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Some Aspects of Quantum Mutual Type Entropies

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Abstract

The mutual entropy (information) denotes an amount of information transmitted correctly from the input system to the output system through a channel. The (semi-classical) mutual entropies for classical input and quantum output were defined by several researchers. The fully quantum mutual entropy, which is called Ohya mutual entropy, for quantum input and output by using the relative entropy was defined by Ohya in 1983.

In this paper, we compare with mutual entropy-type measures and show some resuls for quantum capacity.

1 Introduction

The development of communication theory is closely connected with study of entropy theory. The signal of the input system is carried through a physical device, which is called a channel. The mathematical representation of the channel is a mapping from the input state space to the output state space. In classical communication theory, the mutual entropy was formulated by using the joint probability distribution between the input system and the output system. The (semi-classical) mutual entropies for classical input and quantum output were defined by several researchers [7, 8]. In fully quantum system, there does not exist the joint probability distribution in general. Instead of the joint probability distribution, Ohya took the measure theoretic expression by (KYG) Kolmogorov-Gelfand-Yaglom and defined Ohya mutual entropy [10] by means of quantum relative entropy of Umegaki [24] in 1983, he extended it [11] to general quantum systems by using the relative entropy of Araki [2] and Uhlmann [25].[23] and Bennet et al [3, 4, 21, 22] took the coherent entropy and defined the mutual type entropy to discuss a sort of coding theorem for communication processes.

In this paper, we compare with mutual entropy-type measures and show some resuls for quantum capacity for the attenuation channel. $\mathcal{H}$

2 Quantum Channels

The concept of channel has been carried out an important role in the progress of the quantum communication theory. In particular, an attenuation channel introduced in [10] is one of the most inportaint model for discussing the information transmission in quantum optical communication. Here we review the definition of the quantum channels.

Let $\mathcal{H}_1, \mathcal{H}_2$ be the complex separable Hilbert spaces of an input and an output systems, respectively, and let $\mathbf{B}(\mathcal{H}_k)$ be the set of all bounded linear operators on $\mathcal{H}_k$. We denote the set of all density operators on $\mathcal{H}_k \quad (k = 1, 2)$ by

$$\mathfrak{S}(\mathcal{H}_k) \equiv \{\rho \in \mathbf{B}(\mathcal{H}_k); \rho \geq 0, tr\rho = 1\}. \tag{1}$$

A map Λ^* from the quantum input system to the quantum output system is called a **(fully) quantum channel.**

1. Λ^* is called a **linear channel** if it satisfies the affine property, i.e.,

$$\sum_k \lambda_k = 1\,(\forall \lambda_k \geq 0) \Rightarrow \Lambda^*\left(\sum_k \lambda_k \rho_k\right) = \sum_k \lambda_k \Lambda^*(\rho_k), \forall \rho_k \in S(\mathcal{H}_1).$$

2. $\Lambda^* : \mathfrak{S}(\mathcal{H}_1) \to \mathfrak{S}(\mathcal{H}_2)$ is called a **completely positive (CP) channel** if its dual map Λ satisfies

$$\sum_{j,k=1}^{n} B_j^* \Lambda\left(A_j^* A_k\right) B_k \geq 0 \tag{2}$$

for any $n \in \mathbb{N}$, any $B_j \in \mathbf{B}(\mathcal{H}_1)$ and any $A_k \in \mathbf{B}(\mathcal{H}_2)$, where the dual map $\Lambda : \mathbf{B}(\mathcal{H}_2) \to \mathbf{B}(\mathcal{H}_1)$ of $\Lambda^* : \mathfrak{S}(\mathcal{H}_1) \to \mathfrak{S}(\mathcal{H}_2)$ satisfies $tr\rho\Lambda(A) = tr\Lambda^*(\rho) A$ for any $\rho \in \mathfrak{S}(\mathcal{H}_1)$ and any $A \in \mathbf{B}(\mathcal{H}_2)$.

