when $j > 0$, $\alpha_{i_a i_b}^{(j_a,0,j_c,0)} = 0$ if $j_a \neq 0$ or $j_c \neq 0$, and $\alpha_{i_a i_b}^{(j_a,j,j_c,l)} = 0$ if $j_a = j_c$ or $j_a \neq j, l$ and $j_c \neq j, l$ when $l > j > 0$, or if $j_a \neq j, l$ or $j_c \neq l$ when $j > l > 0$. These settings, (5) and (8) imply that the following statement holds for MRTL's related to weak order 2:

> The expectation of the $((m+1)s+1)$-st element of an elementary numerical weight or the product of those is equal to 0 if the odd number of vertices are of the same color $j (\neq 0)$.

As we have seen in Remark 2.1, the expectation of an elementary weight or the product of those vanishes if the odd number of vertices are of the same color $j(\neq 0)$. Consequently, the above statement ensures that (3) holds for such MRTL's and (4) holds.

Then, let us introduce

$$c_i^{(j)} \stackrel{\text{def}}{=} c_i^{(j,j)}, \qquad \alpha_{i_a i_b}^{(j,j')} \stackrel{\text{def}}{=} \alpha_{i_a i_b}^{(j,j,j',j')}, \qquad A_{i_a}^{(j,j')} \stackrel{\text{def}}{=} \sum_{i_b=1}^{i_a-1} \alpha_{i_a i_b}^{(j,j')} \qquad (j, j' \geq 0),$$

$$A_{i_a}^{(l,j,j,l)} \stackrel{\text{def}}{=} \sum_{i_b=1}^{i_a-1} \alpha_{i_a i_b}^{(l,j,j,l)}, \qquad A_{i_a}^{(j,l,j,j)} \stackrel{\text{def}}{=} \sum_{i_b=1}^{i_a-1} \alpha_{i_a i_b}^{(j,l,j,j)} \qquad (l > j > 0)$$

for ease of notation.

From (9) we obtain

$$\sum c_{i_1}^{(j_1',j)} \tilde{\eta}_{i_1}^{(j_1',j)} = \sum_{i_1} c_{i_1}^{(j)} \Delta W_j + \sum_{\substack{i_1 \\ j_1' > j}} c_{i_1}^{(j_1',j)} \Delta W_j + \sum_{\substack{i_1 \\ j_1' < j}} c_{i_1}^{(j_1',j)} \Delta \tilde{W}_j.$$

Hence, if

$$\sum_{i_1} c_{i_1}^{(j)} = 1, \qquad \sum_{i_1} c_{i_1}^{(l,j)} = 0 \quad (j < l), \qquad \sum_{i_1} c_{i_1}^{(j,l)} = 0 \quad (j < l),$$

then, (5) holds.

Since we also obtain

$$\sum c_{i_1}^{(j_1',j)} \tilde{\eta}_{i_1}^{(j_1',j)} \alpha_{i_1 i_2}^{(j_1',j,j_2',l)} \tilde{\eta}_{i_2}^{(j_2',l)} = \sum_{i_1,i_2} c_{i_1}^{(j)} \Delta W_j \alpha_{i_1 i_2}^{(j,l)} \Delta W_l + \sum_{i_1,i_2} c_{i_1}^{(l,j)} \Delta W_j \alpha_{i_1 i_2}^{(l,j,j,l)} \Delta \tilde{W}_l$$

when $j < l$, (6) is equivalent to

$$\sum_{i_1} c_{i_1}^{(j)} A_{i_1}^{(j,l)} = \frac{1}{2}, \qquad \sum_{i_1} c_{i_1}^{(l,j)} A_{i_1}^{(l,j,j,l)} = \frac{1}{2} \qquad (j < l).$$

Here, note that $\tilde{\gamma}_{i_a i_b}^{(j_a,j_b,j_c,j_d)} = 0$ $(\forall i_a, i_b, j_a, j_b, j_c, j_d)$ since we consider explicit SRK methods. Similarly, (7) is equivalent to

$$\sum_{i_1} c_{i_1}^{(l)} A_{i_1}^{(l,j)} = \frac{1}{2}, \qquad \sum_{i_1} c_{i_1}^{(j,l)} A_{i_1}^{(j,l,j,j)} = -\frac{1}{2} \qquad (j < l).$$

As we have seen, each of (5), (6) and (7) yields at least 2 algebraic equations as a sufficient or equivalent condition. In analogy, each of the following 2 order conditions also yields 2 algebraic equations. The order condition

$$E\left[\bar{\Phi}_{(m+1)s+1}\left([\tau_{A^{(j)}}^{(l)}, [\tau_{\tilde{A}^{(l)}}^{(j)}]_{A^{(j)}}^{(l)}]_{A^{(0)}}^{(j)}\right)\right] = 0 \qquad (j \neq l)$$

yields

$$\sum_{i_1,i_2} c_{i_1}^{(j)} A_{i_1}^{(j,l)} \alpha_{i_1 i_2}^{(j,l)} A_{i_2}^{(l,j)} = 0 \quad (j \neq l), \qquad \sum_{i_1,i_2} c_{i_1}^{(l,j)} A_{i_1}^{(l,j,j,l)} \alpha_{i_1 i_2}^{(l,j,j,l)} A_{i_2}^{(j,l,j,j)} = 0 \quad (j < l),$$

Table 1: The other order conditions for weak order 2

t	Order condition†	t	Order condition†
$[[[\tau^{(l)}_{\tilde{A}^{(l)}}]^{(l)}_{\tilde{A}^{(j)}}]^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{A^{(0)}}$	$E_a(t) = \frac{h^2}{8}$	$[[[\tau^{(l)}_{\tilde{A}^{(j)}}]^{(j)}_{\tilde{A}^{(l)}}]^{(l)}_{\tilde{A}^{(j)}}]^{(j)}_{A^{(0)}}$	$E_a(t) = 0$
$[[[\tau^{(j)}_{\tilde{A}^{(l)}}]^{(l)}_{\tilde{A}^{(l)}}]^{(l)}_{\tilde{A}^{(j)}}]^{(j)}_{A^{(0)}}$	$E_a(t) = 0$	$[[\tau^{(l)}_{A^{(j)}}, \tau^{(l)}_{A^{(j)}}]^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{A^{(0)}}$	$E_a(t) = \frac{h^2}{4}$
$[[\tau^{(l)}_{A^{(l)}}, \tau^{(j)}_{A^{(l)}}]^{(l)}_{\tilde{A}^{(j)}}]^{(j)}_{A^{(0)}}$	$E_a(t) = 0$	$[\tau^{(j)}_{A^{(j)}}, [\tau^{(l)}_{\tilde{A}^{(l)}}]^{(l)}_{A^{(j)}}]^{(j)}_{A^{(0)}}$	$E_a(t) = \frac{h^2}{8}$
$[\tau^{(l)}_{A^{(j)}}, [\tau^{(l)}_{\tilde{A}^{(j)}}]^{(j)}_{A^{(j)}}]^{(j)}_{A^{(0)}}$	$E_a(t) = \frac{h^2}{4}$	$[\tau^{(j)}_{A^{(j)}}, \tau^{(l)}_{A^{(j)}}, \tau^{(l)}_{A^{(j)}}]^{(j)}_{A^{(0)}}$	$E_a(t) = \frac{h^2}{4}$
$[[\tau^{(l)}_{\tilde{A}^{(j)}}]^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{A^{(0)}}$	$E_f(t, \tau^{(l)}_{A^{(0)}}) = \frac{h^2}{4}$	$[[\tau^{(j)}_{\tilde{A}^{(l)}}]^{(l)}_{\tilde{A}^{(j)}}]^{(j)}_{A^{(0)}}$	$E_f(t, \tau^{(l)}_{A^{(0)}}) = 0$
$[[\tau^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{\tilde{A}^{(l)}}]^{(l)}_{A^{(0)}}$	$E_f(t, \tau^{(l)}_{A^{(0)}}) = \frac{h^2}{4}$	$[\tau^{(j)}_{A^{(j)}}, \tau^{(l)}_{A^{(j)}}]^{(j)}_{A^{(0)}}$	$E_f(t, \tau^{(l)}_{A^{(0)}}) = \frac{h^2}{4}$

†$E_a(t)$ and $E_f(t, t_1)$ are defined in Appendix.

and the order condition

$$E\left[\bar{\Phi}_{(m+1)s+1}\left([\tau^{(l)}_{A^{(j)}}, \tau^{(l)}_{A^{(j)}}]^{(j)}_{A^{(0)}}\right)\bar{\Phi}_{(m+1)s+1}\left(\tau^{(j)}_{A^{(0)}}\right)\right] = \frac{h^2}{2} \qquad (j \neq l)$$

yields

$$\sum_{i_1} c^{(j)}_{i_1}\left(A^{(j,l)}_{i_1}\right)^2 = \frac{1}{2} \quad (j \neq l), \qquad \sum_{i_1} c^{(l,j)}_{i_1}\left(A^{(l,j,j,l)}_{i_1}\right)^2 = 0 \quad (j < l).$$

