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R 上の周期係数楕円型作用素のグリーン関数

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In the one dimensional case we shall show that the Green functions of elliptic operators with periodic coefficients are written as a product of an exponential function and a periodic function, and that the limiting absorption principle holds for all λ in the interior of the spectrum. We shall also calculate the resolvent kernel for all $\lambda \in \mathbf{R}$ in the resolvent set. The results are joint work with M. Murata, Tokyo Institute of Technology.

Let

$$L=-\frac{d}{dx}\left(a(x)\frac{d}{dx}\right)+c(x),$$

where $a(x)$ and $c(x)$ are real-valued periodic functions with period 1. Assume that $a \in L^{\infty}(\mathbf{R})$ and $0<\mu\leq a(x)\leq\mu^{-1}$ for some constant μ, and that $c\in L_{loc}^{1}(\mathbf{R})$. Corresponding to this operator, we consider the equation

$$\frac{d}{dx}\begin{pmatrix} y_1(x)\\ y_2(x)\end{pmatrix}=\begin{pmatrix} 0 & a(x)^{-1}\\ c(x)-z & 0\end{pmatrix}\begin{pmatrix} y_1(x)\\ y_2(x)\end{pmatrix} \tag{1}$$

for $z\in \mathbf{C}$. By the standard iteration method of ordinary differential equations, we can find unique solutions to (1), $(c_1(x,z),c_2(x,z))$ and $(s_1(x,z),s_2(x,z))$ with the initial conditions

$$\begin{pmatrix} c_1(0,z)\\ c_2(0,z)\end{pmatrix}=\begin{pmatrix}1\\0\end{pmatrix} \text{ and } \begin{pmatrix} s_1(0,z)\\ s_2(0,z)\end{pmatrix}=\begin{pmatrix}0\\1\end{pmatrix},$$

respectively, in the space of $\mathbf{C}^2$-valued absolutely continuous functions $AC(\mathbf{R})^2$. We can also see that $c_j(x,z)$ and $s_j(x,z)$ are $C([-R,R])$-valued entire functions of z for any R.

For each $\zeta\in \mathbf{C}$, the eigenvalue problem

$$\begin{cases} y\in H_{loc}^{1}(\mathbf{R})\\ Ly=zy\\ y(x+1)=e^{i\zeta}y(x) \quad (\zeta\text{-periodicity})\end{cases} \tag{2}$$

is equivalent to

$$\begin{cases} (y_1,y_2)\in AC(\mathbf{R})^2\\ (y_1,y_2) \text{ satisfies (1) and } y_1 \text{ satisfies the } \zeta\text{-periodicity}\end{cases}$$

under the relation $y_1=y$, $y_2=ay'$. Writing a solution to (2) as $y(x)=\alpha_1c_1(x,z)+\alpha_2s_1(x,z)$, $|\alpha_1|^2+|\alpha_2|^2\neq 0$, by the ζ-periodicity we have $(M(z)-e^{i\zeta}I)\alpha=0$, where

$$M(z):=\begin{pmatrix} c_1(1,z) & s_1(1,z)\\ c_2(1,z) & s_2(1,z)\end{pmatrix},\quad \alpha=\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix}.$$

We see that $\det(M(z)-e^{i\zeta}I)=0$ if and only if

$$D(z)=e^{i\zeta}+e^{-i\zeta}, \tag{3}$$

where $D(z) := c_1(1,z) + s_2(1,z)$ is the discriminant, which is an entire function. Hence the existence of non-trivial solution of (2) is equivalent to (3).

A function y is an eigenfunction of (2) if and only if $u(x) = e^{-ix\zeta}y(x)$ is an eigenfunction of $L(\zeta)$ with the same eigenvalue. Here $L(\zeta) = e^{-ix\zeta}Le^{ix\zeta}$ is an operator on $L^2(\mathbf{T})$ with compact resolvent with the domain $D(L(\zeta)) = \{u \in H^1(\mathbf{T}); L(\zeta)u \in L^2(\mathbf{T})\}$. Regarding L as the selfadjoint operator on $L^2(\mathbf{R})$ with the domain $D(L) = \{u \in H^1(\mathbf{R}); Lu \in L^2(\mathbf{R})\}$, we have the direct integral decomposition $\mathcal{U}L\mathcal{U}^{-1} = \int_{[-\pi,\pi)}^{\oplus} L(\xi)d\xi$, where $\mathcal{U}$ is a unitary operator (cf. [RS]).

We denote the eigenvalues of $L(\xi)$ by $\lambda_1(\xi) \le \lambda_2(\xi) \le \cdots$ for $\xi \in \mathbf{R}$ counted with multiplicities. Each $\lambda_n(\xi)$ is known to be continuous on $\mathbf{R}$. We summarize several facts, which can be proved in ways similar to those in [E], [Ku], [Ma], and [RS]. Each $\lambda_n(\xi)$ is real analytic on $(0,\pi)$, and for $\xi \in (0,\pi)$, $\lambda_n(\xi)$ is a nondegenerate eigenvalue of $L(\xi)$. There exists a sequence of real numbers

$$-\infty < \mu_1 < \nu_1 \le \nu_2 < \mu_2 \le \mu_3 < \nu_3 \le \cdots$$

such that it tends to infinity and has the following properties:

(i) The spectrum $\sigma(L)$ of L is $\cup_{n=1}^{\infty}([\mu_{2n-1},\nu_{2n-1}] \cup [\nu_{2n},\mu_{2n}])$; and $|D(\lambda)| \le 2$, $\lambda \in \mathbf{R}$, if and only if $\lambda \in \sigma(L)$.

(ii) $D(\lambda) = 2$ only at $\lambda = \mu_j$, and $D(\lambda) = -2$ only at $\lambda = \nu_j$.

(iii) $D'(\lambda) < 0$ on $(-\infty,\nu_1)$ and (μ_{2n-1},ν_{2n-1}), and $D'(\lambda) > 0$ on (ν_{2n},μ_{2n}).

(iv) $\lambda'_{2n-1}(\xi) > 0$ and $\lambda'_{2n}(\xi) < 0$ on $(0,\pi)$; in the interval $[0,\pi]$, $\lambda_{2n-1}(\xi)$ increases from μ_{2n-1} to ν_{2n-1}, and $\lambda_{2n}(\xi)$ decreases from μ_{2n} to ν_{2n}; $\lambda_n(k\pi+\xi) = \lambda_n(k\pi-\xi)$ for any integer k and real ξ.

(v) If $\lambda_{2n-1}(\pi) = \lambda_{2n}(\pi)$, then $\lambda_{2n-1}(\pi-0) \ne 0$; if $\lambda_{2n}(0) = \lambda_{2n+1}(0)$, then $\lambda_{2n+1}(0+0) \ne 0$

(vi) If $\nu_{2n-1} \ne \nu_{2n}$, then $D'(\nu_{2n-1}) \ne 0$ and $D'(\nu_{2n}) \ne 0$, and ν_{2n-1} and ν_{2n} are nondegenerate eigenvalues of $L(\pi)$; if $\mu_{2n} \ne \mu_{2n+1}$, then $D'(\mu_{2n}) \ne 0$ and $D'(\mu_{2n+1}) \ne 0$ and μ_{2n} and μ_{2n+1} are nondegenerate eigenvalues of $L(0)$; if $\nu_{2n-1} = \nu_{2n}$ or $\mu_{2n} = \mu_{2n+1}$, then $D' = 0$ at these points, and these are doubly degenerate eigenvalues of $L(\pi)$ or $L(0)$, respectively; if $D(\lambda) \ge 2$ (≤ -2) and $D'(\lambda) = 0$, then $D''(\lambda) < 0$ (> 0).

We denote by $G_z(x,y)$ the integral kernel of the resolvent $R(z) := (L-z)^{-1}$ for z in the resolvent set. We use the notations $(u,v) = \int_0^1 u(x)\overline{v(x)}\,dx$ and $\|u\|^2 = (u,u)$.