2.1 Attenuation channel

Let us consider the communication processes including noise and loss systems. Let $\mathcal{K}_1, \quad \mathcal{K}_2$ be the complex separable Hilbert spaces for the noise and the loss systems, respectively. The quantun communication channel

$$\Lambda_0^*(\rho) \equiv tr_{\mathcal{K}_2} \pi_0^*(\rho \otimes \xi_0), \quad \xi_0 \equiv |0\rangle\langle 0| \text{ and } \pi_0^*(\cdot) \equiv V_0(\cdot) V_0^* \tag{3}$$

is called the attenuation channel, where $|0\rangle\langle 0|$ is vacuum state in $\mathcal{H}_1$ and V_0 is a linear mapping from $\mathcal{H}_1 \otimes \mathcal{K}_1$ to $\mathcal{H}_2 \otimes \mathcal{K}_2$ given by

$$V_0(|n\rangle \otimes |0\rangle) \equiv \sum_{j=0}^{n} C_j^n |j\rangle \otimes |n-j\rangle, \quad C_j^n = \sqrt{\frac{n!}{j!(n-j)!}} \alpha^j \bar{\beta}^{n-j} \tag{4}$$

for any $|n\rangle$ in $\mathcal{H}_1$ and α,β are complex numbers satisfying $|\alpha|^2 + |\beta|^2 = 1$. $\eta = |\alpha|^2$ is the transmission rate of the channel. $\pi_0^*$is called a beam splittings, which means that one beam comes and two beams appear after passing throughπ_0^*. This attenuation channel is generalized by Ohya and Watanabe such as noisy optical channel [17, 18]. After that, Accardi and Ohya [1] reformulated it by using liftings, which is the dual map of the transition expectation by mean of Accardi. It contains the concept of beam splittings, which is extended by Fichtner, Freudenberg and Libsher [6] concerning the mappings on generalized Fock spaces. For the attenuation channel Λ_0^*, one can obtain the following theorem:

Theorem 1 *The attenuation channel Λ_0^* is described by*

$$\Lambda_0^* (\rho) = \sum_{i=0}^{\infty} O_i V_0 Q \rho Q^* V_0^* O_i^*, \tag{5}$$

where $Q \equiv \sum_{l=0}^{\infty} (|y_l\rangle \otimes |0\rangle) \langle y_l|$, $O_i \equiv \sum_{k=0}^{\infty} |z_k\rangle (\langle z_k| \otimes \langle i|)$, $\{|y_l\rangle\}$ is a CONS in$\mathcal{H}_1$, $\{|z_k\rangle\}$ is a CONS in $\mathcal{H}_2$ and $\{|i\rangle\}$ is the set of number states in $\mathcal{K}_2$.

3 Ohya $\mathcal{S}$-Mixing Entropy

The quantum entropy was introduced by von Neumann around 1932 [9], which is defined by

$$S(\rho) \equiv -tr \rho \log \rho$$

for any density operators ρ in $S(\mathcal{H}_1)$. It denotes the amount of information of the quantum state ρ. It was extended by Ohya [12] for general quantum systems as follows.

Let $(\mathcal{A}, S(\mathcal{A}))$ be a C*-system. The entropy of a state $\varphi \in \mathcal{S}$ seen from the reference system, a weak∗-compact convex subset of the whole state space $S(\mathcal{A})$ on the C*-algebra $\mathcal{A}$, was introduced by Ohya, which is called a Ohya $\mathcal{S}$-mixing entropy. This Ohya $\mathcal{S}$-mixing entropy contains von Neumann's entropy and classical entropy as special cases.

