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On the propagation of the homogeneous wavefront set for Schrödinger equations and on the equivalence of the homogeneous and the qsc wavefront sets

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1 Introduction

We consider the Schrödinger equation

$$\left(i\frac{\partial}{\partial t}+\frac{1}{2}\triangle-V\right)u(t,z)=0, \qquad u(0,\cdot)=u_0\in L^2,$$

and study the propagation of singularities, that is, we would like to tell by the information of u_0 where the wavefront set of $u_T=u(T,\cdot)$ disappears for $T>0$.

For the motivation we first deal with the simplest case;

$$\triangle=\triangle_0=\frac{\partial^2}{\partial z_1^2}+\cdots+\frac{\partial^2}{\partial z_n^2}, \qquad V\equiv 0.$$

Let $A(0)=a^w(z,D_z)$ be an observable, then in the Heisenberg picture it moves as

$$A(t):=e^{-\frac{1}{2}it\triangle_0}A(0)e^{\frac{1}{2}it\triangle_0}=a^w(z+tD_z,D_z),$$

where $a^w(z,D_z)$ is the Weyl quantization of a symbol $a(z,\zeta)$:

$$a^w(z,D_z)u(z)=(2\pi)^{-n}\int e^{i(z-w)\zeta}a\left(\frac{z+w}{2},\zeta\right)u(w)\,dwd\zeta.$$

Recall the characterization of the wavefront set; For $u\in\mathcal{S}'(\mathbb{R}^n)$ and $(z_0,\zeta_0)\in\mathbb{R}^n\times(\mathbb{R}^n\setminus\{0\})$ $(z_0,\zeta_0)\notin \mathrm{WF}(u)$ is equivalent to

$$\exists\varphi\in C_0^\infty\left(\mathbb{R}^{2n}\right) \text{ such that } \varphi(z_0,\zeta_0)\neq 0 \text{ and } \|\varphi^w(z,hD_z)u(z)\|_{L^2}=O\left(h^\infty\right).$$

$O\left(h^\infty\right)$ means $O\left(h^N\right)$ as $h\downarrow 0$ for any $N>0$. Then studying the wavefront set of u_T means measuring the decaying rate of $\|\varphi^w(z,hD_z)u_T\|$ as $h\downarrow 0$. Through the Heisenberg picture it means measuring the decaying rate of $\|\varphi^w(z+TD_z,hD_z)u_0\|$. Since φ is compactly supported, we have

$$z=O\left(h^{-1}\right) \text{ and } \zeta=O\left(h^{-1}\right) \text{ on } \operatorname{supp}\varphi(z+T\zeta,h\zeta).$$

Therefore it suffices to measure the decaying rate of u_0 in a $2n$-cone in the (z,ζ)-phase space. With these observations we introduce the *homogeneous wavefront set*:

Definition 1.1 (Nakamura) *Let $u \in S'(\mathbb{R}^n)$ and $(z_0, \zeta_0) \in \mathbb{R}^{2n} \setminus \{0\}$. We denote $(z_0, \zeta_0) \notin$ $\mathrm{HWF}(u)$, if there exists $\varphi \in C_0^\infty(\mathbb{R}^{2n})$ satisfying $\varphi(z_0, \zeta_0) \neq 0$ and*

$$\|\varphi^w(hz, hD_z)u(z)\|_{L^2} = O(h^\infty).$$

The homogeneous wavefront set $\mathrm{HWF}(u)$ is the complement in $\mathbb{R}^{2n} \setminus \{0\}$ of the set of such (z_0, ζ_0)'s.

A more general case is dealt with in this article. Take a scattering metric g on the half sphere S^n_+, and let Δ be defined by

$$\Delta = \sum_{i,j=1}^n \partial_{z^i} g^{ij}(z) \partial_{z^j}, \quad z \in \mathbb{R}^n,$$

Here $\mathbb{R}^n$ is identified with the interior of S^n_+ by the stereographic projection, or the radial compactification:

$$\mathrm{SP}: \ \mathbb{R}^n \to S^n_+ = \left\{w \in \mathbb{R}^{n+1}; |w| = 1, w_n \geq 0\right\}, \quad z \mapsto \left(\frac{z}{\langle z\rangle}, \frac{1}{\langle z\rangle}\right), \quad \langle z\rangle = \sqrt{1+|z|^2}.$$

Then the half sphere is considered as the Euclidean space with spherical boundary at infinity. We refer to the paper [9] by Melrose or Section 2 of this article for the definition of scattering metric. We write $L^2 = L^2(\mathbb{R}^n; dz)$ with the L^2-inner product

$$(u, v)_{L^2} = \int_n u(z)\overline{v(z)}dz.$$

We may have considered the Laplace-Beltrami operator

$$\Delta = \frac{1}{\sqrt{G}} \sum_{i,j=1}^n \partial_i g^{ij} \sqrt{G} \partial_j, \quad G = \det g$$

and $L^2 = L^2\left(\mathbb{R}^n; \sqrt{G}dz\right)$. However, if the metric is a scattering metric on $\mathbb{R}^n$, $\sqrt{G}$ expressed in the standard coordinates is bounded from above and below by positive constants. Thus, whichever is the case, the argument below would be parallel.

Assume V is a smooth potential on $\mathbb{R}^n$ with a subquadratic growth at infinity, that is, there is $\nu < 2$ such that

$$|\partial_z^\alpha V(z)| \leq C_\alpha \langle z\rangle^{\nu - |\alpha|} \quad \forall \alpha \in \mathbb{Z}^n_+.$$

Theorem 1.2 *Let $\omega_- \in \mathbb{R}^n$, $T > 0$ and $u_0 \in L^2$, and assume $(-T\omega_-, \omega_-) \notin \mathrm{HWF}(u_0)$. Then, if $\gamma(t) = (z(t), \zeta(t))$ is a free backward non-trapped classical trajectory with limiting direction ω_-, i.e. if*

$$\dot\gamma(t) = \left(\frac{\partial p}{\partial \zeta}(\gamma(t)), -\frac{\partial p}{\partial z}(\gamma(t))\right), \quad p(z, \zeta) = \frac{1}{2}\sum_{i,j=1}^n g^{ij}(z)\zeta_i\zeta_j,$$

$$\lim_{t \to -\infty} |z(t)| = \infty,$$

$$\omega_- = \lim_{t \to -\infty} \frac{\zeta(t)}{|\zeta(t)|} = -\lim_{t \to -\infty} \frac{z(t)}{|z(t)|},$$

we have

$$\mathrm{WF}(u_T) \cap \{\gamma(t); t \in \mathbb{R}\} = \varnothing$$

Note If the metric is a scattering one on $\mathbb{R}^n$, every free trajectory $\gamma(t)$ is always defined for all $t \in \mathbb{R}$. Moreover, if it is backward non-trapped, there exists $\omega_- \in S^{n-1}$ such that

$$\omega_- = \omega_-(\gamma) = \lim_{t\to-\infty} \frac{\zeta(t)}{|\zeta(t)|} = -\lim_{t\to-\infty} \frac{z(t)}{|z(t)|},$$

that is, the limiting direction exists.

Nakamura [11] proved Theorem 1.2 for asymptotically flat metric on $\mathbb{R}^n$. The following proposition is also from [11].

Proposition 1.3 *If $u \in \mathcal{S}'(\mathbb{R}^n)$ decays rapidly in a conic neighborhood of $z_0 \in \mathbb{R}^n \setminus \{0\}$, that is, if there is a conic neighborhood $\Gamma \subset \mathbb{R}^n$ of z_0 such that $\langle z\rangle^N u|_\Gamma \in L^2(\Gamma)$ for any $N > 0$, then $(z_0, \zeta_0) \notin \mathrm{HWF}(u)$ for any $\zeta_0 \in \mathbb{R}^n$.*

Then the microlocal smoothing property of Craig-Kappeler-Strauss' [1] follows.