First, let λ be in the interior of $\sigma(L)$. Then the only one of the following four cases holds:

(I) $\lambda = \lambda_{2n-1}(\xi) \in (\mu_{2n-1},\nu_{2n-1})$ for some $\xi \in (0,\pi)$,
(II) $\lambda = \lambda_{2n}(\xi) \in (\nu_{2n},\mu_{2n})$ for some $\xi \in (-\pi,0)$,
(III) $\lambda = \lambda_{2n-1}(\pi) = \lambda_{2n}(\pi) = \nu_{2n-1} = \nu_{2n}$,
(IV) $\lambda = \lambda_{2n}(0) = \lambda_{2n+1}(0) = \mu_{2n} = \mu_{2n+1}$.

Theorem 1. *Assume that λ is in the interior of $\sigma(L)$. There exists the limit $\lim_{\varepsilon\downarrow 0}\left(\frac{d}{d\lambda}\right)^m R(\lambda \pm i\varepsilon)f(x)$ in $L^2_{loc}(\mathbf{R})$ for $m \ge 0$ and $f \in L^2(\mathbf{R})$ with compact support, and the convergence is locally uniform in the interior of $\sigma(L)$. The integral kernels $G_{\lambda+i0}(x,y)$ and $G^{(m)}_{\lambda+i0}(x,y)$ of $\lim_{\varepsilon\downarrow 0} R(\lambda+i\varepsilon)$ and $\lim_{\varepsilon\downarrow 0}\left(\frac{d}{d\lambda}\right)^m R(\lambda+i\varepsilon)$, $m \ge 1$, admit the following expressions:*

Case (I).

$$G_{\lambda+i0}(x,y) = G_{\lambda+i0}(y,x) = \frac{ie^{i(x-y)\xi}}{\lambda'_{2n-1}(\xi)} \frac{u_\xi(x)\overline{u_\xi(y)}}{\|u_\xi\|^2}, \qquad y \le x,$$

$$G^{(m)}_{\lambda+i0}(x,y) = G^{(m)}_{\lambda+i0}(y,x)$$
$$= \Big(\frac{i}{\lambda'_{2n-1}(\xi)}\Big)^{m+1}(x-y)^m e^{i(x-y)\xi}\frac{u_\xi(x)\overline{u_\xi(y)}}{\|u_\xi\|^2}(1+O(|x-y|^{-1})), \quad y \le x.$$

Here u_ξ is an eigenfunction corresponding to the eigenvalue $\lambda_{2n-1}(\xi)$.

Case (II). $G_{\lambda+i0}(x,y)$ and $G^{(m)}_{\lambda+i0}(x,y)$ admit the same expressions as in (I) with $\lambda'_{2n-1}(\xi)$ replaced by $\lambda'_{2n}(\xi)$, and with u_ξ being an eigenfunction corresponding to the eigenvalue $\lambda_{2n}(\xi)$.

Case (III). With u_ξ being a $C(\mathbf{T})$-valued holomorphic function in a neighborhood of π such that $\|u_\xi\| \ne 0$, $(L(\xi)-\lambda_{2n-1}(\xi))u_\xi = 0$ for $\xi \le \pi$, and $(L(\xi)-\lambda_{2n}(\xi))u_\xi = 0$ for $\pi < \xi$,

$$G_{\lambda+i0}(x,y) = G_{\lambda+i0}(y,x) = \frac{ie^{i(x-y)\pi}}{\lambda'_{2n-1}(\pi-0)} \frac{u_\pi(x)\overline{u_\pi(y)}}{\|u_\pi\|^2}, \qquad y \le x,$$

$$G^{(m)}_{\lambda+i0}(x,y) = G^{(m)}_{\lambda+i0}(y,x)$$
$$= \Big(\frac{i}{\lambda'_{2n-1}(\pi-0)}\Big)^{m+1}(x-y)^m e^{i(x-y)\pi}\frac{u_\pi(x)\overline{u_\pi(y)}}{\|u_\pi\|^2}(1+O(|x-y|^{-1})), \quad y \le x.$$

Case (IV). With u_ξ being a $C(\mathbf{T})$-valued holomorphic function in a neighborhood of 0 such that $\|u_\xi\| \ne 0$, $(L(\xi)-\lambda_{2n+1}(\xi))u_\xi = 0$ for $0 \le \xi$, and $(L(\xi)-\lambda_{2n}(\xi))u_\xi = 0$ for $\xi < 0$,

$$G_{\lambda+i0}(x,y) = G_{\lambda+i0}(y,x) = \frac{i}{\lambda'_{2n+1}(0+0)} \frac{u_0(x)\overline{u_0(y)}}{\|u_0\|^2}, \qquad y \le x,$$

$$G^{(m)}_{\lambda+i0}(x,y) = G^{(m)}_{\lambda+i0}(y,x)$$
$$= \Big(\frac{i}{\lambda'_{2n+1}(0+0)}\Big)^{m+1}(x-y)^m \frac{u_0(x)\overline{u_0(y)}}{\|u_0\|^2}(1+O(|x-y|^{-1})), \qquad y \le x.$$

Proof. (I) Since $D'(\lambda) < 0$ on (μ_{2n-1}, ν_{2n-1}), there exists a holomorphic inverse function D^{-1} of D on an open set containing $(-2,2)$. Put $\lambda(\zeta) := D^{-1}(e^{i\zeta}+e^{-i\zeta})$ for ζ in an open set containing $(0,\pi)$. We have $\lambda(\xi) = \lambda_{2n-1}(\xi)$ for $\xi \in (0,\pi)$. Let

$$\alpha(\zeta) = (\alpha_1(\zeta), \alpha_2(\zeta)) := (-s_1(1,\lambda(\zeta)), c_1(1,\lambda(\zeta)) - e^{i\zeta}).$$

Since $\alpha(\xi) \ne 0$ for $\xi \in (0,\pi)$, $\alpha(\zeta)$ is an eigenvector of $M(\lambda(\zeta))$ corresponding to the eigenvalue $e^{i\zeta}$ for ζ in an open set containing $(0,\pi)$. Thus $y_\zeta(x) := \alpha_1(\zeta)c_1(x,\lambda(\zeta)) +$

$\alpha_2(\zeta)s_1(x,\lambda(\zeta))$ satisfies (2) with z replaced by $\lambda(\zeta)$. So $u_\zeta(x) := e^{-i\zeta x}y_\zeta(x)$ is a $C(\mathbf{T})$-valued holomorphic eigenfunction of $L(\zeta)$ corresponding to the eigenvalue $\lambda(\zeta)$. Since $\lambda'_{2n-1}(\xi) > 0$ on $(0,\pi)$, the inverse function theorem implies that there exists a holomorphic function $\zeta(z)$ on an open set containing (μ_{2n-1},ν_{2n-1}) such that $\lambda(\zeta(z)) = z$. For each $\lambda \in (\mu_{2n-1},\nu_{2n-1})$, if $\varepsilon > 0$ is small enough, $y_{\zeta(\lambda+i\varepsilon)}(x)$ is a solution to the equation $Ly = (\lambda + i\varepsilon)y$. Taking the complex conjugate of this equation and replacing ε by $-\varepsilon$, we obtain that $\overline{y_{\zeta(\lambda-i\varepsilon)}(x)}$ is also a solution. Since $\zeta'(\lambda) > 0$, we obtain the linearly independent solutions to $Ly = (\lambda + i\varepsilon)y$:

$$y_{\zeta(\lambda+i\varepsilon)}(x) = e^{i\zeta(\lambda+i\varepsilon)x}u_{\zeta(\lambda+i\varepsilon)}(x) = \exp[(i\zeta(\lambda) - \varepsilon\zeta'(\lambda) + O(\varepsilon^2))x]u_{\zeta(\lambda+i\varepsilon)}(x),$$
$$\overline{y_{\zeta(\lambda-i\varepsilon)}(x)} = e^{-i\overline{\zeta(\lambda-i\varepsilon)}x}\overline{u_{\zeta(\lambda-i\varepsilon)}(x)} = \exp[(-i\zeta(\lambda) + \varepsilon\zeta'(\lambda) + O(\varepsilon^2))x]\overline{u_{\zeta(\lambda-i\varepsilon)}(x)}.$$