On the other hand, the other order conditions shown in Table 1 and 12 order conditions of weak order 2 for the following MRTL's yield just 1 algebraic equation, respectively: $[\tau^{(0)}_{\tilde{A}^{(0)}}]^{(0)}_{A^{(0)}}$, $[[\tau^{(0)}_{\tilde{A}^{(j)}}]^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{A^{(0)}}$, $[[\tau^{(j)}_{A^{(0)}}]^{(0)}_{\tilde{A}^{(j)}}]^{(j)}_{A^{(0)}}$, $[[\tau^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{\tilde{A}^{(0)}}]^{(0)}_{A^{(0)}}$, $[\tau^{(0)}_{A^{(j)}}, \tau^{(j)}_{A^{(j)}}]^{(j)}_{A^{(0)}}$, $[\tau^{(j)}_{A^{(0)}}, \tau^{(j)}_{A^{(0)}}]^{(0)}_{A^{(0)}}$, $[[[\tau^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{A^{(0)}}$, $[[\tau^{(j)}_{A^{(j)}}, \tau^{(j)}_{A^{(j)}}]^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{A^{(0)}}$, $[\tau^{(j)}_{A^{(j)}}, [\tau^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{A^{(j)}}]^{(j)}_{A^{(0)}}$, $[\tau^{(j)}_{A^{(j)}}, \tau^{(j)}_{A^{(j)}}, \tau^{(j)}_{A^{(j)}}]^{(j)}_{A^{(0)}}$ $[[\tau^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{A^{(0)}}$ and $[\tau^{(j)}_{A^{(j)}}, \tau^{(j)}_{A^{(j)}}]^{(j)}_{A^{(0)}}$.

Ultimately, in order to find a solution that satisfies the simplifying conditions and the order conditions, all we have to do is to solve the system of equations in Table 2. Here, note that $\alpha^{(j,j')}_{i_a i_b} = 0$ ($i_a \leq i_b$, $\forall j, j'$) and $\tilde{\gamma}^{(j_a,j_b,j_c,j_d)}_{i_a i_b} = 0$ ($\forall i_a, i_b, j_a, j_b, j_c, j_d$) since we consider explicit SRK methods. In the sequel, we suppose $j, l \neq 0$ and omit to write $j \neq l$ as far as it does not cause a confusion. Moreover, we omit all indices $i_1, i_2, \ldots$ in all summations for ease of notation.

Table 2: Simplifying or order conditions

No.	Condition	No.	Condition	No.	Condition
1	$\sum c^{(0)}_{i_1} = 1$	12	$\sum c^{(j)}_{i_1}\alpha^{(j,j)}_{i_1 i_2}\alpha^{(j,j)}_{i_2 i_3}A^{(j,j)}_{i_3} = \frac{1}{24}$	23	$\sum c^{(j)}_{i_1}\alpha^{(j,l)}_{i_1 i_2}\alpha^{(l,l)}_{i_2 i_3}A^{(l,j)}_{i_3} = 0$
2	$\sum c^{(j)}_{i_1} = 1$	13	$\sum c^{(j)}_{i_1}\alpha^{(j,j)}_{i_1 i_2}\left(A^{(j,j)}_{i_2}\right)^2 = \frac{1}{12}$	24	$\sum c^{(j)}_{i_1}\alpha^{(j,j)}_{i_1 i_2}\left(A^{(j,l)}_{i_2}\right)^2 = \frac{1}{4}$
3	$\sum c^{(j)}_{i_1}A^{(j,j)}_{i_1} = \frac{1}{2}$	14	$\sum c^{(j)}_{i_1}A^{(j,j)}_{i_1}\alpha^{(j,j)}_{i_1 i_2}A^{(j,j)}_{i_2} = \frac{1}{8}$	25	$\sum c^{(j)}_{i_1}\alpha^{(j,l)}_{i_1 i_2}A^{(l,l)}_{i_2}A^{(l,j)}_{i_2} = 0$
4	$\sum c^{(0)}_{i_1}A^{(0,j)}_{i_1} = \frac{1}{2}$	15	$\sum c^{(j)}_{i_1}\left(A^{(j,j)}_{i_1}\right)^3 = \frac{1}{4}$	26	$\sum c^{(j)}_{i_1}A^{(j,j)}_{i_1}\alpha^{(j,l)}_{i_1 i_2}A^{(l,l)}_{i_2} = \frac{1}{8}$
5	$\sum c^{(j)}_{i_1}A^{(j,0)}_{i_1} = \frac{1}{2}$	16	$\sum c^{(j)}_{i_1}\alpha^{(j,j)}_{i_1 i_2}A^{(j,j)}_{i_2} = \frac{1}{6}$	27	$\sum c^{(j)}_{i_1}A^{(j,l)}_{i_1}\alpha^{(j,j)}_{i_1 i_2}A^{(j,l)}_{i_2} = \frac{1}{4}$
6	$\sum c^{(0)}_{i_1}A^{(0,0)}_{i_1} = \frac{1}{2}$	17	$\sum c^{(j)}_{i_1}\left(A^{(j,j)}_{i_1}\right)^2 = \frac{1}{3}$	28	$\sum c^{(j)}_{i_1}A^{(j,j)}_{i_1}\left(A^{(j,l)}_{i_1}\right)^2 = \frac{1}{4}$
7	$\sum c^{(j)}_{i_1}\alpha^{(j,j)}_{i_1 i_2}A^{(j,0)}_{i_2} = \frac{1}{4}$	18	$\sum c^{(j)}_{i_1}A^{(j,l)}_{i_1} = \frac{1}{2}$	29	$\sum c^{(j)}_{i_1}\alpha^{(j,j)}_{i_1 i_2}A^{(j,l)}_{i_2} = \frac{1}{4}$
8	$\sum c^{(0)}_{i_1}\alpha^{(0,j)}_{i_1 i_2}A^{(j,j)}_{i_2} = \frac{1}{4}$	19	$\sum c^{(j)}_{i_1}A^{(j,l)}_{i_1}\alpha^{(j,l)}_{i_1 i_2}A^{(l,j)}_{i_2} = 0$	30	$\sum c^{(j)}_{i_1}\alpha^{(j,l)}_{i_1 i_2}A^{(l,l)}_{i_2} = \frac{1}{4}$
9	$\sum c^{(j)}_{i_1}\alpha^{(j,0)}_{i_1 i_2}A^{(0,j)}_{i_2} = 0$	20	$\sum c^{(j)}_{i_1}\left(A^{(j,l)}_{i_1}\right)^2 = \frac{1}{2}$	31	$\sum c^{(j)}_{i_1}\alpha^{(j,l)}_{i_1 i_2}A^{(l,j)}_{i_2} = 0$
10	$\sum c^{(0)}_{i_1}\left(A^{(0,j)}_{i_1}\right)^2 = \frac{1}{2}$	21	$\sum c^{(j)}_{i_1}\alpha^{(j,j)}_{i_1 i_2}\alpha^{(j,l)}_{i_2 i_3}A^{(l,l)}_{i_3} = \frac{1}{8}$	32	$\sum c^{(j)}_{i_1}A^{(j,j)}_{i_1}A^{(j,l)}_{i_1} = \frac{1}{4}$
11	$\sum c^{(j)}_{i_1}A^{(j,0)}_{i_1}A^{(j,j)}_{i_1} = \frac{1}{4}$	22	$\sum c^{(j)}_{i_1}\alpha^{(j,l)}_{i_1 i_2}\alpha^{(l,j)}_{i_2 i_3}A^{(j,l)}_{i_3} = 0$		

No.	Condition	No.	Condition
33	$\sum c^{(l,j)}_{i_1} = 0 \qquad (j < l)$	36	$\sum c^{(j,l)}_{i_1}A^{(j,l,j,j)}_{i_1} = -\frac{1}{2} \qquad (j < l)$
34	$\sum c^{(j,l)}_{i_1} = 0 \qquad (j < l)$	37	$\sum c^{(l,j)}_{i_1}A^{(l,j,j,l)}_{i_1}\alpha^{(l,j,j,l)}_{i_1 i_2}A^{(j,l,j,j)}_{i_2} = 0 \qquad (j < l)$
35	$\sum c^{(l,j)}_{i_1}A^{(l,j,j,l)}_{i_1} = \frac{1}{2} \qquad (j < l)$	38	$\sum c^{(l,j)}_{i_1}\left(A^{(l,j,j,l)}_{i_1}\right)^2 = 0 \qquad (j < l)$