Corollary 1.4 *Let $u_0 \in L^2$ decay rapidly in a conic neighborhood of $-\omega_-$, then for any free trajectory $\gamma(t)$ with limiting direction ω_- we have*

$$\mathrm{WF}(u_T) \cap \{\gamma(t); t \in \mathbb{R}\} = \varnothing \qquad \forall T > 0.$$

Wunsch [12] has obtained a similar results w.r.t. the notion of the *quadratic scattering (qsc) wavefront set.* The qsc wavefront set $\mathrm{WF}_{\mathrm{qsc}}(u)$ in general is defined for a tempered distribution on a scattering manifold M and is a subset of $C_{\mathrm{qsc}}M = \partial\left({}^{\mathrm{qsc}}\overline{T}^*M\right)$. In case of $M = S^n_+ \supset \mathbb{R}^n$ we have an essential identification

$$C_{\mathrm{qsc}}S^n_+ \cong \partial\left(S^n_+ \times S^n_+\right) \cong \left(\mathbb{R}^n \times S^{n-1}\right) \cup \left(S^{n-1} \times S^{n-1}\right) \cup \left(S^{n-1} \times \mathbb{R}^n\right).$$

The intersection $\mathrm{WF}_{\mathrm{qsc}}(u) \cap \left(\mathbb{R}^n \times S^{n-1}\right)$ corresponds to $\mathrm{WF}(u)$, and $\mathrm{WF}_{\mathrm{qsc}}(u) \cap \left(S^{n-1} \times \mathbb{R}^n\right)$ is regarded as a blow-up of the scattering (sc) wavefront set in its corner, where the information on the wavefront sets of u and $\mathcal{F}u$ is mixed up. For a precise definition we refer to [4, 12]. We will also make a brief sketch of the sc and qsc calculus in Section 3. The following theorem implies that $\mathrm{WF}_{\mathrm{qsc}}(u) \cap \left(S^{n-1} \times \mathbb{R}^n\right)$ is equivalent to $\mathrm{HWF}(u)$.

Theorem 1.5 *Define $\Psi : \mathbb{R}^n \setminus \{0\} \to \mathrm{GL}(n; \mathbb{R})$ by*

$$\Psi(z) = \left(\delta_{ij} + \frac{z^i z^j}{|z|^2}\right)_{ij}.$$

Then the following equality holds:

$$\begin{aligned}&\left\{(z, \Psi(z)\zeta) \in \mathbb{R}^{2n}; (z,\zeta) \in \mathrm{HWF}(u) \setminus (\{0\} \times \mathbb{R}^n)\right\} \\ &= \left\{(tz, t\zeta) \in \mathbb{R}^{2n}; (z,\zeta) \in \mathrm{WF}_{\mathrm{qsc}}(u) \cap \left(S^{n-1} \times \mathbb{R}^n\right), t > 0\right\}.\end{aligned}$$

The homogeneous wavefront set is a blow-down of the qsc wavefront set in its wavefront set part $\mathrm{WF}_{\mathrm{qsc}}(u) \cap \left(\mathbb{R}^n \times S^{n-1}\right)$.

If we note that for any $T > 0$

$$(-T\omega_-, \omega_-) \in \mathrm{HWF}(u_T) \iff \left(-\omega_-, \frac{\omega_-}{2T}\right) \in \mathrm{WF}_{\mathrm{qsc}}(u_T),$$

then one of the main results in [12] follows from Theorem 1.2 under a weaker condition on the potential on the Euclidean space. Our condition on the potential is optimally weak in the sense that, if the potential is a quadratic or superquadratic one, the microlocal smoothing property is completely different [2, 13, 14, 15].

The homogeneous wavefront set measure the decaying rate of u in a $2n$-cone in the phase space, while, considering $\|\varphi^w(z+TD_z, hD_z)u_0\|$, that of u in an n-cone transformed by the classical flow must be measured. The homogeneous and the qsc wavefront set are, indeed, a rough scale for investigationtn of $\mathrm{WF}(u_T)$.

Concerning this problem, Hassell and Wunsch [3, 4] obtained more refined results than those in [12].

Cn the other hand in [10] Nakamura independently obtained a necessary and sufficient characterization of $\mathrm{WF}(u_T)$ in terms of u_0 by measuring the decaying rate in a transformed n-cone.

The author has also found it is very easy to see that the above two results by Nakamura and Hassell-Wunsch are equivalent under an appropriate condition.

This article totally depends on [6], [7] by the author, so the proofs of Theorem 1.2 and 1.5 given in Section 2 and 4 are just sketchy. Instead Section 3 is devoted to the explanation of the sc and the qsc calculus, which was omitted in [6], [7] for brevity. This section owes much on [4, 9, 12]. In Appendix A the formulae for the coordinate transformation between the standard and the polar coordinates are gathered for convenience.

2 Proof of Theorem 1.2

2.1 Scattering metric and free trajectories

Let M be a manifold with boundary ∂M, and g a Riemannian metric on the interior M°. If $x \in C^\infty(M)$ (it is C^∞ also on ∂M) satisfies

$$\partial M = \{z \in M; x(z) = 0\}, \quad dx \neq 0 \text{ on } \partial M,$$

x is called a boundary defining function on M. For example $\langle z\rangle^{-1}$, $z \in \mathbb{R}^n$ gives a boundary defining function on S^n_+ under the identification $\mathbb{R}^n = \mathrm{SP}(\mathbb{R}^n) = \left(S^n_+\right)^\circ$. If x is a boundary defining function, there are local coordinates of the form (x, y) such that y gives local coordinates on ∂M when $x = 0$. We say g is a scattering metric on M, if g is of the form

$$g = \frac{dx^2}{x^4} + \frac{h(x, y, dx, dy)}{x^2}$$

near the boundary. Here h is a 2-cotensor on M and, when restricted, or pulled back to the boundary, defines a Riemannian metric on ∂M.

Consider a free trajectory $\gamma(t) = (z(t), \zeta(t))$ w.r.t. a scattering metric on M, that is, $\gamma(t)$ is a solution to the Hamilton equation

$$\dot\gamma(t) = \left(\frac{\partial p}{\partial \zeta}(\gamma(t)), -\frac{\partial p}{\partial z}(\gamma(t))\right), \quad p(z, \zeta) = \frac{1}{2}\sum_{i,j=1}^{n} g^{ij}(z)\zeta_i\zeta_j.$$

Free trajectories on a scattering manifold are always defined for all $t \in \mathbb{R}$. We say γ is backward non-trapped if $\lim_{t\to-\infty} x(z(t)) = 0$.

Proposition 2.1 *Let $\gamma(t) = (z(t), \zeta(t))$ be a free backward non-trapped trajectory, and (x, y) local coordinates near a point on ∂M. Then we have as $t \to -\infty$*

$$x(t) = \left(2p_0t^2 + O\left(t\log t\right)\right)^{-\frac{1}{2}}, \quad \xi(t; z_0, \zeta_0) = -\left(2p_0\right)^{\frac{3}{2}} t^2 + O(t\log t), \quad p_0 \equiv p(\gamma(t)),$$

where (x, y, ξ, η) *is the coordinates of* T^*M *corresponding to* (x, y). *These estimates are independent of the choice of* y. *Moreover*

$$z_- := \lim_{t \to -\infty} z(t) \in \partial M$$

exists, and, with an appropriate choice of coordinates y,

$$y_- := \lim_{t \to -\infty} y(t), \quad \eta_- := \lim_{t \to -\infty} \eta(t)$$

exist.