Every state $\varphi \in \mathcal{S}$ has a maximal measure μ pseudosupported on ex$\mathcal{S}$ (extreme points in $\mathcal{S}$) such that

$$\varphi = \int_{\mathrm{ex}\mathcal{S}} \omega d\mu. \tag{6}$$

The measure μ giving the above decomposition is not unique unless $\mathcal{S}$ is a Choquet simplex, so that we denote the set of all such measures by $M_\varphi(\mathcal{S})$. Take

$$\begin{aligned} D_\varphi(\mathcal{S}) \equiv & \Big\{ \mu \in M_\varphi(\mathcal{S});\ \exists \{\mu_k\} \subset \mathbb{R}^+ \text{ and } \{\varphi_k\} \subset \mathrm{ex}\mathcal{S} \text{ s.t.} \\ & \sum_k \mu_k = 1,\ \mu = \sum_k \mu_k \delta(\varphi_k) \Big\}, \end{aligned} \tag{7}$$

where $\delta(\varphi)$ is the delta measure concentrated on $\{\varphi\}$. Put

$$H(\mu) = -\sum_k \mu_k \log \mu_k \tag{8}$$

for a measure$\mu \in D_\varphi(\mathcal{S})$.

Ohya $\mathcal{S}$-mixing entropy of a general state $\varphi \in \mathcal{S}$ w.r.t. $\mathcal{S}$ is defined by

$$S^{\mathcal{S}}(\varphi) = \begin{cases} \inf\{H(\mu); \mu \in D_\varphi(\mathcal{S})\} & (D_\varphi(\mathcal{S}) \neq \varnothing) \\ \infty & (D_\varphi(\mathcal{S}) = \varnothing) \end{cases} \tag{9}$$

When $\mathcal{S}$ is the total space $S(\mathcal{A})$, we simply denote $S^{\mathcal{S}}(\varphi)$ by $S(\varphi)$. This entropy (mixing $\mathcal{S}$-entropy) of a general state φ satisfies the following properties [12].

Theorem 2 *When $\mathcal{A} = \mathbf{B}(\mathcal{H})$ and $\alpha_t = Ad(U_t)$ (i.e., $\alpha_t(A) = U_t^* A U_t$ for any$A \in \mathcal{A}$) with a unitary operator U_t, for any state φ given by $\varphi(\cdot) = tr\rho\,\cdot$ with a density operator ρ, the following facts hold:*

(1) $S(\varphi) = -tr\rho\log\rho$.

(2) If φ is an α-invariant faithful state and every eigenvalue of ρ is non-degenerate, then $S^{I(\alpha)}(\varphi) = S(\varphi)$, where $I(\alpha)$ is the set of all α-invariant faithful states.

(3) If $\varphi \in K(\alpha)$, then $S^{K(\alpha)}(\varphi) = 0$, where $K(\alpha)$ is the set of all KMS states.

Theorem 3 *For any $\varphi \in K(\alpha)$, we have*

(1) $S^{K(\alpha)}(\varphi) \leq S^{I(\alpha)}(\varphi)$.

(2) $S^{K(\alpha)}(\varphi) \leq S(\varphi)$.

This Ohya$\mathcal{S}$-mixing entropy gives a measure of the uncertainty observed from the reference system $\mathcal{S}$ so that it has the following merits: Even if the total entropy $S(\varphi)$ is infinite, $S^{\mathcal{S}}(\varphi)$ is finite for some $\mathcal{S}$, hence it explains a sort of symmetry breaking in $\mathcal{S}$. Other similar properties as $S(\rho)$ hold for $S^{\mathcal{S}}(\varphi)$. This entropy can be applied to characterize normal states and quantum Markov chains in von Neumann algebras.

The relative entropy for two general states φ and ψ was introduced by Araki and Uhlmann and their relation is considered by Donald and Hiai et al.

4 Quantum Relative Entropy

4.1 Umegaki's definition

Let $\mathbf{B}(\mathcal{H})$ be the set of all bounded linear operators on a Hilbert space $\mathcal{H}$ and ρ, σ be density operators on $\mathcal{H}$. The Umegaki's relative entropy [24] with respect to ρ and σ is defined by

$$S(\rho, \sigma) \equiv \begin{cases} tr\rho(\log\rho - \log\sigma) & (\text{when } \overline{ran\rho} \subset \overline{ran\sigma}) \\ \infty & (\text{otherwise}) \end{cases} \tag{10}$$

It represents a certain difference between two quantum states ρ, σ. There were several trials to extend the relative entropy to more general quantum systems and apply it to some other fields [2, 12, 13, 25].