Table 3: Additional conditions to attain order 4 for ODEs

No.	Condition	No.	Condition
39	$\sum c_{i_1}^{(0)} \alpha_{i_1 i_2}^{(0,0)} \alpha_{i_2 i_3}^{(0,0)} A_{i_3}^{(0,0)} = \frac{1}{24}$	42	$\sum c_{i_1}^{(0)} \left(A_{i_1}^{(0,0)}\right)^3 = \frac{1}{4}$
40	$\sum c_{i_1}^{(0)} \alpha_{i_1 i_2}^{(0,0)} \left(A_{i_2}^{(0,0)}\right)^2 = \frac{1}{12}$	43	$\sum c_{i_1}^{(0)} \alpha_{i_1 i_2}^{(0,0)} A_{i_2}^{(0,0)} = \frac{1}{6}$
41	$\sum c_{i_1}^{(0)} A_{i_1}^{(0,0)} \alpha_{i_1 i_2}^{(0,0)} A_{i_2}^{(0,0)} = \frac{1}{8}$	44	$\sum c_{i_1}^{(0)} \left(A_{i_1}^{(0,0)}\right)^2 = \frac{1}{3}$

Table 4: Conditions equivalent to Conditions 18–32

No.	Condition	No.	Condition	No.	Condition
45	$c_3^{(j)} + c_4^{(j)} = \frac{1}{2}$	47	$c_4^{(j)} \alpha_{43}^{(j,j)} = \frac{1}{4}$	49	$\alpha_{32}^{(j,l)} = \alpha_{42}^{(j,l)}$
46	$c_3^{(j)} A_3^{(j,j)} + c_4^{(j)} A_4^{(j,j)} = \frac{1}{4}$	48	$\alpha_{42}^{(j,l)} A_2^{(l,l)} = \frac{1}{2}$		

The system of Conditions 2, 3, 12, 13, 14, 15, 16 and 17 has the same algebraic structure as that of the order conditions for ordinary Runge-Kutta methods to attain order 4 for ODEs ([4], pp. 90-91). Hence, since the stage number s has to be at least 4, let us suppose $s = 4$ in the sequel.

For SRK schemes, Rößler ([11], p. 99) has proposed taking account of not only weak order but also order for ODEs. Now, for $s = 4$, we can let (2) attain order 4 for ODEs. For this, we add the 6 conditions in Table 3, which come from $[[[\tau_{\tilde{A}^{(0)}}^{(0)}]_{\tilde{A}^{(0)}}^{(0)}]_{\tilde{A}^{(0)}}^{(0)}]_{A^{(0)}}^{(0)}$, $[[\tau_{A^{(0)}}^{(0)}, \tau_{A^{(0)}}^{(0)}]_{\tilde{A}^{(0)}}^{(0)}]_{A^{(0)}}^{(0)}$, $[\tau_{A^{(0)}}^{(0)}, [\tau_{\tilde{A}^{(0)}}^{(0)}]_{A^{(0)}}^{(0)}]_{A^{(0)}}^{(0)}$, $[\tau_{A^{(0)}}^{(0)}, \tau_{A^{(0)}}^{(0)}, \tau_{A^{(0)}}^{(0)}]_{A^{(0)}}^{(0)}$, $[[\tau_{\tilde{A}^{(0)}}^{(0)}]_{\tilde{A}^{(0)}}^{(0)}]_{A^{(0)}}^{(0)}$ and $[\tau_{A^{(0)}}^{(0)}, \tau_{A^{(0)}}^{(0)}]_{A^{(0)}}^{(0)}$.

To find a solution, we first simplify Conditions 18–32. By noting that we can suppose $\alpha_{32}^{(j,l)} = \alpha_{32}^{(l,j)}$, we have $\alpha_{43}^{(j,l)} A_2^{(j,l)} = 0$ from Conditions 21 and 22. If $A_2^{(j,l)} = 0$, by noting that we can suppose $A_i^{(j,l)} = A_i^{(l,j)}$ for any i, we have $\alpha_{43}^{(j,l)} = 0$ from Conditions 29 and 31. Similarly, if $\alpha_{43}^{(j,l)} = 0$, we have $A_2^{(l,j)} = 0$ from Conditions 25 and 30. Hence, $\alpha_{43}^{(j,l)} = A_2^{(j,l)} = 0$. Then, $A_3^{(j,l)} = A_4^{(j,l)} = 1$ holds from Conditions 24, 27 and 29. In summary, we have

$$\alpha_{43}^{(j,l)} = A_2^{(j,l)} = 0, \qquad A_3^{(j,l)} = A_4^{(j,l)} = 1.$$

By substituting these into Conditions 18–32 and rewriting them, we obtain the 5 conditions in Table 4.

The system of Conditions 1–17 and Conditions 39–49 has the same structure as that of all the order conditions in the commutative case [8]. Hence, we can obtain a solution of them by carrying out the calculation steps in [8].

Let us solve the system of Conditions 33–38. When we set

$$c_1^{(j,l)} = c_4^{(j,l)} = c_1^{(l,j)} = c_4^{(l,j)} = A_2^{(j,l,j,j)} = 0 \qquad (j < l),$$

Condition 37 holds automatically, and we obtain

$$A_2^{(l,j,j,l)} = -A_3^{(l,j,j,l)}, \quad c_2^{(l,j)} = -\frac{1}{4A_3^{(l,j,j,l)}}, \quad c_3^{(l,j)} = \frac{1}{4A_3^{(l,j,j,l)}} \qquad (j < l,\ A_3^{(l,j,j,l)} \neq 0)$$

from Conditions 33, 35 and 38 and

$$c_2^{(j,l)} = \frac{1}{2A_3^{(j,l,j,j)}}, \qquad c_3^{(j,l)} = -\frac{1}{2A_3^{(j,l,j,j)}} \qquad (j < l,\ A_3^{(j,l,j,j)} \neq 0)$$

from Conditions 34 and 36. We finally obtain

$$\frac{\left[\alpha_{i_a i_b}^{(j,l,j,j)}\right]}{\left(c^{(j,l)}\right)^{\mathrm{T}}} = \begin{array}{cccc} 0 & & & \\ A_3^{(j,l,j,j)} - \alpha_{32}^{(j,l,j,j)} & \alpha_{32}^{(j,l,j,j)} & & \\ \alpha_{41}^{(j,l,j,j)} & \alpha_{42}^{(j,l,j,j)} & \alpha_{43}^{(j,l,j,j)} & \\ \hline 0 & \frac{1}{2A_3^{(j,l,j,j)}} & -\frac{1}{2A_3^{(j,l,j,j)}} & 0 \end{array} \qquad (j < l),$$

$$
\frac{\left[\alpha_{i_a i_b}^{(l,j,j,l)}\right]}{\left(c^{(l,j)}\right)^{\mathrm{T}}} =
\begin{array}{cccc}
-A_3^{(l,j,j,l)} & & & \\
A_3^{(l,j,j,l)} - \alpha_{32}^{(l,j,j,l)} & \alpha_{32}^{(l,j,j,l)} & & \\
\alpha_{41}^{(l,j,j,l)} & \alpha_{42}^{(l,j,j,l)} & \alpha_{43}^{(l,j,j,l)} & \\
\hline
0 & -\frac{1}{4A_3^{(l,j,j,l)}} & \frac{1}{4A_3^{(l,j,j,l)}} & 0
\end{array}
\qquad (j < l),
$$

$$
\frac{\begin{array}{c|cc}
\left[\alpha_{i_a i_b}^{(0,0)}\right] & \left[\alpha_{i_a i_b}^{(j,0)}\right] & \\
\left[\alpha_{i_a i_b}^{(0,j)}\right] & \left[\alpha_{i_a i_b}^{(j,j)}\right] & \left[\alpha_{i_a i_b}^{(j,l)}\right]
\end{array}}{\begin{array}{c|c}\left(c^{(0)}\right)^{\mathrm{T}} & \left(c^{(j)}\right)^{\mathrm{T}}\end{array}} =
\begin{array}{cccc|ccccccc}
\frac{1}{2} & & & & 2 - 2\alpha_{31}^{(j,0)} & & & & & & \\
0 & \frac{1}{2} & & & \alpha_{31}^{(j,0)} & 0 & & & & & \\
0 & 0 & 1 & & 3\alpha_{31}^{(j,0)} - 2 & 0 & 0 & & & & \\
1 & & & & \frac{2}{3} & & & & 0 & & \\
-\frac{9}{8} & \frac{9}{8} & & & \frac{1}{12} & \frac{1}{4} & & & \frac{1}{4} & \frac{3}{4} & \\
1 & 0 & 0 & & -\frac{5}{4} & \frac{1}{4} & 2 & & \frac{1}{4} & \frac{3}{4} & 0 \\
\hline
\frac{1}{6} & \frac{1}{3} & \frac{1}{3} & \frac{1}{6} & \frac{1}{8} & \frac{3}{8} & \frac{3}{8} & \frac{1}{8} & & &
\end{array}
$$

as a solution of all the order conditions. Note that the set of coefficients for $c_i^{(j)}$'s, $\alpha_{i_a i_b}^{(j,j)}$'s and $\alpha_{i_a i_b}^{(j,l)}$'s in the right-hand side of the last equation is unique with respect to the five cases where a solution surely exists for the system of Conditions 2, 3, 12, 13, 14, 15, 16 and 17 [8].