We now apply Proposition 2.1 to a backward non-trapped trajectory γ on the compactified Euclidean space $S^n_+ \supset \mathbb{R}^n$ with a scattering metric. Take the appropriate coordinates (x, y) as in Proposition 2.1. By exchanging the standard coordinate axes if necessary, we may assume $(x, y) = (x, y_{(+n)})$, which is defined in Appendix A. Then from the existence of $(x_-, y_-, \xi_-, \eta_-)$ and the formulae in Appendix A it follows that

$$\omega_- := -\lim_{t \to -\infty} \frac{z(t)}{|z(t)|} = \lim_{t \to -\infty} \frac{\zeta(t)}{|\zeta(t)|}, \quad \zeta_- := \lim_{t \to -\infty} \zeta(t) = \sqrt{2p_0}\omega_-$$

exist. Thus $(-T\omega_-, \omega_-)$ in Theorem 1.2 can be replaced by $(-T\zeta_-, \zeta_-)$ thanks to the (z, ζ)-homogeneity of the homogeneous wavefront set.

2.2 A sketch of proof of Theorem 1.2

It suffices to show that

$$(z_0, \zeta_0) := \gamma(0) \notin \mathrm{WF}(u_T).$$

Let us given some operator $F(t, h) = \varphi^w(z, D_z; t, h)$ with parameters t and h, then

$$\langle F(0, h)u_T, u_T \rangle = \langle F(-T, h)u_0, u_0 \rangle + \int_{-T}^{0} \langle \delta F(t, h)u_{t+T}, u_{t+T} \rangle \, dt, \tag{2.1}$$

$$\delta F(t, h) = \frac{\partial}{\partial t} F(t, h) + i \left[H, F(t, h) \right].$$

We require the following support properties for $\varphi(z, \zeta; t, h)$; As $h \to 0$, $\operatorname{supp} \varphi(\cdot, \cdot; 0, h)$ moves near $(z_0, h^{-1}\zeta_0)$; and $\operatorname{supp} \varphi(\cdot, \cdot; -T, h)$ moves near $(-h^{-1}T\omega_-, h^{-1}\omega_-)$. Then, in the l.h.s. of (2.1) appears the definition of the wavefront set, and the first term of the r.h.s. gets to be $O(h^\infty)$ by definition of the homogeneous wavefront set. Moreover, if $\delta F \leq O(h^\infty)$, which roughly corresponds to

$$\frac{D}{Dt}\varphi := \frac{\partial \varphi}{\partial t} + \frac{\partial p}{\partial \zeta}\frac{\partial \varphi}{\partial z} - \frac{\partial p}{\partial z}\frac{\partial \varphi}{\partial \zeta} \leq 0, \quad \text{(the *Lagrange derivative* of } \varphi \text{ is non-positive)}$$

then the second term of the r.h.s. of (2.1) is also $O(h^\infty)$. So we have only to construct φ with the above properties. Actually the non-positivity of a symbol results only in an $O(h)$ bound from above in L^2, so our construction needs an asymptotic method, following Nakamura's argument in [11].

Take small $\delta > 0$, large $T_1 > 0$, large $C > 0$, and $\delta_0 \in (0, \frac{\delta}{4})$. Let $\chi \in C^\infty([0, +\infty))$ be such that

$$\chi(r) = \begin{cases} 1, & \text{if } r < \frac{1}{2}, \\ 0, & \text{if } r > 1, \end{cases} \quad \text{and} \quad \frac{d}{dr}\chi(r) \leq 0 \quad \forall r \geq 0,$$

and define $\psi_{-1} : (-\infty, -T_1+1] \times T^*\mathbb{R}^n \to \mathbb{R}$ by

$$\psi_{-1}(t,z,\zeta) = \chi\left(\frac{|x^{-1}-x(t)^{-1}|}{4\delta_0|t|}\right)\chi\left(\frac{|y-y(t)|}{\delta_0-C|t|^{-\kappa}}\right)\chi\left(\frac{|x^2\xi-x(t)^2\xi(t)|}{\delta_0-C|t|^{-\lambda}}\right)\chi\left(\frac{|\eta-\eta(t)|}{\delta_0|t|^{\mu}}\right).$$

The constants κ, λ, μ are supposed to satisfy

$$0<\kappa<1-\mu, \quad 0<\lambda<2-2\mu, \quad \nu-1\le\mu<1.$$

We may assume $\frac{3}{2}\le\nu<2$. Note that, if we put

$$(r,y) = (x^{-1}, y_{(+n)}) = \left(|z|, \frac{z_1}{|z|}, \dots, \frac{z_n}{|z|}\right) \quad \text{(the polar coordinates, cf. Appendix A)}$$

and (r,y,ρ,η) are the corresponding coordinates of $T^*\mathbb{R}^n$, then $x^{-1}=r$ and $x^2\xi=\rho$. Thus the support of ψ_{-1} is designed to move along the trajectory $\gamma(t)$ as t goes to $-\infty$. To define ψ_{-1} for all $t\le 0$ we modify ψ_{-1} for small $|t|$ using the Hamilton flow; Consider the solution ψ_0 to the transport equation

$$\frac{D}{Dt}\psi_0(t,z,\zeta) = \alpha(t)\frac{D}{Dt}\psi_{-1}(t,z,\zeta), \quad \psi_0(-T_1,z,\zeta)=\psi_{-1}(-T_1,z,\zeta),$$

where $\alpha\in C^\infty((-\infty,0])$ satisfies

$$\alpha(t) = \begin{cases} 1, & \text{if } t\le -T_1, \\ 0, & \text{if } t\ge -T_1+1. \end{cases}$$

Lemma 2.2 *ψ_0 satisfies the following:*

1.

$$\psi_0(t,z,\zeta)\ge 0 \quad \textit{for all} \quad (t,z,\zeta)\in\mathbb{R}_-\times T^*\mathbb{R}^n, \qquad \psi_0(t,\gamma(t))=1 \quad \textit{for all} \quad t\le 0.$$

2. If one takes sufficiently small $\delta>0$ and large $C>0$ in the construction of ψ_0, then

$$\frac{D}{Dt}\psi_0(t,z,\zeta)\le 0 \quad \textit{for all} \quad (t,z,\zeta)\in\mathbb{R}_-\times T^*\mathbb{R}^n$$

holds.

3. $\psi_0(t,z,\zeta)$ satisfies the estimates

$$\left|\partial_z^\alpha\partial_\zeta^\beta\partial_t^n\psi_0(t,z,\zeta)\right| \le C_{\alpha\beta n}\langle t\rangle^{-n-\mu|\alpha|+(2-2\mu)|\beta|},$$

that is, $\partial_t^n\psi_0\in S_-\left(\langle t\rangle^{-n}, \langle t\rangle^{-2\mu}dz^2+\langle t\rangle^{2-2\mu}d\zeta^2\right)$.

For the definition of symbol classes we refer to [5]. The subscription $\mathbb{R}_-$ means the set of parameter t, where the uniformity w.r.t. parameter is always supposed in the estimates of symbols. *Proof. 1.* Obvious from the definition.
2., 3. We may use ψ_{-1} instead of ψ_0. Then, compute the differentiations directly. □

Put

$$F_0(t,h) = \tilde{\psi}_0^w(z,D_z;t,h)\circ\tilde{\psi}_0^w(z,D_z;t,h), \quad \tilde{\psi}_0(z,\zeta;t,h) = \psi_0\left(h^{-1}t,z,h\zeta\right),$$

and restrict the parameter $t\in\mathbb{R}_-$ to the interval $[-T,0]$. Then

$$F_0(t,h) = \varphi_0^w(z,D_z;t,h), \quad \exists\varphi_0\in S_{[-T,0]}(1,\tilde{g}_1),$$

where $\tilde{g}_1 = \langle h^{-1}t\rangle^{-2\mu}dz^2 + h^2\langle h^{-1}t\rangle^{2-2\mu}d\zeta^2$.

Lemma 2.3 *There exists* $r_0 \in S_{[-T,0]}\left(\langle h^{-1}t\rangle^{\nu-\mu-1}, \tilde{g}_1\right)$ *such that*

$$\frac{\partial}{\partial t}F_0(t,h) + i[H, F_0(t,h)] \le r_0^w(z, D_z; t, h),$$

and that r_0 *is supported in* $\operatorname{supp}\tilde{\psi}_0$ *modulo* $S_{[-T,0]}\left(h^\infty, dz^2 + d\zeta^2\right)$.