4.2 Araki's definition

Let N be σ-finite von Neumann algebra acting on a Hilbert space $\mathcal{H}$ and φ, ψ be normal states on N given by $\varphi(\cdot) = \langle x, \cdot x\rangle$ and $\psi(\cdot) = \langle y, \cdot y\rangle$ with $x, y \in \mathcal{K}$ (a positive natural cone). The operator $S_{x,y}$ is defined by

$$S_{x,y}(Ay + z) = s^N(y)A^*x, \; A \in N, \; s^{N'}(y)z = 0, \tag{11}$$

on the domain $\mathfrak{N}y + (I - s^{\mathfrak{N}'}(y))\mathcal{H}$, where $s^{\mathfrak{N}}(y)$ is the projection from $\mathcal{H}$ to $\{\mathfrak{N}'y\}^-$, the $\mathfrak{N}$ -support of$y$. Using this $S_{x,y}$, the relative modular operator $\Delta_{x,y}$ is defined as $\Delta_{x,y} = (S_{xy})^* \overline{S_{x,y}}$, whose spectral decomposition is denoted by $\int_0^\infty \lambda de_{x,y}(\lambda)$ ($\overline{S_{x,y}}$ is the closure of $S_{x,y}$). Then the Araki relative entropy [2] is given by

$$S(\psi, \varphi) = \begin{cases} \int_0^\infty \log \lambda d\langle y, e_{x,y}(\lambda) y\rangle & (\psi \ll \varphi) \\ \infty & (\text{otherwise}) \end{cases}, \tag{12}$$

where $\psi \ll \varphi$ means that $\varphi(A^*A) = 0$ implies $\psi(A^*A) = 0$ for $A \in \mathfrak{N}$.

4.3 Uhlmann's definition

Let $\mathcal{L}$ be a complex linear space and p, q be two seminorms on$\mathcal{L}$. Moreover, let $H(L)$ be the set of all positive hermitian forms α on $\mathcal{L}$ satisfying $|\alpha(x, y)| \leq p(x)q(y)$ for all$x, y \in \mathcal{L}$. Then the quadratical mean $QM(p, q)$ of p and q is defined by

$$QM(p, q)(x) = \sup\{\alpha(x, x)^{1/2}; \;\; \alpha \in H(\mathcal{L})\}, \;\; x \in \mathcal{L}, \tag{13}$$

and there exists a function $p_t(x)$ of $t \in [0, 1]$ for each $x \in \mathcal{L}$ satisfying the following conditions:

1. For any$x \in \mathcal{L}$, $p_t(x)$ is continuous in t,
2. $p_{1/2} = QM(p, q)$,
3. $p_{t/2} = QM(p, p_t)$,
4. $p_{(t+1)/2} = QM(p_t, q)$.

This seminorm p_t is denoted by $QI_t(p, q)$ and is called the quadratical interpolation from p to$q$. It is shown that for any positive hermitian formsα, β, there exists a unique function $QF_t(\alpha, \beta)$ of $t \in [0, 1]$ with values in the set $H(\mathcal{L})$ such that $QF_t(\alpha, \beta)(x, x)^{1/2}$ is the quadratical interpolation from $\alpha(x, x)^{1/2}$ to$\beta(x, x)^{1/2}$. The relative entropy functional $S(\alpha, \beta)(x)$ of α and β is defined as

$$S(\alpha, \beta)(x) = -\liminf_{t \to 0} \frac{1}{t} \{QF_t(\alpha, \beta)(x, x) - \alpha(x, x)\} \tag{14}$$

for$x \in \mathcal{L}$. Let $\mathcal{L}$ be a *-algebra $\mathcal{A}$ and φ, ψ be positive linear functionals on $\mathcal{A}$ defining two hermitian forms φ^L, ψ^R such as $\varphi^L(A, B) = \varphi(A^*B)$ and$\psi^R(A, B) = \psi(BA^*)$.