3.3 Numerical experiment

We show the results of a numerical experiment to confirm that the explicit scheme in the previous subsection attains weak order 2 when $\alpha_{31}^{(j,0)} = \alpha_{32}^{(j,l,j,j)} = \alpha_{32}^{(l,j,j,l)} = 0$, $A_3^{(j,l,j,j)} = 1$, $A_3^{(l,j,j,l)} = 1/2$ and $\alpha_{4i_b}^{(j,l,j,j)} = \alpha_{4i_b}^{(l,j,j,l)} = 0$ for $j < l$ and $i_b = 1,2,3$, and to compare it with Platen's scheme ([6], p. 486) or with a scheme for commutative SDEs, which is obtained by setting all $c_i^{(j,l)}$'s $(j \neq l)$, $\alpha_{i_a i_b}^{(j,l,j,j)}$'s and $\alpha_{i_a i_b}^{(l,j,j,l)}$'s $(j < l)$ at 0 in our scheme. This scheme for commutative SDEs satisfies all the order conditions except Conditions 35 and 36.

The following SDE is considered:

$$
\mathrm{d}y(t) = \left(R - \frac{1}{2}\sum_{j=1}^{m} B_j^2 \right) \boldsymbol{y}(t)\mathrm{d}t + \sum_{j=1}^{m} B_j \boldsymbol{y}(t) \circ \mathrm{d}W_j(t), \quad \boldsymbol{y}(0) = \boldsymbol{x}_0, \quad 0 \le t \le 1. \tag{10}
$$

This is non-commutative if $B_j B_l \neq B_l B_j$ $(j \neq l)$.

In (10), we set $m = 2$,

$$
R = \begin{bmatrix} 0 & 1 \\ -3 & -2 \end{bmatrix}, \quad B_1 = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{4} \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 & \frac{1}{4} \\ \frac{1}{4} & 0 \end{bmatrix}, \quad \boldsymbol{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{(w.p.1)}.
$$

Then, we sought $\boldsymbol{y}_M$ by means of the schemes, and calculated the arithmetic variance $\langle y_{M,i}^2 \rangle - \langle y_{M,i} \rangle^2$ of the ith element of $\boldsymbol{y}_M$ and $\langle y_{M,1} y_{M,2} \rangle$ as approximate values of variances $V[y_i(1)]$ $(i = 1,2)$ and $E[y_1(1)y_2(1)]$, respectively. The notation $\langle \cdot \rangle$ stands for an arithmetic mean. On the other hand, their exact values were sought from $\mathrm{d}E[\boldsymbol{y}(t)]/\mathrm{d}t = RE[\boldsymbol{y}(t)]$ and

$$
\frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} E[y_1^2(t)] \\ E[y_1(t)y_2(t)] \\ E[y_2^2(t)] \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & 2 & \frac{1}{16} \\ -3 & -\frac{33}{16} & 1 \\ \frac{1}{16} & -6 & -\frac{63}{16} \end{bmatrix} \begin{bmatrix} E[y_1^2(t)] \\ E[y_1(t)y_2(t)] \\ E[y_2^2(t)] \end{bmatrix}.
$$

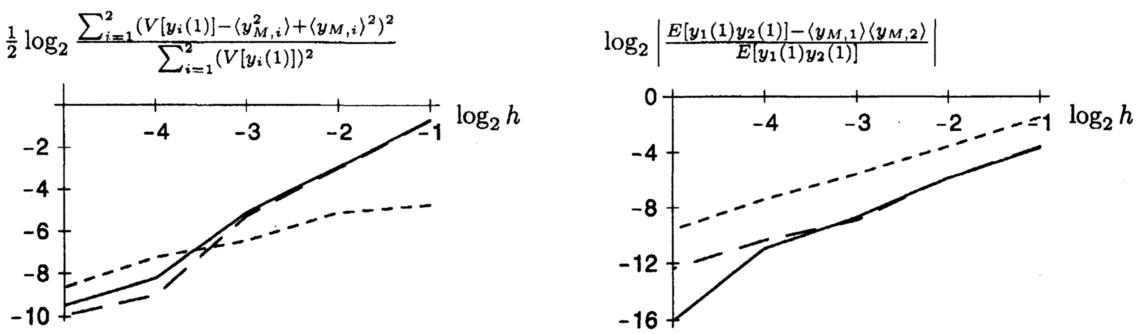


Figure 3: Relative errors in (10)

In the experiment, 1×10^6 sets of independent trajectories were simulated for each step. The results are indicated in Figure 3. The solid, dash or dotted line means our scheme, the scheme for commutative SDEs or Platen's scheme, respectively. The scheme for commutative SDEs is useful to see the influence of non-commutativity of SDEs. The figures illustrate that our scheme is of weak order 2. We can see the influence of non-commutativity in the relative error of the approximation to $E[y_1(1)y_2(1)]$.

4 Summary

First, we have introduced our SRK family and the way of seeking order conditions for it with MRTL's. Second, after introducing the well-chosen simplifying conditions for the non-commutative case, we have found a solution of all the order conditions. Third, we have performed the numerical experiment and have shown that the explicit SRK scheme with 4 stages is of weak order 2. Although lack of space has prevented us from showing other numerical experiments, it is remarkable that the author has obtained similar results in other experiments.

The scheme has the following three features.

- When $m > 2$, it needs random variables less than Platen's scheme does since it has only $m-1$ random variables ($\Delta\tilde{W}_j$'s) except ΔW_j's for one step.
- It is of order 4 for ODEs. For this, it can be expected to show better performance in seeking an approximation to the expectation of a solution for SDEs with small noise.
- It is directly applicable to non-commutative Stratonovich SDEs, whereas Platen's scheme is for non-commutative Itô SDEs.

Appendix

Weak third order conditions

Under the assumption that the statement in Subsection 3.2 holds, we show the order conditions for our SRK family to be of weak third order for multi-dimensional SDEs with a one-dimensional Wiener process. The following symbols are used for ease of notation:

$$\tilde\Phi(t) \stackrel{\text{def}}{=} \tilde\Phi_{(m+1)s+1}(t), \quad E_a(t) \stackrel{\text{def}}{=} E[\tilde\Phi(t)], \quad E_b(t) \stackrel{\text{def}}{=} E[\tilde\Phi(t)\tilde\Phi(\tau^{(j)}_{A(0)})],$$
$$E_c(t) \stackrel{\text{def}}{=} E[\tilde\Phi(t)\tilde\Phi([\tau^{(j)}_{\bar{A}(j)}]^{(j)}_{A(0)})], \quad E_d(t) \stackrel{\text{def}}{=} E[\tilde\Phi(t)\tilde\Phi(\tau^{(0)}_{A(0)})], \quad E_e(t) \stackrel{\text{def}}{=} E[\tilde\Phi(t)\{\tilde\Phi(\tau^{(j)}_{A(0)})\}^2],$$
$$E_f(t,t_1,t_2,\ldots) \stackrel{\text{def}}{=} E[\tilde\Phi(t)\tilde\Phi(t_1)\tilde\Phi(t_2)\cdots], \quad E_g(t,t_1,t_2,\ldots) \stackrel{\text{def}}{=} E[\{\tilde\Phi(t)\}^2\tilde\Phi(t_1)\tilde\Phi(t_2)\cdots].$$

Table 5: Weak third order conditions

Order conditions for weak order 1 at least					
t	Order condition	t	Order condition	t	Order condition
$[\tau^{(j)}_{\bar{A}(j)}]^{(j)}_{A(0)}$	$E_a(t) = \frac{h}{2}$	$\tau^{(0)}_{A(0)}$	$E_a(t) = h$	$\tau^{(j)}_{A(0)}$	$E_b(t) = h$