Let $[y,\tilde{y}](x) := a(x)(y(x)\tilde{y}'(x) - y'(x)\tilde{y}(x))$ be the Wronskian of two solutions y and $\tilde{y}$. Then

$$G_{\lambda+i\varepsilon}(x,y) = \begin{cases} y_{\zeta(\lambda+i\varepsilon)}(x)\overline{y_{\zeta(\lambda-i\varepsilon)}(y)}/[y_{\zeta(\lambda+i\varepsilon)},\overline{y_{\zeta(\lambda-i\varepsilon)}}](0), & y \le x, \\ y_{\zeta(\lambda+i\varepsilon)}(y)\overline{y_{\zeta(\lambda-i\varepsilon)}(x)}/[y_{\zeta(\lambda+i\varepsilon)},\overline{y_{\zeta(\lambda-i\varepsilon)}}](0), & x \le y, \end{cases}$$

(cf. §5.3 in [E]). Since $[y_{\zeta(\lambda+i\varepsilon)},\overline{y_{\zeta(\lambda-i\varepsilon)}}](x)$ is a constant independent of x and $\zeta(\lambda+i\varepsilon) = \overline{\zeta(\lambda-i\varepsilon)}$, it follows that

$$\begin{aligned}&[y_{\zeta(\lambda+i\varepsilon)},\overline{y_{\zeta(\lambda-i\varepsilon)}}](0)\\ &\quad = \int_0^1 \big([u_{\zeta(\lambda+i\varepsilon)},\overline{u_{\zeta(\lambda-i\varepsilon)}}](x) - 2i\zeta(\lambda+i\varepsilon)a(x)u_{\zeta(\lambda+i\varepsilon)}(x)\overline{u_{\zeta(\lambda-i\varepsilon)}(x)}\big)dx.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\int_0^1 [a(x)(\frac{d}{dx} + i\zeta(\lambda+i\varepsilon))u_{\zeta(\lambda+i\varepsilon)}(x)\overline{(\frac{d}{dx} - i\zeta(\lambda+i\varepsilon))u_{\zeta(\lambda-i\varepsilon)}(x)}\\ + c(x)u_{\zeta(\lambda+i\varepsilon)}(x)\overline{u_{\zeta(\lambda-i\varepsilon)}(x)}]dx = (\lambda+i\varepsilon)(u_{\zeta(\lambda+i\varepsilon)},u_{\zeta(\lambda-i\varepsilon)}).\end{aligned}$$

Differentiating both sides of this equation with respect to λ, we have

$$\begin{aligned}&i\zeta'(\lambda+i\varepsilon)\int_0^1 \big([u_{\zeta(\lambda+i\varepsilon)},\overline{u_{\zeta(\lambda-i\varepsilon)}}](x) - 2i\zeta(\lambda+i\varepsilon)a(x)u_{\zeta(\lambda+i\varepsilon)}(x)\overline{u_{\zeta(\lambda-i\varepsilon)}(x)}\big)dx\\ &\quad = (u_{\zeta(\lambda+i\varepsilon)},u_{\zeta(\lambda-i\varepsilon)}).\end{aligned}$$

Thus

$$i\zeta'(\lambda+i\varepsilon)[y_{\zeta(\lambda+i\varepsilon)},\overline{y_{\zeta(\lambda-i\varepsilon)}}](0) = (u_{\zeta(\lambda+i\varepsilon)},u_{\zeta(\lambda-i\varepsilon)}).$$

Therefore we have

$$G_{\lambda+i\varepsilon}(x,y) = G_{\lambda+i\varepsilon}(y,x) = i\zeta'(\lambda+i\varepsilon)e^{i\zeta(\lambda+i\varepsilon)(x-y)}\frac{u_{\zeta(\lambda+i\varepsilon)}(x)\overline{u_{\zeta(\lambda-i\varepsilon)}(y)}}{(u_{\zeta(\lambda+i\varepsilon)},u_{\zeta(\lambda-i\varepsilon)})}, \qquad y \le x.$$

Taking the limit $\varepsilon \downarrow 0$, we have the existence of the limit $\lim_{\varepsilon\downarrow 0} R(\lambda \pm i\varepsilon)f(x)$ and

$$G_{\lambda+i0}(x,y) = \lim_{\varepsilon\downarrow 0} G_{\lambda+i\varepsilon}(x,y) = \frac{ie^{i(x-y)\xi}}{\lambda'_{2n-1}(\xi)} \frac{u_\xi(x)\overline{u_\xi(y)}}{\|u_\xi\|^2}, \qquad y \le x,$$

where $\xi = \zeta(\lambda)$, i.e., $\lambda_{2n-1}(\xi) = \lambda$. Furthermore, we can see that for any integer $m \ge 1$, the limit $\lim_{\varepsilon\downarrow 0} (\frac{d}{d\lambda})^m R(\lambda \pm i\varepsilon)f(x)$ exists and

$$\begin{aligned} G^{(m)}_{\lambda+i0}(x,y) &= \lim_{\varepsilon\downarrow 0} (\frac{d}{d\lambda})^m G_{\lambda+i\varepsilon}(x,y) \\ &= (\frac{i}{\lambda'_{2n-1}(\xi)})^{m+1}(x-y)^m e^{i(x-y)\xi}\frac{u_\xi(x)\overline{u_\xi(y)}}{\|u_\xi\|^2}(1+O(|x-y|^{-1})), \qquad y \le x. \end{aligned}$$

We have thus proved the case (I). The case (II) is proved in the same way as (I).

(III) Assume that $\lambda_{2n-1}(\pi) = \lambda_{2n}(\pi) = \nu_{2n-1} = \nu_{2n}$. Since ν_{2n} is a doubly degenerate eigenvalue and $L(\xi)$ is selfadjoint for ξ real, Theorem XII.13 in [RS] implies that there exist holomorphic eigenvalues $E_1(\zeta)$ and $E_2(\zeta)$ of $L(\zeta)$ near $\zeta = \pi$ such that $E_1(\pi) = E_2(\pi) = \nu_{2n}$. If $\xi \in \mathbf{R}$, each of $\lambda_{2n-1}(\xi)$ and $\lambda_{2n}(\xi)$ must be equal to one of $E_j(\xi)$, $j = 1, 2$. Since $D(E_j(\xi)) = 2\cos\xi$ near $\xi = \pi$, we have

$$D''(E_j(\xi))E'_j(\xi)^2 + D'(E_j(\xi))E''_j(\xi) = -2\cos\xi.$$

So, since $D'(\nu_{2n}) = 0$ and $D''(\nu_{2n}) > 0$, we obtain that $E'_j(\pi) \ne 0$ (which implies the fact (v) stated before Theorem 1). Since

$$\begin{cases} \lambda'_{2n-1}(\xi) > 0, & \xi < \pi, \\ \lambda'_{2n}(\xi) > 0, & \pi < \xi, \end{cases} \quad \text{and} \quad \begin{cases} \lambda'_{2n-1}(\xi) < 0, & \pi < \xi, \\ \lambda'_{2n}(\xi) < 0, & \xi < \pi, \end{cases}$$

we conclude that there exist holomorphic functions $E_1(\zeta)$ and $E_2(\zeta)$ on an open set containing $(0, 2\pi)$ such that

$$E_1(\xi) = \begin{cases} \lambda_{2n-1}(\xi), & 0 \le \xi \le \pi, \\ \lambda_{2n}(\xi), & \pi \le \xi \le 2\pi, \end{cases} \qquad E_2(\xi) = \begin{cases} \lambda_{2n}(\xi), & 0 \le \xi \le \pi, \\ \lambda_{2n-1}(\xi), & \pi \le \xi \le 2\pi. \end{cases}$$

Since $E'_1(\xi) > 0$ on $(0, 2\pi)$, the inverse function theorem implies that there exists a holomorphic function $\zeta(z)$ on an open set containing (μ_{2n-1}, μ_{2n}) such that $E_1(\zeta(z)) = z$.