The Uhlmanns relative entropy [25] of φ and ψ is defined by

$$S(\psi, \varphi) = S(\psi^R, \varphi^L)(I). \tag{15}$$

5 Ohya Mutual Entropy for Genaral Quantum Systems

The classical mutual entropy is determined by an input state and a channel, so that we denote the quantum mutual entropy with respect to the input state φ and the quantum channel Λ^* by $I(\varphi;\Lambda^*)$. This quantum mutual entropy $I(\varphi;\Lambda^*)$ should satisfy the following three conditions:

(1) The quantum mutual entropy is well-matched to the von Neumann entropy. That is, if a channel is trivial, i.e., Λ^* =identity map, then the mutual entropy equals to the von Neumann entropy: $I(\varphi; id) = S(\varphi)$.

(2) When the system is classical, the quantum mutual entropy reduces to classical one.

(3) Shannon's fundamental inequality 0≤ $0 \le I(\varphi;\Lambda^*) \le S(\varphi)$ is held.

In order to define such a quantum mutual entropy, we need the quantum relative entropy and the joint state, which is called a compound state, describing the correlation between an input state φ and the output state $\Lambda^*\varphi$ through a channelΛ^*. For $\varphi \in \mathcal{S} \subset S(\mathcal{A})$ and $\Lambda^* : S(\mathcal{A}) \to S(\overline{\mathcal{A}})$, the compound states are define by

$$\Phi_\mu^{\mathcal{S}} = \int_{\mathcal{S}} \omega \otimes \Lambda^*\omega d\mu \tag{16}$$

and

$$\Phi_0 = \varphi \otimes \Lambda^*\varphi. \tag{17}$$

The first compound state, which is called a Ohya compund state, generalizes the joint probability in classical dynamical system and it exhibits the correlation between the initial state φ and the final state $\Lambda^*\varphi$.

Ohya mutual entropy w.r.t. $\mathcal{S}$ and μ is

$$I_\mu^{\mathcal{S}}(\varphi;\Lambda^*) = S(\Phi_\mu^{\mathcal{S}}, \Phi_0) \tag{18}$$

and Ohya mutual entropy [12] w.r.t. $\mathcal{S}$ is defined by

$$I^{\mathcal{S}}(\varphi;\Lambda^*) = \lim_{\varepsilon\to 0} \sup\left\{ I_\mu^{\mathcal{S}}(\varphi;\Lambda^*); \mu \in F_\varphi^\varepsilon(\mathcal{S})\right\}, \tag{19}$$

where

$$F_\varphi^\varepsilon(\mathcal{S}) = \begin{cases} \{\mu \in D_\varphi(\mathcal{S}); S^{\mathcal{S}}(\varphi) \le H(\mu) \le S^{\mathcal{S}}(\varphi) + \varepsilon < +\infty\} & (S^{\mathcal{S}}(\varphi) < +\infty) \\ M_\varphi(\mathcal{S}) & (S^{\mathcal{S}}(\varphi) = +\infty) \end{cases} . \tag{20}$$

The following fundamental inequality is satisfied for almost all physical cases [13].

$$0 \le I^{\mathcal{S}}(\varphi;\Lambda^*) \le S^{\mathcal{S}}(\varphi) \tag{21}$$

In the case that the C*-algebra is $\mathbf{B}(\mathcal{H})$ and $\mathcal{S}$ is the set of all density operators, the above Ohya mutual entropy goes to

$$I(\rho;\Lambda^*) = \sup\left\{ \sum_n S(\Lambda^* E_n, \Lambda^*\rho),\ \rho = \sum_n \lambda_n E_n \right\}, \tag{22}$$

where ρ is a density operator (state), $S(\Lambda^* E_n, \Lambda^* \rho)$is Umegaki's relative entropy and $\rho = \sum_{\sum} nE_n$ is Schatten-von Neumann (one dimensional spectral) decomposition. As was mentioned above, it satisfies the Shannon's type inequality as follows:$0 \leq I(\rho, \Lambda^*) \leq \min\{S(\rho), S(\Lambda^*\rho)\}$. It is easily shown that we can take orthogonal decomposition instead of the Schatten-von Neumann decomposition [20].