Proof. The symbol of $\frac{\partial}{\partial t}F_0(t,h) + i[H, F_0(t,h)]$ is given by

$$2\tilde{\psi}_0 \frac{D}{Dt}\tilde{\psi}_0 + r, \quad \exists r \in S_{[-t_0,0]}\left(\langle h^{-1}t\rangle^{\nu-\mu-1}, \tilde{g}_1\right).$$

Then apply the sharp Gårding inequality to the principal part $2\tilde{\psi}_0 \frac{D}{Dt}\tilde{\psi}_0 \le 0$. □

Take an increasing sequence

$$0 < \delta_0 < \delta_1 < \delta_2 < \cdots < \frac{\delta}{4}.$$

Using C, T_1, and these δ_j construct ψ_j similarly to ψ_0, that is, consider

$$\psi_{-1}(t,z,\zeta) = \chi\left(\frac{|x^{-1} - x(t)^{-1}|}{4\delta_j|t|}\right)\chi\left(\frac{|y - y(t)|}{\delta_j - C|t|^{-\kappa}}\right)\chi\left(\frac{|x^2\xi - x(t)^2\xi(t)|}{\delta_j - C|t|^{-\lambda}}\right)\chi\left(\frac{|\eta - \eta(t)|}{\delta_j|t|^\mu}\right),$$

$$\frac{D}{Dt}\psi_j(t,z,\zeta) = \alpha(t)\frac{D}{Dt}\psi_{-1}(t,z,\zeta), \quad \psi_j(-T_1,z,\zeta) = \psi_{-1}(-T_1,z,\zeta).$$

We put

$$\tilde{\psi}_j(z,\zeta;t,h) = \psi_j\left(h^{-1}t, z, h\zeta\right).$$

Since $\tilde{\psi}_1$ is bounded from below by a positive constant on $\operatorname{supp}\tilde{\psi}_0$, there is $C_1 > 0$ such that

$$r_0(z,\zeta;t,h) \le C_1\tilde{\psi}_1(z,\zeta;t,h) \mod S_{[-T,0]}\left(h^\infty, dz^2 + d\zeta^2\right).$$

Put

$$F_1(t,h) = \varphi_1^w(z, D_z; t, h), \quad \varphi_1 = C_1|t|\tilde{\psi}_1 \in S_{[-T,0]}\left(|t|, \tilde{g}_1\right).$$

Then similarly to the proof of Lemma 2.3 there is $r_1 \in S_{[-T,0]}\left(h^{\mu+1-\nu}, \tilde{g}_1\right)$ that is supported in $\operatorname{supp}\tilde{\psi}_1$ modulo $S_{[-T,0]}\left(h^\infty, dz^2 + d\zeta^2\right)$ and satisfies

$$\frac{\partial}{\partial t}F_1(t,h) + i\left[H, F_1(t,h)\right] \le r_1^w(z, D_z; t, h) - r_0^w(z, D_z; t, h).$$

Thus

$$\frac{\partial}{\partial t}(F_0(t,h) + F_1(t,h)) + i\left[H, F_0(t,h) + F_1(t,h)\right] \le r_1^w(z, D_z; t, h).$$

We repeat this procedure to get $F_j(t,h) = \varphi_j^w(z, D_z; t, h)$ for $j = 1, 2, \ldots$. Suppose $\varphi_1, \ldots, \varphi_k$ are given such that

$$\frac{\partial}{\partial t}\sum_{j=0}^{k} F_j(t,h) + i\left[H, \sum_{j=0}^{k} F_j(t,h)\right] \le r_k^w(z, D_z; t, h), \tag{2.2}$$

where $r_k \in S_{[-T,0]}\left(h^{k(\mu+1-\nu)}, \tilde{g}_1\right)$ is supported in $\operatorname{supp}\tilde{\psi}_k$ modulo $S_{[-T,0]}\left(h^{\infty}, dz^2+d\zeta^2\right)$. Then one finds $C_{k+1}>0$ such that

$$r_k(z,\zeta;t,h) \leq C_{k+1}h^{k(\mu+1-\nu)}\tilde{\psi}_{k+1}\left(z,\zeta;t,h\right) \qquad \bmod S_{[-T,0]}\left(h^{\infty}, dz^2+d\zeta^2\right).$$

Put

$$F_{k+1}(t,h)=\varphi^w_{k+1}(z,D_z;t,h), \quad \varphi_{k+1}=C_{k+1}h^{k(\mu+1-\nu)}|t|\tilde{\psi}_{k+1}\in S_{[-T,0]}\left(h^{k(\mu+1-\nu)}|t|, \tilde{g}_1\right).$$

There exists $r_{k+1}\in S_{[-T,0]}\left(h^{(k+1)(\mu+1-\nu)}, \tilde{g}_1\right)$ with support contained in $\operatorname{supp}\tilde{\psi}_{k+1}$ modulo $S_{[-T,0]}\left(h^{\infty}, dz^2+\zeta^2\right)$ satisfying

$$\frac{\partial}{\partial t}F_{k+1}+i\left[H,F_{k+1}\right] \leq r^w_{k+1}(z,D_z;t,h)-r^w_k(z,D_z;t,h),$$

so that

$$\frac{\partial}{\partial t}\sum_{j=0}^{k+1}F_j(t,h)+i\left[H,\sum_{j=0}^{k+1}F_j(t,h)\right] \leq r^w_{k+1}(z,D_z;t,h).$$

φ_{k+1} is constructed.

Lemma 2.4 *There exists an operator $F(t,h)=\varphi\left(z,D_z;t,h\right)$, $\varphi\in S_{[-T,0]}\left(1,dz^2+d\zeta^2\right)$ such that*

1. *$F(t,h)\in\mathcal{L}\left(L^2\right)$ is differentiable in $t\in[-T,0]$ and*

$$F(0,h)=F_0(0,h)=\psi^w_0(0,z,hD_z)^2. \tag{2.3}$$

2. *For any $\varepsilon>0$, choose small $\delta>0$, then the support of $\varphi(z,\zeta;-T,h)$ is contained in*

$$\left\{(z,\zeta)\in T^*\mathbb{R}^n; \left|z+\zeta_-h^{-1}T\right|<\varepsilon h^{-1}T,\ \left|\zeta-h^{-1}\zeta_-\right|<\varepsilon h^{-1}\right\},$$

modulo $S\left(h^{\infty},dz^2+d\zeta^2\right)$.

3. *The Heisenberg derivative of $F(t,h)$ satisfies*

$$\delta F(t,h):=\frac{\partial}{\partial t}F(t,h)+i\left[H,F(t,h)\right]\leq R(t),$$

where $R(t)$ is an L^2-bounded operator with $\sup_{-T\leq t\leq 0}\|R(t)\|=O(h^{\infty})$.

Proof. The asymptotic sum $\varphi\sim\sum_{j=0}^{\infty}\varphi_j$ satisfies the required properties. □

Then we have

$$\begin{aligned}\langle F(0,h)u_T,u_T\rangle_{L^2} &= \langle F(-T,h)u_0,u_0\rangle_{L^2}+\int_{-T}^{0}\langle\delta F(t,h)u_t,u_t\rangle_{L^2}\,dt\\ &\leq \langle F(-T,h)u_0,u_0\rangle_{L^2}+T\sup_{-T\leq t\leq 0}\|R(t,h)\|.\end{aligned}$$

The r.h.s. is $O(h^{\infty})$ by the assumption and Lemma 2.4. Thus Theorem 1.2 is proved.

3 The scattering and the quadratic scattering calculus

Before going to the proof of Theorem 1.5, we give an introduction to the sc and the qsc calculus. This section depends much on [4, 9, 12].