Table 5: Continued

Order conditions for weak order 2 at least			
t	Order condition	t	Order condition
$[[[\tau^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{A^{(0)}}$	$E_a(t) = \frac{h^2}{8}$	$[[\tau^{(j)}_{A^{(j)}}, \tau^{(j)}_{A^{(j)}}]^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{A^{(0)}}$	$E_a(t) = \frac{h^2}{4}$
$[\tau^{(j)}_{A^{(j)}}, [\tau^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{A^{(j)}}]^{(j)}_{A^{(0)}}$	$E_a(t) = \frac{3h^2}{8}$	$[\tau^{(j)}_{A^{(j)}}, \tau^{(j)}_{A^{(j)}}, \tau^{(j)}_{A^{(j)}}]^{(j)}_{A^{(0)}}$	$E_a(t) = \frac{3h^2}{4}$
$[[\tau^{(0)}_{\tilde{A}^{(j)}}]^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{A^{(0)}}$	$E_a(t) = \frac{h^2}{4}$	$[[\tau^{(j)}_{A^{(0)}}]^{(0)}_{\tilde{A}^{(j)}}]^{(j)}_{A^{(0)}}$	$E_a(t) = 0$
$[[\tau^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{\tilde{A}^{(0)}}]^{(0)}_{A^{(0)}}$	$E_a(t) = \frac{h^2}{4}$	$[[\tau^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{A^{(0)}}$	$E_b(t) = \frac{h^2}{2}$
$[\tau^{(j)}_{A^{(j)}}, \tau^{(0)}_{A^{(j)}}]^{(j)}_{A^{(0)}}$	$E_a(t) = \frac{h^2}{4}$	$[\tau^{(j)}_{A^{(0)}}, \tau^{(j)}_{A^{(0)}}]^{(0)}_{A^{(0)}}$	$E_a(t) = \frac{h^2}{2}$
$[\tau^{(j)}_{A^{(j)}}, \tau^{(j)}_{A^{(j)}}]^{(j)}_{A^{(0)}}$	$E_b(t) = h^2$		
$[\tau^{(0)}_{\tilde{A}^{(0)}}]^{(0)}_{A^{(0)}}$	$E_a(t) = \frac{h^2}{2}$	$[\tau^{(0)}_{\tilde{A}^{(j)}}]^{(j)}_{A^{(0)}}$	$E_b(t) = \frac{h^2}{2}$
$[\tau^{(j)}_{\tilde{A}^{(0)}}]^{(0)}_{A^{(0)}}$	$E_b(t) = \frac{h^2}{2}$		
t	Order condition	Order condition	Order condition
$[\tau^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{A^{(0)}}$	$E_c(t) = \frac{3h^2}{4}$	$E_d(t) = \frac{h^2}{2}$	$E_e(t) = \frac{3h^2}{2}$
$\tau^{(0)}_{A^{(0)}}$	$E_d(t) = h^2$	$E_e(t) = h^2$	
$\tau^{(j)}_{A^{(0)}}$	$E[\{\tilde{\Phi}(t)\}^4] = 3h^2$		