5.1 Semi-classical mutual entropy

When the input system is classical, the state φ is a probability distribution and the Schatten-von Neumann decomposition is unique with delta measures δ_n such that $\varphi = \sum_n \lambda_n \delta_n$. In this case we need to code the classical state φ by a quantum state ψ, whose process is a quantum coding described by a channel Γ^* such that $\Gamma^*\delta_n = \psi_n$ (quantum state) and$\psi \equiv \Gamma^*\varphi = \sum_n \lambda_n \psi_n$. Then Ohya mutual entropy $I(\varphi; \Lambda^* \circ \Gamma^*)$ becomes Holevo's one, that is,

$$I(\varphi; \Lambda^* \circ \Gamma^*) = S(\Lambda^*\psi) - \sum_n \lambda_n S(\Lambda^*\psi_n) \tag{23}$$

when $\sum_n \lambda_n S(\Lambda^*\psi_n)$ is finite. These Ohya mutual entropy (ME) are completely quantum, namely, it represents the information transmission from a quantum input to a quantum output. The quantum system is described by a noncommutative structure. The classical system is expressed by a commutative construction. In the mathematical point of view, the commutative systems are contained in the noncommutative framework. One can obtain the following diagram.

$$\begin{array}{ccccc} & \swarrow & \text{Semi-classical ME} & \longleftarrow & \text{Ohya ME} \\ \text{Shannon's ME} & & & & \uparrow \\ & \nwarrow & \text{ME (GKY)} & \longleftarrow & \text{Ohya ME for GQS} \end{array}$$

6 Quantum Mutual Type Entropies

Recently Shor [23] and Bennet et al [3, 4] took the coherent entropy and defined the mutual type entropy to discuss a sort of coding theorem for quantum communication. In this section, we compare these mutual types entropy.

Let us discuss the entropy exchange [21]. For a stateρ, a channel Λ^* is denoted by using an operator valued measure $\{A_j\}$ such as

$$\Lambda^*(\cdot) \equiv \sum_j A_j^* \cdot A_j, \tag{24}$$

which is called a Stinespring-Sudarshan-Kraus form. Then one can define a matrix $W = (W_{ij})_{i,j}$ with

$$W_{ij} \equiv trA_i^*\rho A_j, \tag{25}$$

by which the entropy exchange is defined by

$$S_e(\rho, \Lambda^*) = -trW \log W. \tag{26}$$

By using the entropy exchange, two mutual type entropies are defined as follows:

$$I_C(\rho;\Lambda^*) \equiv S(\Lambda^*\rho) - S_e(\rho,\Lambda^*), \tag{27}$$

$$I_L(\rho;\Lambda^*) \equiv S(\rho) + S(\Lambda^*\rho) - S_e(\rho,\Lambda^*). \tag{28}$$

The first one is called the coherent entropy $I_C(\rho;\Lambda^*)$ [22] and the second one is called the Lindblad entropy $I_L(\rho;\Lambda^*)$ [4]. By comparing these mutual entropies for quantum information communication processes, we have the following theorem [19]:

Theorem 4 *Let $\{A_j\}$ be a projection valued measure with $dimA_j = 1$. For arbitrary state ρ and the quantum channel $\Lambda^*(\cdot) \equiv \sum_j A_j \cdot A_j^*$, one has*
(1) $0 \le I(\rho;\Lambda^*) \le \min\{S(\rho), S(\Lambda^*\rho)\}$ *(Ohya mutual entropy),*
(2) $I_C(\rho;\Lambda^*) = 0$ *(coherent entropy),*
(3) $I_L(\rho;\Lambda^*) = S(\rho)$ *(Lindblad entropy).*

For the attenuation channel Λ_0^*, one can obatain the following theorems [19]:

Theorem 5 *For any state $\rho = \sum_n \lambda_n |n\rangle\langle n|$ and the attenuation channel Λ_0^* with $|\alpha|^2 = |\beta|^2 = \frac{1}{2}$, one has*
(1) $0 \le I(\rho;\Lambda_0^*) \le \min\{S(\rho), S(\Lambda_0^*\rho)\}$ *(Ohya mutual entropy),*
(2) $I_C(\rho;\Lambda_0^*) = 0$ *(coherent entropy),*
(3) $I_L(\rho;\Lambda_0^*) = S(\rho)$ *(Lindblad entropy).*