Let $p(\xi)$ be the eigenprojection for the eigenvalue $e^{i\xi}$ of $M(E_1(\xi))$ for $\xi \in (0,\pi)\cup(\pi, 2\pi)$:

$$\begin{aligned} p(\xi) &:= (-2\pi i)^{-1} \oint_{|z-e^{i\xi}|=\delta} (M(E_1(\xi)) - z)^{-1} dz \\ &= \frac{-1}{e^{i\xi} - e^{-i\xi}} \begin{pmatrix} s_2(1, E_1(\xi)) - e^{i\xi} & -s_1(1, E_1(\xi)) \\ -c_2(1, E_1(\xi)) & c_1(1, E_1(\xi)) - e^{i\xi} \end{pmatrix}, \end{aligned}$$

where $\delta > 0$ is taken so that $e^{i\xi}$ is the only eigenvalue of $M(E_1(\xi))$ inside the circle $|z - e^{i\xi}| = \delta$. Since $s_2(1,\nu_{2n}) + 1 = c_1(1,\nu_{2n}) + 1 = s_1(1,\nu_{2n}) = c_2(1,\nu_{2n}) = 0$ (cf. [E, p.7 and p.29]), $\xi = \pi$ is a removable singularity of $p(\xi)$. We have $(p(\xi))_{11} \neq 0$ on $(0, 2\pi)$, since

$$(p(\pi))_{11} = (2i)^{-1}\partial_\xi(s_2(1, E_1(\xi)) - e^{i\xi})|_{\xi=\pi} = (2i)^{-1}(\partial_z s_2(1,\nu_{2n})E_1'(\pi) + i) \neq 0.$$

Thus $p(\xi)$ is a real analytic rank one matrix on $(0, 2\pi)$. Note that the holomorphically extended $p(\zeta)$ to an open set containing $(0, 2\pi)$ is the eigenprojection for the eigenvalue $e^{i\zeta}$ of $M(E_1(\zeta))$. Thus the function $y_\zeta(x) := (p(\zeta))_{11}c_1(x, E_1(\zeta)) + (p(\zeta))_{21}s_1(x, E_1(\zeta))$ is a solution to (2) with z replaced by $E_1(\zeta)$; and so $u_\zeta(x) = e^{-i\zeta x}y_\zeta(x)$ is a $C(\mathbf{T})$-valued holomorphic eigenfunction of $L(\zeta)$ corresponding to $E_1(\zeta)$ on an open set containing $(0, 2\pi)$. Thus as in the case (I), since $\zeta'(\lambda) > 0$ for $\lambda \in (\mu_{2n-1}, \mu_{2n})$, $y_{\zeta(\lambda+i\varepsilon)}(x)$ and $\overline{y_{\zeta(\lambda-i\varepsilon)}(x)}$ are linearly independent solutions to $Ly = (\lambda + i\varepsilon)y$. Hence, as in the proof of (I) we have

$$G_{\nu_{2n}+i0}(x,y) = \lim_{\varepsilon\downarrow 0} G_{\nu_{2n}+i\varepsilon}(x,y) = \frac{ie^{i(x-y)\pi}}{E_1'(\pi)} \frac{u_\pi(x)\overline{u_\pi(y)}}{\|u_\pi\|^2}, \qquad y \le x,$$

and for any integer $m \ge 1$,

$$G^{(m)}_{\nu_{2n}+i0}(x,y) = \lim_{\varepsilon\downarrow 0}\Big(\frac{d}{d\lambda}\Big)^m G_{\nu_{2n}+i\varepsilon}(x,y)$$

$$= \Big(\frac{i}{E_1'(\pi)}\Big)^{m+1}(x-y)^m e^{i(x-y)\pi}\frac{u_\pi(x)\overline{u_\pi(y)}}{\|u_\pi\|^2}(1 + O(|x-y|^{-1})), \qquad y \le x.$$

Note that $E_1'(\pi) = \lambda_{2n-1}'(\pi - 0)$. We have thus proved (III). (IV) is proved similarly. From the proof above it follows that the covergence $\lim_{\varepsilon\downarrow 0}\big(\frac{d}{d\lambda}\big)^m R(\lambda \pm i\varepsilon)f(x)$ is locally uniform with respect to λ. □

The following is a direct consequence of Theorem 1.

Corollary 2. *Let λ be in the interior of $\sigma(L)$. Then $\big(\frac{d}{d\lambda}\big)^m R(\lambda \pm i0)$, $m \ge 0$, is bounded from $B_{\frac{1}{2}+m}$ to $B^*_{\frac{1}{2}+m}$.*

Proof. Let $f \in C_0^\infty(\mathbf{R})$. Since Theorem 1 yields that

$$\Big|\Big(\frac{d}{d\lambda}\Big)^m R(\lambda+i0)f(x)\Big| \le C_m(1+|x|)^m \int_{\mathbf{R}}(1+|y|)^m|f(y)|dy \le C_m(1+|x|)^m\|f\|_{B_{\frac{1}{2}+m}},$$

it follows that

$$\Big\|\Big(\frac{d}{d\lambda}\Big)^m R(\lambda+i0)f(x)\Big\|_{B^*_{\frac{1}{2}+m}} \le C_m\|(1+|x|)^m\|_{B^*_{\frac{1}{2}+m}}\|f\|_{B_{\frac{1}{2}+m}} \le C_m\|f\|_{B_{\frac{1}{2}+m}}.$$

□

Next we study the case that the parameter $\lambda \in \mathbf{R}$ is in the resolvent set of L. This case is equivalent to $|D(\lambda)| > 2$. $D(\lambda) > 2$ if and only if $\lambda \in A_+ := (-\infty, \mu_1) \cup [\cup_{n=1}^\infty(\mu_{2n}, \mu_{2n+1})]$; and $D(\lambda) < -2$ if and only if $\lambda \in A_- := \cup_{n=1}^\infty(\nu_{2n-1}, \nu_{2n})$. Consider a function $e^\eta + e^{-\eta}$ on $(0, \infty)$, and solve the equation

$$e^\eta + e^{-\eta} = D(\lambda)$$

with respect to η, where $\lambda \in A_+$. By the implicit function theorem, we have a unique solution $\eta(\lambda)$ which is real analytic on A_+. Similarly, define $\eta(\lambda)$ on A_- by $e^\eta + e^{-\eta} = -D(\lambda)$. Note that $\dim\mathrm{Ker}\,(L(\pm i\eta(\lambda)) - \lambda) = 1$ for $\lambda \in A_+$ and $\dim\mathrm{Ker}\,(L(\pi \pm i\eta(\lambda)) - \lambda) = 1$ for $\lambda \in A_-$ (cf. [E, p.6]).