Order conditions for weak order 3			
t	Order condition	t	Order condition
$[[[[[\tau^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{A^{(0)}}$	$E_a(t) = \frac{h^3}{48}$	$[[[[\tau^{(j)}_{A^{(j)}}, \tau^{(j)}_{A^{(j)}}]^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{A^{(0)}}$	$E_a(t) = \frac{h^3}{24}$
$[[[\tau^{(j)}_{A^{(j)}}, [\tau^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{A^{(j)}}]^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{A^{(0)}}$	$E_a(t) = \frac{h^3}{16}$	$[[[\tau^{(j)}_{A^{(j)}}, \tau^{(j)}_{A^{(j)}}, \tau^{(j)}_{A^{(j)}}]^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{A^{(0)}}$	$E_a(t) = \frac{h^3}{8}$
$[[\tau^{(j)}_{A^{(j)}}, [[\tau^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{A^{(j)}}]^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{A^{(0)}}$	$E_a(t) = \frac{h^3}{12}$	$[[\tau^{(j)}_{A^{(j)}}, [\tau^{(j)}_{A^{(j)}}, \tau^{(j)}_{A^{(j)}}]^{(j)}_{A^{(j)}}]^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{A^{(0)}}$	$E_a(t) = \frac{h^3}{6}$
$[[[\tau^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{A^{(j)}}, [\tau^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{A^{(j)}}]^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{A^{(0)}}$	$E_a(t) = \frac{h^3}{8}$	$[[\tau^{(j)}_{A^{(j)}}, \tau^{(j)}_{A^{(j)}}, [\tau^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{A^{(j)}}]^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{A^{(0)}}$	$E_a(t) = \frac{h^3}{4}$
$[[\tau^{(j)}_{A^{(j)}}, \tau^{(j)}_{A^{(j)}}, \tau^{(j)}_{A^{(j)}}, \tau^{(j)}_{A^{(j)}}]^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{A^{(0)}}$	$E_a(t) = \frac{h^3}{2}$	$[\tau^{(j)}_{A^{(j)}}, [\tau^{(j)}_{A^{(j)}}, \tau^{(j)}_{A^{(j)}}, \tau^{(j)}_{A^{(j)}}]^{(j)}_{A^{(j)}}]^{(j)}_{A^{(0)}}$	$E_a(t) = \frac{5h^3}{8}$
$[\tau^{(j)}_{A^{(j)}}, [\tau^{(j)}_{A^{(j)}}, [\tau^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{A^{(j)}}]^{(j)}_{A^{(j)}}]^{(j)}_{A^{(0)}}$	$E_a(t) = \frac{5h^3}{16}$	$[\tau^{(j)}_{A^{(j)}}, [[\tau^{(j)}_{A^{(j)}}, \tau^{(j)}_{A^{(j)}}]^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{A^{(j)}}]^{(j)}_{A^{(0)}}$	$E_a(t) = \frac{5h^3}{24}$
$[\tau^{(j)}_{A^{(j)}}, [[[\tau^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{A^{(j)}}]^{(j)}_{A^{(0)}}$	$E_a(t) = \frac{5h^3}{48}$	$[[\tau^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{A^{(j)}}, [[\tau^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{A^{(j)}}]^{(j)}_{A^{(0)}}$	$E_a(t) = \frac{5h^3}{24}$
$[[\tau^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{A^{(j)}}, [\tau^{(j)}_{A^{(j)}}, \tau^{(j)}_{A^{(j)}}]^{(j)}_{A^{(j)}}]^{(j)}_{A^{(0)}}$	$E_a(t) = \frac{5h^3}{12}$	$[\tau^{(j)}_{A^{(j)}}, [\tau^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{A^{(j)}}, [\tau^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{A^{(j)}}]^{(j)}_{A^{(0)}}$	$E_a(t) = \frac{5h^3}{8}$
$[\tau^{(j)}_{A^{(j)}}, \tau^{(j)}_{A^{(j)}}, [[\tau^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{A^{(j)}}]^{(j)}_{A^{(0)}}$	$E_a(t) = \frac{5h^3}{12}$	$[\tau^{(j)}_{A^{(j)}}, \tau^{(j)}_{A^{(j)}}, [\tau^{(j)}_{A^{(j)}}, \tau^{(j)}_{A^{(j)}}]^{(j)}_{A^{(j)}}]^{(j)}_{A^{(0)}}$	$E_a(t) = \frac{5h^3}{6}$
$[\tau^{(j)}_{A^{(j)}}, \tau^{(j)}_{A^{(j)}}, \tau^{(j)}_{A^{(j)}}, [\tau^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{A^{(j)}}]^{(j)}_{A^{(0)}}$	$E_a(t) = \frac{5h^3}{4}$	$[\tau^{(j)}_{A^{(j)}}, \tau^{(j)}_{A^{(j)}}, \tau^{(j)}_{A^{(j)}}, \tau^{(j)}_{A^{(j)}}, \tau^{(j)}_{A^{(j)}}]^{(j)}_{A^{(0)}}$	$E_a(t) = \frac{5h^3}{2}$
$[[[[\tau^{(0)}_{\tilde{A}^{(j)}}]^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{A^{(0)}}$	$E_a(t) = \frac{h^3}{24}$	$[[[[\tau^{(j)}_{\tilde{A}^{(0)}}]^{(0)}_{\tilde{A}^{(j)}}]^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{A^{(0)}}$	$E_a(t) = 0$
$[[[[\tau^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{\tilde{A}^{(0)}}]^{(0)}_{\tilde{A}^{(j)}}]^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{A^{(0)}}$	$E_a(t) = \frac{h^3}{24}$	$[[[[\tau^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{\tilde{A}^{(0)}}]^{(0)}_{\tilde{A}^{(j)}}]^{(j)}_{A^{(0)}}$	$E_a(t) = 0$
$[[[[\tau^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{\tilde{A}^{(0)}}]^{(0)}_{A^{(0)}}$	$E_a(t) = \frac{h^3}{24}$	$[[[[\tau^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{A^{(0)}}$	$E_b(t) = \frac{h^3}{8}$
$[[[\tau^{(j)}_{A^{(j)}}, \tau^{(0)}_{A^{(j)}}]^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{A^{(0)}}$	$E_a(t) = \frac{h^3}{24}$	$[[[\tau^{(j)}_{A^{(0)}}, \tau^{(j)}_{A^{(0)}}]^{(0)}_{\tilde{A}^{(j)}}]^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{A^{(0)}}$	$E_a(t) = \frac{h^3}{12}$
$[[[\tau^{(j)}_{A^{(j)}}, \tau^{(j)}_{A^{(j)}}]^{(j)}_{\tilde{A}^{(0)}}]^{(0)}_{\tilde{A}^{(j)}}]^{(j)}_{A^{(0)}}$	$E_a(t) = 0$	$[[[\tau^{(j)}_{A^{(j)}}, \tau^{(j)}_{A^{(j)}}]^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{\tilde{A}^{(0)}}]^{(0)}_{A^{(0)}}$	$E_a(t) = \frac{h^3}{12}$
$[[[\tau^{(j)}_{A^{(j)}}, \tau^{(j)}_{A^{(j)}}]^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{A^{(0)}}$	$E_b(t) = \frac{h^3}{4}$	$[[\tau^{(j)}_{A^{(j)}}, [\tau^{(0)}_{\tilde{A}^{(j)}}]^{(j)}_{A^{(j)}}]^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{A^{(0)}}$	$E_a(t) = \frac{h^3}{12}$
$[[\tau^{(j)}_{A^{(j)}}, [\tau^{(j)}_{\tilde{A}^{(0)}}]^{(0)}_{A^{(j)}}]^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{A^{(0)}}$	$E_a(t) = \frac{h^3}{12}$	$[[\tau^{(0)}_{A^{(j)}}, [\tau^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{A^{(j)}}]^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{A^{(0)}}$	$E_a(t) = \frac{h^3}{12}$
$[[\tau^{(j)}_{A^{(0)}}, [\tau^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{A^{(0)}}]^{(0)}_{\tilde{A}^{(j)}}]^{(j)}_{A^{(0)}}$	$E_a(t) = 0$	$[[\tau^{(j)}_{A^{(j)}}, [\tau^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{A^{(j)}}]^{(j)}_{\tilde{A}^{(j)}}]^{(0)}_{A^{(0)}}$	$E_a(t) = \frac{h^3}{8}$
$[[\tau^{(j)}_{A^{(j)}}, [\tau^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{A^{(j)}}]^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{A^{(0)}}$	$E_b(t) = \frac{3h^3}{8}$	$[[\tau^{(j)}_{A^{(j)}}, \tau^{(j)}_{A^{(j)}}, \tau^{(0)}_{A^{(j)}}]^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{A^{(0)}}$	$E_a(t) = \frac{h^3}{6}$
$[[\tau^{(j)}_{A^{(0)}}, \tau^{(j)}_{A^{(0)}}, \tau^{(j)}_{A^{(0)}}]^{(0)}_{\tilde{A}^{(j)}}]^{(j)}_{A^{(0)}}$	$E_a(t) = 0$	$[[\tau^{(j)}_{A^{(j)}}, \tau^{(j)}_{A^{(j)}}, \tau^{(j)}_{A^{(j)}}]^{(j)}_{\tilde{A}^{(0)}}]^{(0)}_{A^{(0)}}$	$E_a(t) = \frac{h^3}{4}$
$[[\tau^{(j)}_{A^{(j)}}, \tau^{(j)}_{A^{(j)}}, \tau^{(j)}_{A^{(j)}}]^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{A^{(0)}}$	$E_b(t) = \frac{3h^3}{4}$	$[\tau^{(j)}_{A^{(j)}}, [[\tau^{(0)}_{\tilde{A}^{(j)}}]^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{A^{(j)}}]^{(j)}_{A^{(0)}}$	$E_a(t) = \frac{h^3}{8}$
$[\tau^{(j)}_{A^{(j)}}, [[\tau^{(j)}_{\tilde{A}^{(0)}}]^{(0)}_{\tilde{A}^{(j)}}]^{(j)}_{A^{(j)}}]^{(j)}_{A^{(0)}}$	$E_a(t) = \frac{h^3}{12}$	$[\tau^{(j)}_{A^{(j)}}, [[\tau^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{\tilde{A}^{(0)}}]^{(0)}_{A^{(j)}}]^{(j)}_{A^{(0)}}$	$E_a(t) = \frac{h^3}{24}$
$[\tau^{(0)}_{A^{(j)}}, [[\tau^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{A^{(j)}}]^{(j)}_{A^{(0)}}$	$E_a(t) = \frac{h^3}{12}$	$[\tau^{(j)}_{A^{(0)}}, [[\tau^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{A^{(0)}}]^{(0)}_{A^{(0)}}$	$E_a(t) = \frac{h^3}{6}$
$[\tau^{(j)}_{A^{(j)}}, [[\tau^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{A^{(j)}}]^{(j)}_{A^{(0)}}$	$E_b(t) = \frac{h^3}{2}$	$[\tau^{(j)}_{A^{(j)}}, [\tau^{(j)}_{A^{(j)}}, \tau^{(0)}_{A^{(j)}}]^{(j)}_{A^{(j)}}]^{(j)}_{A^{(0)}}$	$E_a(t) = \frac{5h^3}{24}$
$[\tau^{(j)}_{A^{(j)}}, [\tau^{(j)}_{A^{(0)}}, \tau^{(j)}_{A^{(0)}}]^{(0)}_{A^{(j)}}]^{(j)}_{A^{(0)}}$	$E_a(t) = \frac{h^3}{12}$	$[\tau^{(0)}_{A^{(j)}}, [\tau^{(j)}_{A^{(j)}}, \tau^{(j)}_{A^{(j)}}]^{(j)}_{A^{(j)}}]^{(j)}_{A^{(0)}}$	$E_a(t) = \frac{h^3}{6}$
$[\tau^{(j)}_{A^{(0)}}, [\tau^{(j)}_{A^{(j)}}, \tau^{(j)}_{A^{(j)}}]^{(j)}_{A^{(0)}}]^{(0)}_{A^{(0)}}$	$E_a(t) = \frac{h^3}{3}$	$[\tau^{(j)}_{A^{(j)}}, [\tau^{(j)}_{A^{(j)}}, \tau^{(j)}_{A^{(j)}}]^{(j)}_{A^{(j)}}]^{(j)}_{A^{(0)}}$	$E_b(t) = h^3$
$[[\tau^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{A^{(j)}}, [\tau^{(0)}_{\tilde{A}^{(j)}}]^{(j)}_{A^{(j)}}]^{(j)}_{A^{(0)}}$	$E_a(t) = \frac{h^3}{6}$	$[[\tau^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{A^{(j)}}, [\tau^{(j)}_{\tilde{A}^{(0)}}]^{(0)}_{A^{(j)}}]^{(j)}_{A^{(0)}}$	$E_a(t) = \frac{h^3}{12}$
$[[\tau^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{A^{(0)}}, [\tau^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{A^{(0)}}]^{(0)}_{A^{(0)}}$	$E_a(t) = \frac{h^3}{4}$	$[[\tau^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{A^{(j)}}, [\tau^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{A^{(j)}}]^{(j)}_{A^{(0)}}$	$E_b(t) = \frac{3h^3}{4}$