3.1 The scattering calculus

Let M be a manifold with boundary, and x a boundary defining function. We put

$$\mathcal{V}_{\mathrm{b}}(M) = \{v \in \mathfrak{X}(M); v \text{ is tangent to } \partial M\}, \quad \mathcal{V}_{\mathrm{sc}}(M) = x\mathcal{V}_{\mathrm{b}}$$

If (x, y) are local coordinates of M near ∂M, $x\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ span $\mathcal{V}_{\mathrm{b}}(M)$ near ∂M, and $x^2\frac{\partial}{\partial x}$ and $x\frac{\partial}{\partial y}$ span $\mathcal{V}_{\mathrm{sc}}(M)$ near ∂M.

Lemma 3.1 *Let $M = S^n_+$ be the compactified Euclidean space. Then $\mathcal{V}_{\mathrm{b}}\left(S^n_+\right)$ is spanned by linear vector fields $z^i\frac{\partial}{\partial z^j}$, $z \in \mathbb{R}^n$, $i, j = 1, \ldots, n$ over $C^\infty\left(S^n_+\right)$, and $\mathcal{V}_{\mathrm{sc}}\left(S^n_+\right)$ is spanned by constant vectors $\frac{\partial}{\partial z^i}$, $i = 1, \ldots, n$ over $C^\infty\left(S^n_+\right)$.*

Here we note that the functions in $C^\infty\left(S^n_+\right)$ are required to be smooth also on the boundary. This implies, for example, that derivatives of any function in this class are bounded, which is not the case for functions in $C^\infty\left(\mathbb{R}^n\right)$.

Let ${}^{\mathrm{sc}}TM$ be a vector bundle over M whose sections form $\mathcal{V}_{\mathrm{sc}}(M)$;

$$\mathcal{V}_{\mathrm{sc}}(M) = \Gamma\left(M; {}^{\mathrm{sc}}TM\right).$$

Then, of course, $x^2\frac{\partial}{\partial x}$ and $x\frac{\partial}{\partial y}$ make a local frame near ∂M. Note that there exist fibers of ${}^{\mathrm{sc}}TM$ also on ∂M. (This frame seems to vanish on ∂M thanks to the coefficient x. They indeed vanish as vector fields, but they do not as sections of ${}^{\mathrm{sc}}TM$. We just use them as a notation.) We make some explanations for why we consider such a vector bundle. We would like to treat the boundary ∂M as infinities of an open manifold M°, the interior of M. Since M° is an open manifold, there are infinitely many ways of taking frames near the boundary; Some frames may grow *longer* as they approach the boundary, and others may *shrink* to vanish on the boundary. Here we standardize the growing rate of frames of the vector fields by attaching ∂M to M° and allowing only the vector fields that is tangent of degree 2 to ∂M to be a frame. Since M consists of local charts which are diffeomorphic to open sets in S^n_+, $\mathcal{V}_{\mathrm{sc}}(M)$ is considered as the vector fields on M° of bounded length by Lemma 3.1. Then only the vector fields that approach constant vector fields are allowed to be frames of ${}^{\mathrm{sc}}TM$. Similarly $\mathcal{V}_{\mathrm{b}}(M)$ is considered as vector fields on M of linear growth at infinity.

Let ${}^{\mathrm{sc}}T^*M$ be the dual bundle of ${}^{\mathrm{sc}}TM$. We can take $\frac{dx}{x^2}$ and $\frac{dy}{x}$ as a frame near the boundary. Note that then a scattering metric is a Riemannian metric on the vector bundle ${}^{\mathrm{sc}}TM$ the coefficient of whose cross terms $\frac{dx}{x^2} \otimes \frac{dy}{x}$ and $\frac{dy}{x} \otimes \frac{dx}{x^2}$ vanish on ∂M. We compactify each fiber of ${}^{\mathrm{sc}}T^*M$ by SP and write it ${}^{\mathrm{sc}}\overline{T}^*M$. If $M = S^n_+$, it is the same as $S^n_+ \times S^n_+$ which is obtained by compactifying the each component of $T^*\mathbb{R}^n \cong \mathbb{R}^{2n}_{(z,\zeta)}$. This isomorphism is essential, since a frame of ${}^{\mathrm{sc}}TS^n_+$ near the boundary must be written by a linear combination of constant vector field on $\mathbb{R}^n$ over bounded functions. If we compactify the fibers of the tangent bundle $T^*S^n_+$, then it is also isomorphic to $S^n_+ \times S^n_+$. However, we can say it is not essential in the sense that a frame of $T^*S^n_+$ near the boundary corresponds to growing vector fields on $\mathbb{R}^n$ at quadratic rate near infinity; The proportion of compactified spheres of ${}^{\mathrm{sc}}T^*S^n_+$ and $T^*S^n_+$ are different as approaching ∂M.

We now want to define an appropriate symbol class on M, which is a natural extension of the class $S\left(\langle z\rangle^{-l}\langle\zeta\rangle^{m}, \langle z\rangle^{-2} dz^2 + \langle\zeta\rangle^{-2} d\zeta^2\right)$ on the Euclidean space. Let

$$a \in S\left(\langle z\rangle^{-l}\langle\zeta\rangle^{m}, \langle z\rangle^{-2} dz^2 + \langle\zeta\rangle^{-2} d\zeta^2\right), \tag{3.1}$$

that is, $a \in C^\infty\left(\mathbb{R}^{2n}\right)$ satisfies

$$\left|\partial_z^\alpha \partial_\zeta^\beta a(z,\zeta)\right| \le C_{\alpha\beta}\langle z\rangle^{-l-|\alpha|}\langle\zeta\rangle^{m+|\beta|}.$$

Put $\rho_N = \langle z\rangle^{-1}$, $\rho_\sigma = \langle\zeta\rangle^{-1}$, which are boundary defining functions of two faces of ${}^{\mathrm{sc}}\overline{T}^* S^n_+$. (${}^{\mathrm{sc}}\overline{T}^* S^n_+$ is actually a manifold with corners. We define boundary defining functions on such a manifold in a natural way. Other notions are also extended naturally.) Then (3.1) is equivalent to $a \in \mathcal{A}^{m,l}$, where

$$\begin{aligned}&\mathcal{A}^{m,l}\left(S^n_+ \times S^n_+\right)\\ &= \left\{u \in \rho_N^l \rho_\sigma^{-m} L^\infty\left(S^n_+ \times S^n_+\right); \mathrm{Diff}_{\mathrm{b}}^*\left(S^n_+ \times S^n_+\right) u \subset \rho_N^l \rho_\sigma^{-m} L^\infty\left(S^n_+ \times S^n_+\right)\right\}.\end{aligned}$$

$\mathrm{Diff}_{\mathrm{b}}^*\left(S^n_+ \times S^n_+\right)$ is the set of all the differential operators generated by $\mathcal{V}_{\mathrm{b}}\left(S^n_+ \times S^n_+\right)$ over $C^\infty\left(S^n_+\right)$. The smoothness of functions in $\mathcal{A}^{m,l}\left(S^n_+ \times S^n_+\right)$ on $\mathbb{R}^n \times \mathbb{R}^n = \left(S^n_+ \times S^n_+\right)^\circ$ follows from the Sobolev embedding theorem. Recalling that b means linear growth at infinity, it is natural that the two sets above coincide. We use this characterization for the required symbol class, that is, define

$$\mathcal{A}^{m,l}\left({}^{\mathrm{sc}}\overline{T}^* M\right) = \left\{u \in \rho_N^l \rho_\sigma^{-m} L^\infty\left({}^{\mathrm{sc}}\overline{T}^* M\right); \mathrm{Diff}_{\mathrm{b}}^*\left({}^{\mathrm{sc}}\overline{T}^* M\right) u \subset \rho_N^l \rho_\sigma^{-m} L^\infty\left({}^{\mathrm{sc}}\overline{T}^* M\right)\right\},$$

where ρ_N and ρ_σ are boundary defining functions of ${}^{\mathrm{sc}}\overline{T}^* M$ for two faces respectively. Since M is defined by coordinate patches that are diffeomorphic to open sets of S^n_+, we can define the quantization of symbols in $\mathcal{A}^{m,l}\left({}^{\mathrm{sc}}\overline{T}^* M\right)$ by patching the pseudodifferential operators on each chart using a partition of unity, as in the case of open manifolds. The reader might feel anxiety for the treatment near the boundary, however, it is well-defined, because our operators are first supposed to act on the Schwartz class $\dot{C}^\infty(M) = \cap_{N=0}^\infty x^N C^\infty(M)$. Using a partition of unity, the Schwartz function on M are mapped to Schwartz functions in each chart, and on each chart the Schwartz functions are mapped into themselves by locally defined pseudodifferential operators. Once the pseudodifferential operators are defined on the Schwartz class, they can be extended to on the tempered distributions $C^{-\infty}(M) = \left(\dot{C}^\infty(M)\right)'$ by the duality argument. Of course, the quantization is not unique.