Theorem 3. *(i) Let* $\lambda \in A_+$. *Let* u_λ *and* v_λ *be real-valued eigenfunctions of* $L(i\eta(\lambda))$ *and* $L(-i\eta(\lambda))$ *corresponding to the eigenvalue* λ, *respectively.*

Suppose $D'(\lambda) \neq 0$. *Then* $(u_\lambda, v_\lambda) \neq 0$ *and*

$$G_\lambda(x,y) = G_\lambda(y,x) = -\eta'(\lambda)e^{-\eta(\lambda)(x-y)}\frac{u_\lambda(x)v_\lambda(y)}{(u_\lambda, v_\lambda)}, \quad y \leq x. \tag{4}$$

Suppose $D'(\lambda) = 0$. *Then there exists a solution* $\psi_{v_\lambda} \in H^1(\mathbf{T})$ *of the equation* $(L(-i\eta(\lambda)) - \lambda)\psi = v_\lambda$ *such that* $(u_\lambda, \psi_{v_\lambda}) \neq 0$, *and*

$$G_\lambda(x,y) = G_\lambda(y,x) = -\frac{\eta''(\lambda)}{2}e^{-\eta(\lambda)(x-y)}\frac{u_\lambda(x)v_\lambda(y)}{(u_\lambda, \psi_{v_\lambda})}, \quad y \leq x. \tag{5}$$

(ii) Let $\lambda \in A_-$. *Let* u_λ *and* v_λ *be eigenfunctions of* $L(\pi + i\eta(\lambda))$ *and* $L(\pi - i\eta(\lambda))$ *corresponding to the eigenvalue* λ, *respectively.*

Suppose $D'(\lambda) \neq 0$. *Then* $(u_\lambda, v_\lambda) \neq 0$ *and*

$$G_\lambda(x,y) = G_\lambda(y,x) = -\eta'(\lambda)e^{(i\pi-\eta(\lambda))(x-y)}\frac{u_\lambda(x)\overline{v_\lambda(y)}}{(u_\lambda, v_\lambda)}, \quad y \leq x.$$

Suppose $D'(\lambda) = 0$. *Then there exists a solution* $\psi_{v_\lambda} \in H^1(\mathbf{T})$ *of the equation* $(L(\pi - i\eta(\lambda)) - \lambda)\psi = v_\lambda$ *such that* $(u_\lambda, \psi_{v_\lambda}) \neq 0$, *and*

$$G_\lambda(x,y) = G_\lambda(y,x) = -\frac{\eta''(\lambda)}{2}e^{(i\pi-\eta(\lambda))(x-y)}\frac{u_\lambda(x)\overline{v_\lambda(y)}}{(u_\lambda, \psi_{v_\lambda})}, \quad y \leq x.$$

Proof. Let $\lambda \in A_+$. Since $c_1(1,\lambda) - e^{\pm\eta(\lambda)}$ and $s_2(1,\lambda) - e^{\pm\eta(\lambda)} = e^{\mp\eta(\lambda)} - c_1(1,\lambda)$ do not vanish simultaneously on a neighborhood of each $\lambda \in A_+$, there exist nonzero real analytic eigenvectors $\alpha_\pm(\lambda) = (\alpha_{\pm,1}(\lambda), \alpha_{\pm,2}(\lambda))$ of $M(\lambda)$ corresponding to the eigenvalues $e^{\eta(\lambda)}$ and $e^{-\eta(\lambda)}$, respectively. Then $y_\lambda(x) := \alpha_{-,1}(\lambda)c_1(x,\lambda) + \alpha_{-,2}(\lambda)s_1(x,\lambda)$ and $z_\lambda(x) := \alpha_{+,1}(\lambda)c_1(x,\lambda) + \alpha_{+,2}(\lambda)s_1(x,\lambda)$ are solutions to (2) with ζ replaced by $i\eta(\lambda)$ and $-i\eta(\lambda)$. Thus $u_\lambda(x) := e^{\eta(\lambda)x}y_\lambda(x)$ and $v_\lambda(x) := e^{-\eta(\lambda)x}z_\lambda(x)$ are $C(\mathbf{T})$-valued real analytic eigenfunctions on A_+ of $L(i\eta(\lambda))$ and $L(i\eta(\lambda))^* = L(-i\eta(\lambda))$ corresponding to the eigenvalue λ, respectively. Hence $y_\lambda(x) = e^{-\eta(\lambda)x}u_\lambda(x)$ and $z_\lambda(x) = e^{\eta(\lambda)x}v_\lambda(x)$ are linearly independent solutions, and so

$$G_\lambda(x,y) = \begin{cases} y_\lambda(x)z_\lambda(y)/[y_\lambda, z_\lambda](0), & y \leq x, \\ y_\lambda(y)z_\lambda(x)/[y_\lambda, z_\lambda](0), & x \leq y. \end{cases}$$

Since $[y_\lambda, z_\lambda](x)$ is a constant independent of x, it follows that

$$[y_\lambda, z_\lambda](0) = \int_0^1 ([u_\lambda, v_\lambda](x) + 2\eta(\lambda)a(x)u_\lambda(x)v_\lambda(x))dx.$$

On the other hand, we have

$$\int_0^1 [a(x)(\frac{d}{dx} - \eta(\lambda))u_\lambda(x)(\frac{d}{dx} + \eta(\lambda))v_\lambda(x) + c(x)u_\lambda(x)v_\lambda(x)]dx = \lambda(u_\lambda, v_\lambda).$$

Differentiating both sides of this equation with respect to λ, we have

$$-\eta'(\lambda)\int_0^1([u_\lambda,v_\lambda](x)+2\eta(\lambda)a(x)u_\lambda(x)v_\lambda(x))dx=(u_\lambda,v_\lambda).$$

Hence

$$-\eta'(\lambda)[y_\lambda,z_\lambda](0)=(u_\lambda,v_\lambda). \tag{6}$$

Suppose $D'(\lambda)\neq 0$. Then $\eta'(\lambda)=D'(\lambda)/(e^{\eta(\lambda)}-e^{-\eta(\lambda)})\neq 0$ and

$$G_\lambda(x,y)=-\eta'(\lambda)e^{-\eta(\lambda)(x-y)}u_\lambda(x)v_\lambda(y)/(u_\lambda,v_\lambda),\quad y\leq x.$$

Suppose $D'(\lambda)=0$. Then $\eta'(\lambda)=0$ and $\eta''(\lambda)=D''(\lambda)/(e^{\eta(\lambda)}-e^{-\eta(\lambda)})<0$. Differentiating (6), we have

$$\eta''(\lambda)[y_\lambda,z_\lambda](0)=-(u_\lambda,v_\lambda)'. \tag{7}$$

Therefore

$$G_\lambda(x,y)=-\eta''(\lambda)e^{-\eta(\lambda)(x-y)}u_\lambda(x)v_\lambda(y)/(u_\lambda,v_\lambda)',\quad y\leq x.$$

By (6), $(u_\lambda,v_\lambda)=0$. Moreover, since $\eta'(\lambda)=0$,

$$(L(i\eta(\lambda))-\lambda)\partial_\lambda u_\lambda=u_\lambda \text{ and } (L(-i\eta(\lambda))-\lambda)\partial_\lambda v_\lambda=v_\lambda. \tag{8}$$

Put $\psi_{v_\lambda}=\partial_\lambda v_\lambda$. Then ψ_{v_λ} is a solution of $(L(-i\eta(\lambda))-\lambda)\psi=v_\lambda$. By (8), we have

$$(\partial_\lambda u_\lambda,v_\lambda)=(\partial_\lambda u_\lambda,(L(-i\eta(\lambda))-\lambda)\partial_\lambda v_\lambda)=((L(i\eta(\lambda))-\lambda)\partial_\lambda u_\lambda,\partial_\lambda v_\lambda)=(u_\lambda,\partial_\lambda v_\lambda).$$

Thus $(u_\lambda,v_\lambda)'=2(u_\lambda,\psi_{v_\lambda})$, which together with (7) implies that $(u_\lambda,\psi_{v_\lambda})\neq 0$. Therefore we have (5). The assertion (ii) is proved similarly. □

We have seen that in the formula (4) and (5) the different factor $\dfrac{u_\lambda(x)v_\lambda(y)}{(u_\lambda,v_\lambda)}$ or $\dfrac{u_\lambda(x)v_\lambda(y)}{(u_\lambda,\psi_{v_\lambda})}$ appears according to whether $D'(\lambda)$ does not vanish or not. This is related to the Laurent expansion of $(L(i\eta(\lambda))-z)^{-1}$ with respect to z around λ.