Theorem 6 *For the attenuation channel Λ_0^* and the input state$\rho = \lambda|0\rangle\langle 0| + (1-\lambda)|\theta\rangle\langle\theta|$, we have*
(1) $0 \le I(\rho;\Lambda_0^*) \le \min\{S(\rho), S(\Lambda_0^*\rho)\}$ *(Ohya mutual entropy),*
(2) $-S(\rho) \le I_C(\rho;\Lambda_0^*) \le S(\rho)$ *(coherent entropy),*
(3) $0 \le I_L(\rho;\Lambda_0^*) \le 2S(\rho)$ *(Lindblad entropy).*

Therem 4.3 shows that the coherent entropy $I_C(\rho;\Lambda_0^*)$ takes a minus value for $|\alpha|^2 < |\beta|^2$ and the Lindblad entropy $I_L(\rho;\Lambda_0^*)$ is grater than the von Neumann entropy of the input state ρ for $|\alpha|^2 > |\beta|^2$.

From these theorems, Ohya mutual entropy $I(\rho;\Lambda^*)$ only satisfies the inequality held in classical systems, so that Ohya mutual entropy can be a most suitable candidate as quantum extension of the classical mutual entropy.

7 Quantum Capacity

The capacity means the ability of the information transmission of the channel, which is used as a measure for construction of channels. The fully quantum capacity is formulated by taking the supremum of the fully quantum mutual entropy with respect to a certain subset of the initial state space. The capacity of purely quantum channel was studied in [14, 15, 16, 17].

Let $\mathcal{S}$ be the set of all input states satisfying some physical conditions. Let us consider the ability of information transmission for the quantum channelΛ^*.

The answer of this question is the capacity of quantum channel Λ^* for a certain set $\mathcal{S} \subset S(\mathcal{H}_1)$ defined by

$$C_q^{\mathcal{S}}(\Lambda^*) \equiv \sup \{I(\rho; \Lambda^*) ; \rho \in \mathcal{S}\}. \tag{29}$$

When $\mathcal{S} = S(\mathcal{H}_1)$, the capacity of quantum channel Λ^* is denoted by $C_q(\Lambda^*)$. Then the following theorem for the attenuation channel was proved in [19].

Theorem 7 *For a subset $\mathcal{S}_n \equiv \{\rho \in S(\mathcal{H}_1) ; \dim s(\rho) = n\}$, the capacity of the attenuation channel Λ_0^* satisfies*

$$C_q^{\mathcal{S}_n}(\Lambda_0^*) = \log n,$$

where $s(\rho)$ is the support projection of ρ.

When the mean energy of the input state vectors $\{|\tau\theta_k\rangle\}$ can be taken infinite, i.e., $\lim_{\tau\to\infty} |\tau\theta_k|^2 = \infty$, the above theorem tells that the quantum capacity for the attenuation channel Λ_0^* with respect to $\mathcal{S}_n$ becomes $\log n$. It is a natural result, however it is impossible to take the mean energy of input state vector infinite. Therefore we have to compute the quantum capacity

$$C_q^{\mathcal{S}_e}(\Lambda^*) = \sup \{I(\rho; \Lambda^*) ; \rho \in \mathcal{S}_e\} \tag{30}$$

under some constraint $\mathcal{S}_e \equiv \{\rho \in \mathcal{S}; E(\rho) < e\}$ on the mean energy $E(\rho)$ of the input state ρ. In [11, 14], we also considered the **pseudo-quantum capacity** $C_{pq}^{\mathcal{S}_e}(\Lambda^*)$ defined by

$$C_{pq}^{\mathcal{S}_e}(\Lambda^*) = \sup \{I_p(\rho; \Lambda^*) ; \rho \in \mathcal{S}_e\} \tag{31}$$