The class

$$S^{m,l}\left({}^{\mathrm{sc}}\overline{T}^* M\right) = \rho_N^l \rho_\sigma^{-m} C^\infty\left({}^{\mathrm{sc}}\overline{T}^* M\right) \subset \mathcal{A}^{m,l}\left({}^{\mathrm{sc}}\overline{T}^* M\right)$$

corresponds to the classical symbols, since the asymptotic expansion corresponds to the Taylor expansion around the boundary by Borel's lemma. The inclusion relation above is due to the good behavior of functions that are smooth also on the boundary. Put

$$\Psi^{m,l}_{\mathrm{scc}}(M) = \mathrm{Op}\,\mathcal{A}^{m,l}\left({}^{\mathrm{sc}}\overline{T}^* M\right), \quad \Psi^{m,l}_{\mathrm{sc}}(M) = \mathrm{Op}\,S^{m,l}\left({}^{\mathrm{sc}}\overline{T}^* M\right).$$

We denote

$$C_{\mathrm{sc}}M = \partial\left({}^{\mathrm{sc}}\overline{T}^*M\right), \quad \mathcal{A}^{\{m,l\}}\left(C_{\mathrm{sc}}M\right) = \mathcal{A}^{m,l}\left({}^{\mathrm{sc}}\overline{T}^*M\right) \Big/ \mathcal{A}^{m-1,l+1}\left({}^{\mathrm{sc}}\overline{T}^*M\right).$$

Since $\mathcal{A}^{\{m,l\}}\left(C_{\mathrm{sc}}M\right)$ forms a sheaf over $C_{\mathrm{sc}}M$, the notation is compatible. (The behavior of symbols in the interior of ${}^{\mathrm{sc}}\overline{T}^*M$ is irrelevant.) Let $a \in \mathcal{A}^{m,l}\left({}^{\mathrm{sc}}\overline{T}^*M\right)$ and its quantization be $A \in \Psi^{m,l}_{\mathrm{scc}}(M)$. The equivalence class in $\mathcal{A}^{\{m,l\}}\left(C_{\mathrm{sc}}M\right)$ of a is determined uniquely by A regardless of the quantization. Therefore define $j_{\mathrm{sc},m,l} : \Psi^{m,l}_{\mathrm{scc}}(M) \to \mathcal{A}^{\{m,l\}}\left(C_{\mathrm{sc}}M\right)$ by $A \mapsto a$. Then we clearly have an exact sequence for sc category:

$$0 \longrightarrow \Psi^{m-1,l+1}_{\mathrm{sc}}(M) \longrightarrow \Psi^{m,l}_{\mathrm{sc}}(M) \longrightarrow \mathcal{A}^{\{m,l\}}\left(C_{\mathrm{sc}}\right) \xrightarrow{j_{\mathrm{sc},m,l}} 0.$$

Put for $A \in \Psi^{m,l}_{\mathrm{sc}}(M)$

$$\mathrm{ell}^{m,l}_{\mathrm{sc}}(A) = \left\{p \in C_{\mathrm{sc}}M; j_{\mathrm{sc},m,l}(A)(p) \neq 0\right\}, \quad \Sigma^{m,l}_{\mathrm{sc}}(A) = C_{\mathrm{sc}}M \setminus \mathrm{ell}^{m,l}_{\mathrm{sc}}(A),$$

and for $u \in C^{-\infty}(M) = \left(\dot{C}^{\infty}(M)\right)'$

$$\mathrm{WF}_{\mathrm{sc}}(u) = \bigcap\left\{\Sigma^{m,l}_{\mathrm{sc}}(A); A \in \Psi^{m,l}_{\mathrm{sc}}(M),\ Au \in \dot{C}^{\infty}(M)\right\}.$$

We are using the classical operators $\Psi^{m,l}_{\mathrm{sc}}(M)$, while $j_{\mathrm{sc},m,l}$ is defined on $\Psi^{m,l}_{\mathrm{scc}}(M)$. This is because we can not use $\{p \in C_{\mathrm{sc}}M; j_{\mathrm{sc},m,l}(A)(p) \neq 0\}$ for the definition of $\mathrm{ell}^{m,l}_{\mathrm{sc}}(A)$, since there might be, for example, a logarithmic growth near the boundary, or at infinity.

It should be noted that, if $M = S^n_+$, then

$$C_{\mathrm{sc}}S^n_+ = \partial\left(S^n_+ \times S^n_+\right) = \left(\mathbb{R}^n \times S^{n-1}\right) \cup \left(S^{n-1} \times \mathbb{R}^n\right) \cup \left(S^{n-1} \times S^{n-1}\right),$$

and, since $\mathcal{F}a^w\left(z, D_z\right)\mathcal{F}^{-1} = a^w\left(-D_\zeta, \zeta\right)$, we have a correspondence

$$\mathrm{WF}_{\mathrm{sc}}(u) \cap \mathbb{R}^n \times S^{n-1} \longleftrightarrow \mathrm{WF}(u), \quad \mathrm{WF}_{\mathrm{sc}}(u) \cap S^{n-1} \times \mathbb{R}^n \longleftrightarrow \mathrm{WF}\left(\mathcal{F}u\right).$$

Observing this, as long as we restrict ourselves to the case $M = S^n_+$, we could have defined, for example, for $A = \operatorname{Op} a$, $a \in \mathcal{A}^{0,0}\left(S^n_+ \times S^n_+\right) = S\left(1, \langle z\rangle^{-2}dz^2 + \langle\zeta\rangle^{-2}d\zeta^2\right)$

$$\begin{aligned}\Sigma^{m,l}_{\mathrm{sc}}(A) = &\left\{(z,\zeta) \in S^{n-1} \times \mathbb{R}^n; \liminf_{t\to+\infty} |a(tz,\zeta)| = 0\right\} \\ &\bigsqcup\left\{(z,\zeta) \in \mathbb{R}^n \times S^{n-1}; \liminf_{t\to+\infty} |a(z,t\zeta)| = 0\right\}.\end{aligned} \qquad \text{(disjoint union)}$$

However, then the ellipticity on the corner of $C_{\mathrm{sc}}S^n_+$ can not be defined naturally, since there are many ways to approach the corner, although the corner part of $\mathrm{WF}_{\mathrm{sc}}$ contains less information than the qsc or the homogeneous wavefront set do.