Proposition 4. *Let $\lambda\in A_+$. If $D'(\lambda)\neq 0$, the eigenvalue λ of $L(i\eta(\lambda))$ is nondegenerate and its eigenprojection has the integral kernel $\dfrac{u_\lambda(x)v_\lambda(y)}{(u_\lambda,v_\lambda)}$; and if $D'(\lambda)=0$, the eigenvalue λ of $L(i\eta(\lambda))$ is degenerate and its eigennilpotent has the integral kernel $\dfrac{u_\lambda(x)v_\lambda(y)}{(u_\lambda,\psi_{v_\lambda})}$. Similar statement holds for $\lambda\in A_-$.*

Proof. We shall represent the integral kernel $R(\zeta,z;x,y)$ of the resolvent $R(\zeta,z):=(L(\zeta)-z)^{-1}$, by using $c_j(x,z)$ and $s_j(x,z)$. Let $(\zeta,z)\in\Gamma:=\{(\zeta,z)\in\mathbf{C}^2; z\notin\sigma(L(\zeta))\}$. Put

$$k(z;x,y):=\begin{cases} c_1(x,z)s_1(y,z), & y\leq x,\\ s_1(x,z)c_1(y,z), & x\leq y.\end{cases}$$

For $f \in C_0^\infty(0,1)$, put

$$K_z f(x) := \int k(z;x,y)f(y)dy.$$

Since $(L-z)K_z f(x) = f(x)$ and $(L-z)e^{ix\zeta}R(\zeta,z)e^{-ix\zeta}f(x) = f(x)$ on $(0,1)$, $e^{ix\zeta}R(\zeta,z)e^{-ix\zeta}f(x) - K_z f(x)$ is a solution to $Ly = zy$. Thus

$$e^{ix\zeta}R(\zeta,z)e^{-ix\zeta}f(x) - K_z f(x) = \alpha c_1(x,z) + \beta s_1(x,z) \tag{9}$$

for some α and β. Since $R(\zeta,z)e^{-ix\zeta}f(x) \in D(L(\zeta))$ has the periodicity, we get

$$K_z f(x) + \alpha c_1(x,z) + \beta s_1(x,z) = e^{-i\zeta}(K_z f(x+1) + \alpha c_1(x+1,z) + \beta s_1(x+1,z)), \tag{10}$$

so putting $x = 0$, we have

$$\alpha = e^{-i\zeta}[c_1(1,z)\int_0^1 s_1(y,z)f(y)dy + \alpha c_1(1,z) + \beta s_1(1,z)]. \tag{11}$$

Differentiating both sides of (10) with respect to x and putting $x = 0$, we have

$$\int_0^1 c_1(y,z)f(y)dy + \beta = e^{-i\zeta}[c_2(1,z)\int_0^1 s_1(y,z)f(y)dy + \alpha c_2(1,z) + \beta s_2(1,z)]. \tag{12}$$

Note that $(\zeta, z) \in \Gamma$ if and only if $\delta(\zeta,z) := D(z) - e^{i\zeta} - e^{-i\zeta} \neq 0$. Solving (11) and (12) with respect to (α, β), we have

$$\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \delta(\zeta,z)^{-1}\int_0^1 [\begin{pmatrix} s_1(1,z) \\ e^{i\zeta} - c_1(1,z)\end{pmatrix} c_1(y,z) + \begin{pmatrix} e^{-i\zeta} - c_1(1,z) \\ -c_2(1,z)\end{pmatrix} s_1(y,z)]f(y)dy.$$

Combining this with (9), we obtain that

$$R(\zeta,z;x,y) = e^{i\zeta(y-x)}k(z;x,y) + \frac{e^{i\zeta(y-x)}s(\zeta,z;x,y)}{D(z) - e^{i\zeta} - e^{-i\zeta}},$$

where

$$\begin{aligned} s(\zeta,z;x,y) :=& [s_1(1,z)c_1(x,z) + (e^{i\zeta} - c_1(1,z))s_1(x,z)]c_1(y,z) \\ &+ [(e^{-i\zeta} - c_1(1,z))c_1(x,z) - c_2(1,z)s_1(x,z)]s_1(y,z). \end{aligned}$$

Suppose $D'(\lambda) \neq 0$. For z near λ, we have $D(z) - e^{\eta(\lambda)} - e^{-\eta(\lambda)} = (z-\lambda)F_\lambda(z)$ for some $F_\lambda(z)$ such that $F_\lambda(\lambda) = D'(\lambda) \neq 0$. Thus $R(i\eta(\lambda), z; x, y)$ has a pole λ of order one with the residue

$$r_1(\lambda;x,y) := D'(\lambda)^{-1}e^{(x-y)\eta(\lambda)}s(i\eta(\lambda),\lambda;x,y).$$

This implies that the eigenvalue λ of $L(i\eta(\lambda))$ is nondegenerate and its eigenprojection has the integral kernel $-r_1(\lambda;x,y)$. On the other hand, the eigenprojection and its adjoint are projections onto the spaces Ker $(L(i\eta(\lambda))-\lambda)$ and Ker $(L(-i\eta(\lambda))-\lambda)$, respectively, so the eigenprojection has the integral kernel $\dfrac{u_\lambda(x)v_\lambda(y)}{(u_\lambda, v_\lambda)}$. Therefore $\dfrac{u_\lambda(x)v_\lambda(y)}{(u_\lambda, v_\lambda)} = -r_1(\lambda; x, y)$.

Let $\lambda_0 \in \mathbf{R}$ satisfy $D'(\lambda_0) = 0$. For z near λ_0, we have $D(z) - e^{\eta(\lambda_0)} - e^{-\eta(\lambda_0)} = (z-\lambda_0)^2 H(z)$ for some $H(z)$ such that $H(\lambda_0) = D''(\lambda_0)/2 \neq 0$. Thus $R(i\eta(\lambda_0), z; x, y)$ has a pole λ_0 of order two:

$$R(i\eta(\lambda_0), z; x, y) = r_2(x, y)(z-\lambda_0)^{-2} + O((z-\lambda_0)^{-1}),$$

where

$$r_2(x, y) := 2D''(\lambda_0)^{-1} e^{(x-y)\eta(\lambda_0)} s(i\eta(\lambda_0), \lambda_0; x, y).$$

Hence the eigenvalue λ_0 of $L(i\eta(\lambda_0))$ is degenerate and its eigennilpotent has the integral kernel $-r_2(x, y)$. We shall show that $\dfrac{u_\lambda(x)v_\lambda(y)}{(u_\lambda, \psi_{v_\lambda})} = -r_2(x, y)$ at $\lambda = \lambda_0$. Since

$$\begin{aligned} \partial_z c_1(x, z) &= \int_0^x (c_1(x, z)s_1(t, z) - s_1(x, z)c_1(t, z))c_1(t, z)\, dt, \\ \partial_z s_2(x, z) &= \int_0^x (c_2(x, z)s_1(t, z) - s_2(x, z)c_1(t, z))s_1(t, z)\, dt \end{aligned}$$

(cf. [E]), we have for $\lambda \in A_+$

$$\begin{aligned} D'(\lambda) &= \partial_\lambda c_1(1, \lambda) + \partial_\lambda s_2(1, \lambda) \\ &= \int_0^1 [c_2(1, \lambda)s_1(x, \lambda)^2 + (c_1(1, \lambda) - s_2(1, \lambda))c_1(x, \lambda)s_1(x, \lambda) - s_1(1, \lambda)c_1(x, \lambda)^2]dx \\ &= -\int_0^1 s(i\eta(\lambda), \lambda; x, x)dx. \end{aligned}$$

As eigenfunctions of $L(i\eta(\lambda))$ and $L(-i\eta(\lambda))$ for $\lambda \in A_+$ near λ_0, we can choose u_λ and v_λ as follows: (i) when $c_1(1, \lambda_0) - e^{-\eta(\lambda_0)} \neq 0$,

$$\begin{aligned} u_\lambda(x) &:= e^{\eta(\lambda)x}[-s_1(1, \lambda)c_1(x, \lambda) + (c_1(1, \lambda) - e^{-\eta(\lambda)})s_1(x, \lambda)], \\ v_\lambda(x) &:= e^{-\eta(\lambda)x}[(c_1(1, \lambda) - e^{-\eta(\lambda)})c_1(x, \lambda) + c_2(1, \lambda)s_1(x, \lambda)]; \end{aligned}$$