Table 5: Continued

t	Order condition	t	Order condition
$[\tau^{(j)}_{A^{(j)}}, \tau^{(j)}_{A^{(j)}}, [\tau^{(0)}_{\tilde{A}^{(j)}}]^{(j)}_{A^{(j)}}]^{(j)}_{A^{(0)}}$	$E_a(t) = \frac{h^3}{3}$	$[\tau^{(j)}_{A^{(j)}}, \tau^{(j)}_{A^{(j)}}, [\tau^{(j)}_{\tilde{A}^{(0)}}]^{(0)}_{A^{(j)}}]^{(j)}_{A^{(0)}}$	$E_a(t) = \frac{h^3}{6}$
$[\tau^{(j)}_{A^{(j)}}, \tau^{(0)}_{A^{(j)}}, [\tau^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{A^{(j)}}]^{(j)}_{A^{(0)}}$	$E_a(t) = \frac{h^3}{4}$	$[\tau^{(j)}_{A^{(0)}}, \tau^{(j)}_{A^{(0)}}, [\tau^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{A^{(0)}}]^{(0)}_{A^{(0)}}$	$E_a(t) = \frac{h^3}{2}$
$[\tau^{(j)}_{A^{(j)}}, \tau^{(j)}_{A^{(j)}}, [\tau^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{A^{(j)}}]^{(j)}_{A^{(0)}}$	$E_b(t) = \frac{3h^3}{2}$	$[\tau^{(j)}_{A^{(j)}}, \tau^{(j)}_{A^{(j)}}, \tau^{(j)}_{A^{(j)}}, \tau^{(0)}_{A^{(j)}}]^{(j)}_{A^{(0)}}$	$E_a(t) = \frac{h^3}{2}$
$[\tau^{(j)}_{A^{(0)}}, \tau^{(j)}_{A^{(0)}}, \tau^{(j)}_{A^{(0)}}, \tau^{(j)}_{A^{(0)}}]^{(0)}_{A^{(0)}}$	$E_a(t) = h^3$	$[\tau^{(j)}_{A^{(j)}}, \tau^{(j)}_{A^{(j)}}, \tau^{(j)}_{A^{(j)}}, \tau^{(j)}_{A^{(j)}}]^{(j)}_{A^{(0)}}$	$E_b(t) = 3h^3$
$[[[\tau^{(0)}_{\tilde{A}^{(0)}}]^{(0)}_{\tilde{A}^{(j)}}]^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{A^{(0)}}$	$E_a(t) = \frac{h^3}{12}$	$[[[\tau^{(0)}_{\tilde{A}^{(j)}}]^{(j)}_{\tilde{A}^{(0)}}]^{(0)}_{\tilde{A}^{(j)}}]^{(j)}_{A^{(0)}}$	$E_a(t) = 0$
$[[[\tau^{(0)}_{\tilde{A}^{(j)}}]^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{\tilde{A}^{(0)}}]^{(0)}_{A^{(0)}}$	$E_a(t) = \frac{h^3}{12}$	$[[[\tau^{(j)}_{\tilde{A}^{(0)}}]^{(0)}_{\tilde{A}^{(0)}}]^{(0)}_{\tilde{A}^{(j)}}]^{(j)}_{A^{(0)}}$	$E_a(t) = 0$
$[[[\tau^{(j)}_{\tilde{A}^{(0)}}]^{(0)}_{\tilde{A}^{(j)}}]^{(j)}_{\tilde{A}^{(0)}}]^{(0)}_{A^{(0)}}$	$E_a(t) = 0$	$[[[\tau^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{\tilde{A}^{(0)}}]^{(0)}_{\tilde{A}^{(0)}}]^{(0)}_{A^{(0)}}$	$E_a(t) = \frac{h^3}{12}$
$[[[\tau^{(0)}_{\tilde{A}^{(j)}}]^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{A^{(0)}}$	$E_b(t) = \frac{h^3}{6}$	$[[[\tau^{(j)}_{\tilde{A}^{(0)}}]^{(0)}_{\tilde{A}^{(j)}}]^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{A^{(0)}}$	$E_b(t) = \frac{h^3}{12}$
$[[[\tau^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{\tilde{A}^{(0)}}]^{(0)}_{\tilde{A}^{(j)}}]^{(j)}_{A^{(0)}}$	$E_b(t) = \frac{h^3}{12}$	$[[[\tau^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{\tilde{A}^{(0)}}]^{(0)}_{A^{(0)}}$	$E_b(t) = \frac{h^3}{6}$
$[[\tau^{(0)}_{A^{(j)}}, \tau^{(0)}_{A^{(j)}}]^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{A^{(0)}}$	$E_a(t) = \frac{h^3}{6}$	$[[\tau^{(j)}_{A^{(0)}}, \tau^{(0)}_{A^{(0)}}]^{(0)}_{\tilde{A}^{(j)}}]^{(j)}_{A^{(0)}}$	$E_a(t) = 0$
$[[\tau^{(j)}_{A^{(j)}}, \tau^{(0)}_{A^{(j)}}]^{(j)}_{\tilde{A}^{(0)}}]^{(0)}_{A^{(0)}}$	$E_a(t) = \frac{h^3}{12}$	$[[\tau^{(j)}_{A^{(0)}}, \tau^{(j)}_{A^{(0)}}]^{(0)}_{\tilde{A}^{(0)}}]^{(0)}_{A^{(0)}}$	$E_a(t) = \frac{h^3}{6}$
$[[\tau^{(j)}_{A^{(j)}}, \tau^{(0)}_{A^{(j)}}]^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{A^{(0)}}$	$E_b(t) = \frac{h^3}{4}$	$[[\tau^{(j)}_{A^{(0)}}, \tau^{(j)}_{A^{(0)}}]^{(0)}_{\tilde{A}^{(j)}}]^{(j)}_{A^{(0)}}$	$E_b(t) = \frac{h^3}{6}$
$[[\tau^{(j)}_{A^{(j)}}, \tau^{(j)}_{A^{(j)}}]^{(j)}_{\tilde{A}^{(0)}}]^{(0)}_{A^{(0)}}$	$E_b(t) = \frac{h^3}{3}$	$[\tau^{(j)}_{A^{(j)}}, [\tau^{(0)}_{\tilde{A}^{(0)}}]^{(0)}_{A^{(j)}}]^{(j)}_{A^{(0)}}$	$E_a(t) = \frac{h^3}{12}$
$[\tau^{(j)}_{A^{(0)}}, [\tau^{(0)}_{\tilde{A}^{(j)}}]^{(j)}_{A^{(0)}}]^{(0)}_{A^{(0)}}$	$E_a(t) = \frac{h^3}{6}$	$[\tau^{(0)}_{A^{(j)}}, [\tau^{(0)}_{\tilde{A}^{(j)}}]^{(j)}_{A^{(j)}}]^{(j)}_{A^{(0)}}$	$E_a(t) = \frac{h^3}{6}$
$[\tau^{(0)}_{A^{(j)}}, [\tau^{(j)}_{\tilde{A}^{(0)}}]^{(0)}_{A^{(j)}}]^{(j)}_{A^{(0)}}$	$E_a(t) = 0$	$[\tau^{(j)}_{A^{(0)}}, [\tau^{(j)}_{\tilde{A}^{(0)}}]^{(0)}_{A^{(0)}}]^{(0)}_{A^{(0)}}$	$E_a(t) = \frac{h^3}{6}$
$[\tau^{(0)}_{A^{(0)}}, [\tau^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{A^{(0)}}]^{(0)}_{A^{(0)}}$	$E_a(t) = \frac{h^3}{6}$	$[\tau^{(j)}_{A^{(j)}}, [\tau^{(0)}_{\tilde{A}^{(j)}}]^{(j)}_{A^{(j)}}]^{(j)}_{A^{(0)}}$	$E_b(t) = \frac{5h^3}{12}$
$[\tau^{(j)}_{A^{(j)}}, [\tau^{(j)}_{\tilde{A}^{(0)}}]^{(0)}_{A^{(j)}}]^{(j)}_{A^{(0)}}$	$E_b(t) = \frac{h^3}{4}$	$[\tau^{(0)}_{A^{(j)}}, [\tau^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{A^{(j)}}]^{(j)}_{A^{(0)}}$	$E_b(t) = \frac{h^3}{3}$
$[\tau^{(j)}_{A^{(0)}}, [\tau^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{A^{(0)}}]^{(0)}_{A^{(0)}}$	$E_b(t) = \frac{h^3}{2}$	$[\tau^{(j)}_{A^{(j)}}, \tau^{(0)}_{A^{(j)}}, \tau^{(0)}_{A^{(j)}}]^{(j)}_{A^{(0)}}$	$E_a(t) = \frac{h^3}{6}$
$[\tau^{(j)}_{A^{(0)}}, \tau^{(j)}_{A^{(0)}}, \tau^{(0)}_{A^{(0)}}]^{(0)}_{A^{(0)}}$	$E_a(t) = \frac{h^3}{3}$	$[\tau^{(j)}_{A^{(j)}}, \tau^{(j)}_{A^{(j)}}, \tau^{(0)}_{A^{(j)}}]^{(j)}_{A^{(0)}}$	$E_b(t) = \frac{2h^3}{3}$
$[\tau^{(j)}_{A^{(0)}}, \tau^{(j)}_{A^{(0)}}, \tau^{(j)}_{A^{(0)}}]^{(0)}_{A^{(0)}}$	$E_b(t) = h^3$		
$[[\tau^{(0)}_{\tilde{A}^{(0)}}]^{(0)}_{\tilde{A}^{(0)}}]^{(0)}_{A^{(0)}}$	$E_a(t) = \frac{h^3}{6}$	$[[\tau^{(0)}_{\tilde{A}^{(0)}}]^{(0)}_{\tilde{A}^{(j)}}]^{(j)}_{A^{(0)}}$	$E_b(t) = \frac{h^3}{6}$
$[[\tau^{(0)}_{\tilde{A}^{(j)}}]^{(j)}_{\tilde{A}^{(0)}}]^{(0)}_{A^{(0)}}$	$E_b(t) = \frac{h^3}{6}$	$[[\tau^{(j)}_{\tilde{A}^{(0)}}]^{(0)}_{\tilde{A}^{(0)}}]^{(0)}_{A^{(0)}}$	$E_b(t) = \frac{h^3}{6}$
$[\tau^{(0)}_{A^{(0)}}, \tau^{(0)}_{A^{(0)}}]^{(0)}_{A^{(0)}}$	$E_a(t) = \frac{h^3}{3}$	$[\tau^{(0)}_{A^{(j)}}, \tau^{(0)}_{A^{(j)}}]^{(j)}_{A^{(0)}}$	$E_b(t) = \frac{h^3}{3}$
$[\tau^{(j)}_{A^{(0)}}, \tau^{(0)}_{A^{(0)}}]^{(0)}_{A^{(0)}}$	$E_b(t) = \frac{h^3}{3}$		