3.2 The quadratic scattering calculus

Let M_q be a copy of M as a topological manifold with boundary, and $\Theta = \mathrm{id} : M \to M_q$ the identity as sets. We introduce new coordinates on M_q such that x^2, the square of a boundary defining function on M gives a boundary defining function on M_q. If $z \in M_q = M$ is near the boundary, we can take a coordinate neighborhood U in M in which z is expressed by $(x(z), y(z))$. Put $U_q = U$ and we give coordinates in U_q by

$$(q(z), y(z)) = (x(z)^2, y(z)).$$

For a point in M_q far from the boundary we use the same coordinates as in M. This gives M_q a C^∞ structure, and, indeed, $q = x^2$ is a boundary defining function on M_q. Θ is C^∞ and bijective. $\Theta|_{M^\circ}$ is a diffeomorphism onto M_q°. Θ^{-1} is not smooth on ∂M_q, but

$$\Theta^* : \dot{C}^\infty(M_q) \to \dot{C}^\infty(M), \quad \text{or} \quad C^{-\infty}(M_q) \to C^{-\infty}(M)$$

are isomorphisms. Using this isomorphism, we define

$$\Psi^{m,l}_{\mathrm{qsc}}(M) = \Theta^* \circ \Psi^{m,(l-m)/2}_{scc}(M_q) \circ (\Theta^*)^{-1}.$$

The seemingly eccentric indexing is intended to indicate the index of the scattering calculus, for example,

$$\begin{aligned} \Theta^* \circ q^2 \frac{\partial}{\partial q} \circ (\Theta^*)^{-1} &= \frac{1}{2} x \cdot x^2 \frac{\partial}{\partial x} && (m = 1,\ l = 1), \\ \Theta^* \circ q^k \circ (\Theta^*)^{-1} &= x^{2k} && (m = 0,\ l = 2k), \\ \Theta^* \circ q \frac{\partial}{\partial y} \circ (\Theta^*)^{-1} &= x \cdot x \frac{\partial}{\partial x} && (m = 1,\ l = 1). \end{aligned}$$

For $A \in \Psi^{m,l}_{\mathrm{qsc}}(M)$ there is $P \in \Psi^{m,(l-m)/2}_{\mathrm{scc}}(M_q)$ with $A = \Theta^* \circ P \circ (\Theta^*)^{-1}$. We define

$$\begin{aligned} j_{\mathrm{qsc},m,l}(A) &= (\Theta^{**})^{-1} \left[j_{\mathrm{sc},m,(l-m)/2}(P) \right] \\ &\in (\Theta^{**})^{-1} \left[\mathcal{A}^{m,(l-m)/2} \left({}^{\mathrm{sc}}\overline{T}^* M_q \right) \Big/ \mathcal{A}^{m-1,(l-m)/2+1} \left({}^{\mathrm{sc}}\overline{T}^* M_q \right) \right] \\ &= \mathcal{A}^{m,l-m} \left({}^{\mathrm{qsc}}\overline{T}^* M \right) \Big/ \mathcal{A}^{m-1,l-m+2} \left({}^{\mathrm{qsc}}\overline{T}^* M \right), \end{aligned}$$

where $\Theta^* : {}^{\mathrm{sc}}\overline{T}^* M_q \to {}^{\mathrm{qsc}}\overline{T}^* M$ is a naturally defined pull-back, and Θ^{**} is a pull-back from functions on ${}^{\mathrm{qsc}}\overline{T}^* M$ to functions on ${}^{\mathrm{sc}}\overline{T}^* M_q$. The compactified qsc cotangent bundle ${}^{\mathrm{qsc}}\overline{T}^* M$ is defined similarly to the sc case;

$\mathcal{V}_{\mathrm{qsc}}(M) = x\mathcal{V}_{\mathrm{sc}}(M)$,
${}^{\mathrm{qsc}}T^* M$: a vector bundle on M such that $\Gamma(M; {}^{\mathrm{qsc}}T^* M) = \mathcal{V}_{\mathrm{qsc}}(M)$,
${}^{\mathrm{qsc}}\overline{T}^* M$: the fiber-compactified ${}^{\mathrm{qsc}}T^* M$.

The rest argument can be done similarly to the sc case, so we just write down

$$\mathrm{WF}_{\mathrm{qsc}} = \bigcap \left\{ \Sigma^{m,l}_{\mathrm{qsc}}(A); A \in \Theta^* \circ \Psi^{m,(l-m)/2}_{\mathrm{sc}}(M_q) \circ (\Theta^*)^{-1},\ Au \in \dot{C}^\infty(M) \right\}.$$

The qsc wavefront set is nothing but the sc wavefront set of a coordinate-changed function. So, away from ∂M, we have

Lemma 3.2 *For any $u \in C^{-\infty}(M)$ we have a correspondence*

$$\mathrm{WF}_{\mathrm{sc}}(u) \cap ({}^{\mathrm{sc}}S^* M)^\circ \longleftrightarrow \mathrm{WF}_{\mathrm{qsc}}(u) \cap ({}^{\mathrm{qsc}}S^* M)^\circ \longleftrightarrow \mathrm{WF}(u),$$

where ${}^{\mathrm{sc}}S^* M$, ${}^{\mathrm{qsc}}S^* M$ are sphere bundles over M defined by

$${}^{\mathrm{sc}}S^* M = C_{\mathrm{sc}} M \setminus \left({}^{\mathrm{sc}}\overline{T}^*_{\partial M} M \right)^\circ, \quad {}^{\mathrm{qsc}}S^* M = C_{\mathrm{qsc}} M \setminus \left({}^{\mathrm{qsc}}\overline{T}^*_{\partial M} M \right)^\circ.$$

However, on ∂M there is a crucial difference between the sc and the qsc wavefront sets. The corner part of the sc wavefront set corresponds to one face of the qsc wavefront set:

Proposition 3.3 *Let $u \in C^\infty(M)$ and assume $\mathrm{WF}_{\mathrm{sc}}(u) \cap {}^{\mathrm{sc}}S^*_{\partial M}M = \varnothing$. Then $\mathrm{WF}_{\mathrm{qsc}}(u) \subset 0$ (the zero section).*

4 Proof of Theorem 1.5

Let $M = M_q = S^n_+$ and define the mapping $q : M \to M_q$ by

$$q = q(z) = \left(2 + |z|^2\right)^{\frac{1}{2}} z.$$

q is a bijection between M and M_q and designed to satisfy $\langle q\rangle^{-1} = \langle z\rangle^{-2}$. Thus M_q is thought to be M whose C^∞ structure near the boundary is generated by the new boundary defining function $\langle z\rangle^{-2}$.

Let $u \in \mathcal{S}'(\mathbb{R}^n) = C^{-\infty}(M)$, and we first assume

$$(z_0, \zeta_0) \in \left(S^{n-1} \times \mathbb{R}^n\right) \setminus \mathrm{WF}_{\mathrm{qsc}}(u),$$

which is equivalent to

$$(z_0, \zeta_0) \in \left(S^{n-1} \times \mathbb{R}^n\right) \setminus \mathrm{WF}_{\mathrm{sc}}\left((q^*)^{-1} u\right).$$

Then there exists $\varphi \in C_0^\infty\left(\mathbb{R}^{2n}\right)$ such that $\varphi(z_0, \zeta_0) \neq 0$ and

$$\left\|\varphi(hq, D_q)(q^*)^{-1} u\right\| = O(h^\infty), \tag{4.1}$$

where $\varphi(hq, D_q)$ is the standard [left] quantization of $\varphi(hq, \tau)$.