(ii) when $c_1(1, \lambda_0) - e^{\eta(\lambda_0)} \neq 0$,

$$\begin{aligned} u_\lambda(x) &:= e^{\eta(\lambda)x}[(c_1(1, \lambda) - e^{\eta(\lambda)})c_1(x, \lambda) + c_2(1, \lambda)s_1(x, \lambda)], \\ v_\lambda(x) &:= e^{-\eta(\lambda)x}[-s_1(1, \lambda)c_1(x, \lambda) + (c_1(1, \lambda) - e^{\eta(\lambda)})s_1(x, \lambda)]. \end{aligned}$$

Let us treat the former case. (The latter is done similarly.) We have

$$\begin{aligned} &s_1(1, \lambda)c_2(1, \lambda) = c_1(1, \lambda)s_2(1, \lambda) - 1 \\ &= c_1(1, \lambda)(e^{\eta(\lambda)x} + e^{-\eta(\lambda)x} - c_1(1, \lambda)) - 1 = (e^{\eta(\lambda)x} - c_1(1, \lambda))(c_1(1, \lambda) - e^{-\eta(\lambda)x}). \end{aligned}$$

Thus

$$u_\lambda(x)v_\lambda(y) = -e^{\eta(\lambda)(x-y)}(c_1(1,\lambda) - e^{-\eta(\lambda)})s(i\eta(\lambda),\lambda;x,y),$$
$$(u_\lambda, v_\lambda) = (c_1(1,\lambda) - e^{-\eta(\lambda)})D'(\lambda).$$

So $(u_\lambda, v_\lambda)' = (c_1(1,\lambda) - e^{-\eta(\lambda)})D''(\lambda)$ at $\lambda = \lambda_0$. Therefore

$$\frac{u_\lambda(x)v_\lambda(y)}{(u_\lambda, \psi_{v_\lambda})} = 2\frac{u_\lambda(x)v_\lambda(y)}{(u_\lambda, v_\lambda)'} = -2\frac{e^{\eta(\lambda)(x-y)}s(i\eta(\lambda),\lambda;x,y)}{D''(\lambda)} = -r_2(x,y)$$

at $\lambda = \lambda_0$. We have thus shown the proposition. □

Finally, we give an asymptotic expansion of the Green function $G_z(x,y)$ as the spectral parameter z approaches one of edges of the spectrum of L. We show it in a direct and elementary way, although the expansion of resolvents for Schrödinger operators with periodic potentials is given by [G, Corollary 4.2]. Let $\Delta_+ := \mathbf{C}\setminus[0,\infty)$. We denote by $z^{\frac{1}{2}}$ a branch of the square root of $z \in \Delta_+$ such that $z^{\frac{1}{2}} = \sqrt{r}e^{i\theta/2}$ for $z = re^{i\theta}$, $0 < \theta < 2\pi$, $r > 0$. Note that λ is an edge of the spectrum of L if and only if $|D(\lambda)| = 2$ and $D'(\lambda) \neq 0$. If $D(\lambda) = 2$ and $D'(\lambda) \neq 0$, there exist real-valued linearly independent solutions u and ψ of $Ly = \lambda y$ such that u is a real-valued periodic function with period 1 and $\psi(x) = xu(x) + v(x)$ for some real-valued periodic function v with period 1; if $D(\lambda) = -2$ and $D'(\lambda) \neq 0$, there exist real-valued linearly independent solutions u and ψ of $Ly = \lambda y$ such that u is a real-valued semi-periodic function with semi-period 1, i.e., $u(x+1) = -u(x)$, and $\psi(x) = xu(x) + v(x)$ for some real-valued semi-periodic function v with semi-period 1 (cf. [E, p.7 and p.29]).

Theorem 5. *Assume that μ_{2n-1} is an edge of the spectrum of L. Then for any integer $m \geq -1$ one has the expansion for small $z - \mu_{2n-1} \in \Delta_+$*

$$G_z(x,y) = \sum_{j=-1}^{m} (z - \mu_{2n-1})^{\frac{j}{2}} q_j(x,y) + r_m(z;x,y),$$

where $r_m(z;x,y)$ satisfies the estimate: for any $0 \leq \theta \leq 1$

$$|r_m(z;x,y)| \leq C_m|z - \mu_{2n-1}|^{(m+\theta)/2}(|x-y|+1)^{m+1+\theta}.$$

Furthermore, $q_j(x,y)$ is of the form

$$q_j(x,y) = q_j(y,x) = \sum_{k=0}^{j+1}(x-y)^k q_{j,k}(x,y), \quad y \leq x,$$

for some $q_{j,k}(x,y) \in C(\mathbf{T}\times\mathbf{T})$. In particular,

$$q_{-1}(x,y) = \frac{i}{\sqrt{2\lambda''_{2n-1}(0)}}\frac{u(x)u(y)}{\|u\|^2},$$
$$q_0(x,y) = q_0(y,x) = \lambda''_{2n-1}(0)^{-1}(u(x)\psi(y) - \psi(x)u(y))/\|u\|^2, \quad y \leq x,$$

where $\lambda''_{2n-1}(0) > 0$, and u and ψ are real-valued linearly independent solutions of $Ly = \mu_{2n-1}y$ such that u is a periodic function with period 1 and $\psi(x) = xu(x) + v(x)$ for some periodic function v with period 1.

Remark 6. If ν_{2n-1}, ν_{2n}, or μ_{2n} is an edge of the spectrum, a similar expansion holds around it.

Proof. Since $D(\mu_{2n-1}) = 2$ and $D'(\mu_{2n-1}) < 0$, there exists a holomorphic inverse function D^{-1} of D near $D = 2$. Put $\lambda(\zeta) = D^{-1}(e^{i\zeta} + e^{-i\zeta})$ near $\zeta = 0$. Then $\lambda(\xi) = \lambda_{2n-1}(\xi) \geq \mu_{2n-1}$ for small $\xi \in \mathbf{R}$ and $\lambda'(0) = 0$. Furthermore, since $D(\lambda(\xi)) = 2\cos\xi$, we have

$$D''(\lambda(\xi))\lambda'(\xi)^2 + D'(\lambda(\xi))\lambda''(\xi) = -2\cos\xi.$$

This implies that $\lambda''(0) = -2/D'(\mu_{2n-1}) > 0$. Therefore we can choose a sufficiently small positive number R such that the set $\{\lambda(\zeta); \operatorname{Im}\zeta > 0, |\zeta| < R\}$ is a subdomain of $\mathbf{C} \setminus [\mu_{2n-1}, \infty)$. We have also that $s_1(1, \mu_{2n-1})$ and $c_2(1, \mu_{2n-1})$ are not both zero (cf. [E, p.29]). So we can choose a holomorphic eigenvector $(\alpha_1(\zeta), \alpha_2(\zeta))$ of $M(\lambda(\zeta))$ corresponding to the eigenvalue $e^{i\zeta}$ near $\zeta = 0$. Put $y_\zeta(x) := \alpha_1(\zeta)c_1(x, \lambda(\zeta)) + \alpha_2(\zeta)s_1(x, \lambda(\zeta))$. Then $u_\zeta(x) := e^{-i\zeta x}y_\zeta(x)$ is a holomorphic eigenfunction of $L(\zeta)$ corresponding to the eigenvalue $\lambda(\zeta)$ near $\zeta = 0$. Let $\mathbf{C}_+ := \{\zeta \in \mathbf{C}; \operatorname{Im}\zeta > 0\}$. For small $\zeta \in \mathbf{C}_+$, since $\overline{\lambda(\zeta)} = \lambda(\bar\zeta)$, it follows that $y_\zeta = e^{i\zeta x}u_\zeta$ and $\overline{y_{\bar\zeta}} = e^{-i\zeta x}\overline{u_{\bar\zeta}}$ are linearly independent solutions to $Ly = \lambda(\zeta)y$. Hence as in the proof of Theorem 1, since $i[y_\zeta, \overline{y_{\bar\zeta}}](0) = \lambda'(\zeta)(u_\zeta, u_{\bar\zeta})$, we have for small $\zeta \in \mathbf{C}_+$