t	Order condition	Order condition	Order condition
$[[[\tau^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{A^{(0)}}$	$E_c(t) = \frac{5h^3}{16}$	$E_d(t) = \frac{h^3}{8}$	$E_e(t) = \frac{5h^3}{8}$
$[[\tau^{(j)}_{A^{(j)}}, \tau^{(j)}_{A^{(j)}}]^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{A^{(0)}}$	$E_c(t) = \frac{5h^3}{8}$	$E_d(t) = \frac{h^3}{4}$	$E_e(t) = \frac{5h^3}{4}$
$[\tau^{(j)}_{A^{(j)}}, [\tau^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{A^{(j)}}]^{(j)}_{A^{(0)}}$	$E_c(t) = \frac{15h^3}{16}$	$E_d(t) = \frac{3h^3}{8}$	$E_e(t) = \frac{15h^3}{8}$
$[\tau^{(j)}_{A^{(j)}}, \tau^{(j)}_{A^{(j)}}, \tau^{(j)}_{A^{(j)}}]^{(j)}_{A^{(0)}}$	$E_c(t) = \frac{15h^3}{8}$	$E_d(t) = \frac{3h^3}{4}$	$E_e(t) = \frac{15h^3}{4}$
$[[\tau^{(0)}_{\tilde{A}^{(j)}}]^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{A^{(0)}}$	$E_c(t) = \frac{7h^3}{24}$	$E_d(t) = \frac{h^3}{4}$	$E_e(t) = \frac{7h^3}{12}$
$[[\tau^{(j)}_{\tilde{A}^{(0)}}]^{(0)}_{\tilde{A}^{(j)}}]^{(j)}_{A^{(0)}}$	$E_c(t) = \frac{h^3}{6}$	$E_d(t) = 0$	$E_e(t) = \frac{h^3}{3}$
$[[\tau^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{\tilde{A}^{(0)}}]^{(0)}_{A^{(0)}}$	$E_c(t) = \frac{7h^3}{24}$	$E_d(t) = \frac{h^3}{4}$	$E_e(t) = \frac{7h^3}{12}$
$[\tau^{(j)}_{A^{(j)}}, \tau^{(0)}_{A^{(j)}}]^{(j)}_{A^{(0)}}$	$E_c(t) = \frac{11h^3}{24}$	$E_d(t) = \frac{h^3}{4}$	$E_e(t) = \frac{11h^3}{12}$
$[\tau^{(j)}_{A^{(0)}}, \tau^{(j)}_{A^{(0)}}]^{(0)}_{A^{(0)}}$	$E_c(t) = \frac{7h^3}{12}$	$E_d(t) = \frac{h^3}{2}$	$E_e(t) = \frac{7h^3}{6}$
$[\tau^{(0)}_{\tilde{A}^{(0)}}]^{(0)}_{A^{(0)}}$	$E_c(t) = \frac{h^3}{4}$	$E_d(t) = \frac{h^3}{2}$	$E_e(t) = \frac{h^3}{2}$

t	Order condition	Order condition	Order condition
$[[\tau^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{A^{(0)}}$	$E[\{\tilde{\Phi}(t)\}^2] = \frac{5h^3}{12}$	$E_f(t, [\tau^{(j)}_{A^{(j)}}, \tau^{(j)}_{A^{(j)}}]^{(j)}_{A^{(0)}}) = \frac{5h^3}{6}$	$E_f(t, [\tau^{(0)}_{\tilde{A}^{(j)}}]^{(j)}_{A^{(0)}}) = \frac{h^3}{4}$
	$E_f(t, [\tau^{(j)}_{\tilde{A}^{(0)}}]^{(0)}_{A^{(0)}}) = \frac{h^3}{4}$	$E_f(t, [\tau^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{A^{(0)}}, \tau^{(j)}_{A^{(0)}}) = \frac{5h^3}{4}$	$E_f(t, \tau^{(0)}_{A^{(0)}}, \tau^{(j)}_{A^{(0)}}) = \frac{h^3}{2}$
		$E[\tilde{\Phi}(t)\{\tilde{\Phi}(\tau^{(j)}_{A^{(0)}})\}^3] = \frac{5h^3}{2}$	
$[\tau^{(j)}_{A^{(j)}}, \tau^{(j)}_{A^{(j)}}]^{(j)}_{A^{(0)}}$	$E[\{\tilde{\Phi}(t)\}^2] = \frac{5h^3}{3}$	$E[\tilde{\Phi}(t)\{\tilde{\Phi}(\tau^{(j)}_{A^{(0)}})\}^3] = 5h^3$	$E_f(t, [\tau^{(0)}_{\tilde{A}^{(j)}}]^{(j)}_{A^{(0)}}) = \frac{h^3}{2}$
	$E_f(t, [\tau^{(j)}_{\tilde{A}^{(0)}}]^{(0)}_{A^{(0)}}) = \frac{h^3}{2}$	$E_f(t, [\tau^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{A^{(0)}}, \tau^{(j)}_{A^{(0)}}) = \frac{5h^3}{2}$	$E_f(t, \tau^{(0)}_{A^{(0)}}, \tau^{(j)}_{A^{(0)}}) = h^3$
$[\tau^{(0)}_{\tilde{A}^{(j)}}]^{(j)}_{A^{(0)}}$	$E[\{\tilde{\Phi}(t)\}^2] = \frac{h^3}{3}$	$E[\tilde{\Phi}(t)\{\tilde{\Phi}(\tau^{(j)}_{A^{(0)}})\}^3] = \frac{3h^3}{2}$	
	$E_f(t, [\tau^{(j)}_{\tilde{A}^{(0)}}]^{(0)}_{A^{(0)}}) = \frac{h^3}{6}$	$E_f(t, [\tau^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{A^{(0)}}, \tau^{(j)}_{A^{(0)}}) = \frac{3h^3}{4}$	$E_f(t, \tau^{(0)}_{A^{(0)}}, \tau^{(j)}_{A^{(0)}}) = \frac{h^3}{2}$

Table 5: Continued

t	Order condition	Order condition	Order condition
$[\tau^{(j)}_{\tilde{A}^{(0)}}]^{(0)}_{A^{(0)}}$	$E[\{\tilde{\Phi}(t)\}^2] = \frac{h^3}{3}$	$E[\tilde{\Phi}(t)\{\tilde{\Phi}(\tau^{(j)}_{A^{(0)}})\}^3] = \frac{3h^3}{2}$	
		$E_f(t, [\tau^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{A^{(0)}}, \tau^{(j)}_{A^{(0)}}) = \frac{3h^3}{4}$	$E_f(t, \tau^{(0)}_{A^{(0)}}, \tau^{(j)}_{A^{(0)}}) = \frac{h^3}{2}$
$[\tau^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{A^{(0)}}$	$E[\{\tilde{\Phi}(t)\}^3] = \frac{15h^3}{8}$	$E_f(t, \tau^{(0)}_{A^{(0)}}, \tau^{(j)}_{A^{(0)}}, \tau^{(j)}_{A^{(0)}}) = \frac{3h^3}{2}$	$E_g(t, \tau^{(0)}_{A^{(0)}}) = \frac{3h^3}{4}$
	$E_f(t, \tau^{(0)}_{A^{(0)}}, \tau^{(0)}_{A^{(0)}}) = \frac{h^3}{2}$	$E[\tilde{\Phi}(t)\{\tilde{\Phi}(\tau^{(j)}_{A^{(0)}})\}^4] = \frac{15h^3}{2}$	$E_g(t, \tau^{(j)}_{A^{(0)}}, \tau^{(j)}_{A^{(0)}}) = \frac{15h^3}{4}$
$\tau^{(0)}_{A^{(0)}}$	$E[\{\tilde{\Phi}(t)\}^3] = h^3$	$E[\tilde{\Phi}(t)\{\tilde{\Phi}(\tau^{(j)}_{A^{(0)}})\}^4] = 3h^3$	$E_g(t, \tau^{(j)}_{A^{(0)}}, \tau^{(j)}_{A^{(0)}}) = h^3$
$\tau^{(j)}_{A^{(0)}}$	$E[\{\tilde{\Phi}(t)\}^6] = 15h^3$		

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