with the **pseudo-mutual entropy** $I_{pq}(\rho; \Lambda^*)$

$$I_{pq}(\rho; \Lambda^*) = \sup \left\{ \sum_k \lambda_k S(\Lambda^* \rho_k, \Lambda^* \rho) ; \rho = \sum_k \lambda_k \rho_k, \text{finite decomposition} \right\}, \tag{32}$$

where the supremum is taken over all finite decompositions instead of all orthogonal pure decompositions for purely quantum mutual entropy. A **pseudo-quantum code** is a probability distribution on $\mathfrak{S}(\mathcal{H})$ with finite support in the set of product states. So $\{(\lambda_k), (\rho_k)\}$ is a pseudo-quantum code if (λ_k) is a probability vector and ρ_k are product states of $B(\mathcal{H})$. The quantum states ρ_k are sent over the quantum mechanical media, for example, optical fiber, and yield the output quantum states $\Lambda^* \rho_k$. The performance of coding and transmission is measured by the **pseudo-mutual entropy (information)**

$$I_{pq}((\lambda_k), (\rho_k); \Lambda^*) = I_{pq}(\rho; \Lambda^*) \tag{33}$$

with $\rho = \sum_k \lambda_k \rho_k$. Taking the supremum over certain classes of pseudo-quantum codes, we obtain various capacities of the channel. The supremum is over product states because we have mainly product (that is, memoryless) channels in our mind. Here we consider a subclass of pseudo-quantum codes. A **quantum code** is defined by the additional requirement that $\{\rho_k\}$ is a set of pairwise orthogonal pure states [10]. However the pseudo-mutual entropy is not well-matched to the conditions explained in Sec.3, and it is difficult to compute numerically [15]. From the monotonicity of the mutual entropy [13], we have

$$0 \leq C_q^{\mathcal{S}_0}(\Lambda^*) \leq C_{pq}^{\mathcal{S}_0}(\Lambda^*) \leq \sup\{S(\rho); \rho \in \mathcal{S}_0\}.$$

In order to estimate the quantum mutual entropy , we introduce the concept of **divergence center**. Let $\{\omega_i : i \in I\}$ be a family of states and $R > 0$. We say that the state ω is a divergence center for $\{\omega_i : i \in I\}$ with radius $\leq R$ if

$$S(\omega_i, \omega) \leq R \qquad \text{for every } i \in I.$$

In the following discussion about the geometry of relative entropy (or divergence as it is called in information theory) the ideas of [5] can be recognized very well.

Lemma 8 *[14] Let $((\lambda_k), (\rho_k))$ be a quantum code for the channel Λ^* and ω a divergence center with radius $\leq R$ for $\{\Lambda^* \rho_k\}$. Then*

$$I_{pq}((\lambda_k), (\rho_k); \Lambda^*) \leq R.$$

Lemma 9 *[14] Let ψ_0, ψ_1 and ω be states of $B(\mathcal{K})$ such that the Hilbert space $\mathcal{K}$ is finite dimensional and set $\psi_\lambda = (1-\lambda)\psi_0 + \lambda\psi_1$ $(0 \leq \lambda \leq 1)$. If $S(\psi_0, \omega)$, $S(\psi_1, \omega)$ are finite and*

$$S(\psi_\lambda, \omega) \geq S(\psi_1, \omega) \quad (0 \leq \lambda \leq 1)$$

then

$$S(\psi_1, \omega) + S(\psi_0, \psi_1) \leq S(\psi_0, \omega).$$

Lemma 10 *[14] Let $\{\omega_i : i \in I\}$ be a finite set of states of $B(\mathcal{K})$ such that the Hilbert space $\mathcal{K}$ is finite dimensional. Then the exact divergence center is unique and it is in the convex hull on the states ω_i.*

Theorem 11 *[14] Let $\Lambda^* : \mathfrak{S}(\mathcal{H}) \to \mathfrak{S}(\mathcal{K})$ be a channel with finite dimensional $\mathcal{K}$. Then the capacity $C_p(\ \Lambda^*)$ is the divergence radius of the range of Λ^*.*

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