$$G_{\lambda(\zeta)}(x,y) = G_{\lambda(\zeta)}(y,x) = y_\zeta(x)\overline{y_{\bar\zeta}(y)}/[y_\zeta, \overline{y_{\bar\zeta}}](0) = i\lambda'(\zeta)^{-1}e^{i(x-y)\zeta}p_\zeta(x,y), \quad y \leq x, \tag{13}$$

where $p_\zeta(x,y) := u_\zeta(x)\overline{u_{\bar\zeta}(y)}/(u_\zeta, u_{\bar\zeta})$ is a $C(\mathbf{T} \times \mathbf{T})$-valued holomorphic function near $\zeta = 0$. Let $y \leq x$. We write the Taylor expansion of $e^{i(x-y)\zeta}p_\zeta(x,y)$ with respect to ζ as follows:

$$e^{i(x-y)\zeta}p_\zeta(x,y) = \sum_{j=0}^{m} \tilde q_j(x,y)\zeta^j + \tilde r_m(\zeta; x, y), \tag{14}$$

where

$$\tilde q_j(x,y) = \sum_{k=0}^{j} (x-y)^k \tilde q_{j,k}(x,y) \tag{15}$$

for some $\tilde q_{j,k}(x,y) \in C(\mathbf{T} \times \mathbf{T})$, and $\tilde r_m(\zeta; x, y)$ satisfies the estimate: for any $0 \leq \theta \leq 1$

$$|\tilde r_m(\zeta; x, y)| \leq C_m |\zeta|^{m+\theta}(|x-y|+1)^{m+\theta}. \tag{16}$$

Let us show this remainder estimate. We have

$$e^{i(x-y)\zeta} = \sum_{j=0}^{m} \frac{(i(x-y)\zeta)^j}{j!} + \frac{(i(x-y)\zeta)^{m+1}}{m!}\int_0^1 (1-t)^m e^{it(x-y)\zeta}dt.$$

Thus

$$\left| e^{i(x-y)\zeta} - \sum_{j=0}^{m} \frac{(i(x-y)\zeta)^j}{j!} \right| \leq \frac{(|x-y||\zeta|)^{m+1}}{(m+1)!},$$

since $\operatorname{Re}[it(x-y)\zeta] \leq 0$. This implies that

$$|\tilde r_m(\zeta; x, y)| \leq C_m |\zeta|^{m+1}(|x-y|+1)^{m+1}.$$

On the other hand, since

$$\tilde{r}_m(\zeta;x,y)=\tilde{r}_{m-1}(\zeta;x,y)-\tilde{q}_m(x,y)\zeta^m,$$

we have

$$|\tilde{r}_m(\zeta;x,y)|\le C_m|\zeta|^m(|x-y|+1)^m.$$

Hence we get the desired estimate (16). We see that $\tilde{q}_0(x,y)=p_0(x,y)$ and $\tilde{q}_1(x,y)=i(x-y)p_0(x,y)+\partial_\zeta p_\zeta(x,y)|_{\zeta=0}$. We shall show that $\tilde{q}_1(x,y)=i(\psi(x)u(y)-u(x)\psi(y))/\|u\|^2$, where $u(x)$ and $\psi(x)=xu(x)+v(x)$ are linearly independent solutions stated in the theorem. We have

$$\begin{aligned}\partial_\zeta y_\zeta|_{\zeta=0}&=\alpha_1'(0)c_1(x,\mu_{2n-1})+\alpha_2'(0)s_1(x,\mu_{2n-1})=ixu_0+\partial_\zeta u_\zeta|_{\zeta=0},\\ \partial_\zeta\overline{y_{\bar\zeta}}|_{\zeta=0}&=\overline{\alpha_1'(0)}c_1(x,\mu_{2n-1})+\overline{\alpha_2'(0)}s_1(x,\mu_{2n-1})=-ix\overline{u_0}+\partial_\zeta\overline{u_{\bar\zeta}}|_{\zeta=0}.\end{aligned}$$

So $\partial_\zeta y_\zeta|_{\zeta=0}$ and $\partial_\zeta\overline{y_{\bar\zeta}}|_{\zeta=0}=\overline{\partial_\zeta y_\zeta}|_{\zeta=0}$ are solutions of $Ly=\mu_{2n-1}y$, and we have $u_0=cu$ and $\partial_\zeta y_\zeta|_{\zeta=0}=ic\psi+c'u$ for some $c,c'\in\mathbf{C}$. Hence

$$\partial_\zeta u_\zeta|_{\zeta=0}=icv(x)+c'u(x),\quad \partial_\zeta\overline{u_{\bar\zeta}}|_{\zeta=0}=-i\bar{c}v(x)+\bar{c}'u(x).$$

Using this we have

$$\begin{aligned}\tilde{q}_1(x,y)&=i(x-y)p_0(x,y)+\partial_\zeta p_\zeta(x,y)|_{\zeta=0}\\ &=i(x-y)p_0(x,y)+\frac{\partial_\zeta(u_\zeta(x)\overline{u_{\bar\zeta}(y)})|_{\zeta=0}}{\|u_0\|^2}-p_0(x,y)\frac{(u_\zeta,u_{\bar\zeta})'|_{\zeta=0}}{\|u_0\|^2}\\ &=i(x-y)\frac{u(x)u(y)}{\|u\|^2}+\frac{(icv(x)+c'u(x))\bar{c}u(y)+cu(x)(-i\bar{c}v(y)+\bar{c}'u(y))}{|c|^2\|u\|^2}\\ &\quad-\frac{u(x)u(y)}{\|u\|^2}\frac{2\mathrm{Re}(icv+c'u,cu)}{|c|^2\|u\|^2}\\ &=i(x-y)u(x)u(y)/\|u\|^2+i(v(x)u(y)-u(x)v(y))/\|u\|^2\\ &=i(\psi(x)u(y)-u(x)\psi(y))/\|u\|^2.\end{aligned}$$

There exists an entire function $F(z)$ such that $F(\zeta^2)=e^{i\zeta}+e^{-i\zeta}-2$; $F(z)$ is real for real z, $F(0)=0$, and $F'(0)=-1$. So there exists an inverse function F^{-1} of F near the origin. Thus for $\delta>0$ small, the map $z\in\{z\in\Delta_++\mu_{2n-1};|z-\mu_{2n-1}|<\delta\}\mapsto\zeta(z):=(F^{-1}(D(z)-2))^{\frac{1}{2}}\in\mathbf{C}_+$ is conformal from the disc with the cut to the intersection of a neighborhood of the origin and $\mathbf{C}_+$. Note that $\lambda(\zeta(z))=z$. Noting that $D(z)-2=D'(\mu_{2n-1})(z-\mu_{2n-1})+O((z-\mu_{2n-1})^2)$ and $F^{-1}(w)=-w+O(w^2)$, we have the Puiseux series

$$\zeta(z)=\sum_{j=0}^{\infty}a_j(z-\mu_{2n-1})^{j+\frac{1}{2}},\tag{17}$$

where $a_0 = \sqrt{|D'(\mu_{2n-1})|} = \sqrt{2/\lambda''_{2n-1}(0)}$. Note that $\lambda'(\zeta(z))^{-1} = \zeta'(z)$. By (13), (14) and (17),

$$\begin{aligned} G_z(x,y) &= i\zeta'(z)e^{i(x-y)\zeta(z)}p_{\zeta(z)}(x,y) \\ &= i[\sum_{j=0}^{\infty} a_j(j+\frac{1}{2})(z-\mu_{2n-1})^{j-1/2}][\sum_{j=0}^{m} \tilde{q}_j(x,y)\zeta(z)^j + \tilde{r}_m(\zeta(z);x,y)] \\ &= \sum_{j=-1}^{m}(z-\mu_{2n-1})^{j/2}q_j(x,y) + r_m(z;x,y). \end{aligned}$$

This together with (15) and (16) yields the desired expansion. □

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