By the change of variables, we have

$$\left\|\varphi(hq, D_q)(q^*)^{-1} u\right\|^2 = 8 \int \left|\int e^{i(z-w)\zeta} \tilde{\varphi}(z, w, \zeta; h) u(w) dw đ\zeta\right|^2 dz,$$

where

$$\begin{aligned}\tilde{\varphi}(z, w, \zeta; h) = {} & \langle z\rangle \left(2 + |z|^2\right)^{\frac{n-2}{4}} \langle w\rangle^2 \left(2 + |w|^2\right)^{\frac{n-2}{2}} \\ & \cdot \varphi\left(h\left(2 + |z|^2\right)^{\frac{1}{2}} z, \Phi(z, w)\zeta\right) \det \Phi(z, w), \\ \Phi_{ij}(z, w) = {} & \delta_{ij} \int_0^1 \frac{dt}{\left(1 + \langle p + t(q-p)\rangle\right)^{\frac{1}{2}}} - \frac{1}{2} \int_0^1 \frac{\left(p^i + t\left(q^i - p^i\right)\right)\left(p^j + t\left(q^j - p^j\right)\right)}{\left(1 + \langle p + t(q-p)\rangle\right)^{\frac{3}{2}} \langle p + t(q-p)\rangle} dt.\end{aligned}$$

We wrote $q = q(z)$ and $p = q(w)$. $\tilde{\varphi}$ belongs to a very good class;

Lemma 4.1

$$\tilde{\varphi} \in S\left(\frac{\langle z\rangle^{\frac{n}{2}} \langle w\rangle^n}{\langle z; w\rangle^n}, \frac{dz^2}{\langle z\rangle^2} + \frac{dw^2}{\langle w\rangle^2} + \frac{d\zeta^2}{\langle z; w\rangle^2}.\right).$$

Proof. We first claim that there is $C_n > 0$ such that

$$\langle z;w\rangle^{-n} \le \int_0^1 \frac{dt}{\langle p+t\,(q-p)\rangle^{\frac{n}{2}}} \le C_n \langle z;w\rangle^{-n}$$

for each positive odd integer n. The first inequality is easily obtained, and for the second consider the four cases:

$$\text{(i) } |q-p| \le \frac{1}{2}|q|, \qquad \text{(ii) } |q-p| \le \frac{1}{2}|p|,$$
$$\text{(iii) } |q-p| \ge \frac{1}{4}(|q|+|p|) \ge 1, \qquad \text{(iv) } \frac{1}{4}(|q|+|p|) \le 1.$$

It then follows from the claimed inequality that

$$\Phi_{ij}(z,w) \in S\left(\langle z;w\rangle^{-1}; \langle z;w\rangle^{-2} dz^2 + \langle z;w\rangle^{-2} dw^2\right).$$

As a polynomial in $\Phi_{ij}(z,w)$ of degree n

$$\det\Phi(z,w) \in S\left(\langle z;w\rangle^{-n}; \langle z;w\rangle^{-2} dz^2 + \langle z;w\rangle^{-2} dw^2\right).$$

Considering $\operatorname{supp}\tilde{\varphi}$, the lemma follows. □

Since $\tilde{\varphi}$ is in a good class, we can get the principal part of $\tilde{\varphi}$ by substituting $w = z$:

$$\tilde{\varphi}(z,z,\zeta;h) = \langle z\rangle^3 \left(2+|z|^2\right)^{\frac{3n-6}{4}} \varphi\left(h\left(2+|z|^2\right)^{\frac{1}{2}} z, \Phi(z,z)\zeta\right) \det\Phi(z,z)$$

with

$$\Phi_{ij}(z,z) = \delta_{ij}\frac{1}{(1+\langle q\rangle)^{\frac{1}{2}}} - \frac{1}{2}\frac{q^iq^j}{(1+\langle q\rangle)^{\frac{3}{2}}\langle q\rangle}, \qquad q = q(z).$$

Note that $z \sim h^{-1}$ and $\zeta \sim h^{-1}$ on $\operatorname{supp}\varphi\left(h\left(2+|z|^2\right)^{\frac{1}{2}} z, \Phi(z,z)\zeta\right)$ as $h \downarrow 0$, that is, the support of $\tilde{\varphi}(z,z,\zeta;h)$ moves towards the homogeneous direction in the phase space. This argument is verified, and actually we have the ellipticity:

$$\tilde{\varphi}\left(h^{-\frac{1}{2}}z_0, h^{-\frac{1}{2}}z_0, h^{-\frac{1}{2}}\Psi(z_0)\zeta_0; h\right) \ge Ch^{-\frac{n}{4}}$$

uniformly in small $h > 0$, where for $z \neq 0$

$$\Psi(z) = \left(\delta_{ij} - \frac{z^iz^j}{2|z|^2}\right)^{-1}_{ij} = \left(\delta_{ij} + \frac{z^iz^j}{|z|^2}\right)_{ij}.$$

Thus

$$(z_0, \Psi(z_0)\zeta_0) \notin \mathrm{HWF}(u). \tag{4.2}$$

One inclusion relation is obtained and the other is similarly obtained.

A Formulae for Coordinate Transformation

For a point $z = (z^1, \dots, z^n) \in \mathbb{R}^n \subset M$, $z \neq 0$ we set

$$x = \frac{1}{|z|}, \quad \omega = \left(\omega^1, \dots, \omega^n\right) = \frac{z}{|z|}.$$

Since $z \neq 0$, there exists non-zero ω^k, and so, when $\pm\omega^k > 0$, we can get rid of ω^k to make local coordinates $\left(x, y_{(\pm k)}\right) = \left(x, y^1_{(\pm k)}, \dots, y^{n-1}_{(\pm k)}\right)$ of $M(\supset \mathbb{R}^n)$ near the boundary respectively:

$$y^j_{(\pm k)} = \begin{cases} \omega^j, & \text{for } 1 \leq j \leq k-1, \\ \omega^{j+1}, & \text{for } k \leq j \leq n-1. \end{cases}$$

We denote $y_{(\pm k)}$ simply by y if there is no confusion. We introduce local coordinates (z, ζ) and (x, y, ξ, η) of the cotangent bundle T^*M corresponding to z and (x, y) respectively. In the following we write down formulae for the coordinate change between the above coordinates that are needed in the article. We consider only the case where $x^n > 0$, i.e., $y = y_{(+n)}$. Introducing a notation

$$y^n = \sqrt{1 - (y^1)^2 - \dots - (y^{n-1})^2},$$

we have

$$x = \frac{1}{|z|}, \quad y^i = \frac{z^i}{|z|} \quad (i = 1, \dots, n-1); \quad z^i = \frac{y^i}{x} \quad (i = 1, \dots, n),$$

and thus

$$\begin{aligned} \partial_{z^i} &= -\frac{z^i}{|z|^3}\partial_x + \sum_{j=1}^{n-1}\left(\frac{\delta^j_i}{|z|} - \frac{z^i z^j}{|z|^3}\right)\partial_{y^j} \\ &= -x^2 y^i \partial_x + x\sum_{j=1}^{n-1}\left(\delta^j_i - y^i y^j\right)\partial_{y^j} \quad (i = 1, \dots, n), \end{aligned} \tag{A.1}$$

$$\partial_x = -\frac{1}{x^2}\sum_{i=1}^{n} y^i \partial_{z^i} = -|z|\sum_{i=1}^{n} z^i \partial_{z^i}, \tag{A.2}$$

$$\partial_{y^i} = \frac{1}{x}\partial_{z^i} - \frac{1}{x}\frac{y^i}{y^n}\partial_{z^n} = |z|\partial_{z^i} - |z|\frac{z^i}{z^n}\partial_{z^n} \quad (i = 1, \dots, n-1), \tag{A.3}$$

$$dz^i = -\frac{y^i}{x^2}dx + \frac{1}{x}dy^i = -|z|z^i dx + |z|dy^i \quad (i = 1, \dots, n), \tag{A.4}$$

$$dx = -\sum_{i=1}^{n}\frac{z^i}{|z|^3}dz^i = -x^2\sum_{i=1}^{n} y^i dz^i, \tag{A.5}$$

$$dy^i = \sum_{j=1}^{n}\left(\frac{\delta^i_j}{|z|} - \frac{z^i z^j}{|z|^3}\right)dz^j = x\sum_{j=1}^{n}\left(\delta^i_j - y^i y^j\right)dz^j. \tag{A.6}$$

Then for the same point in $T^*\left(\mathbb{R}^n \setminus \{0\}\right) \subset T^*M$

$$\sum_{i=1}^{n}\zeta_i dz^i = \xi dx + \sum_{i=1}^{n-1}\eta_i dy^i,$$

we obtain

$$\begin{aligned}\zeta_i &= -\frac{z^i}{|z|^3}\xi + \sum_{j=1}^{n-1}\left(\frac{\delta_i^j}{|z|} - \frac{z^i z^j}{|z|^3}\right)\eta_j \\ &= -x^2 y^i \xi + x\sum_{j=1}^{n-1}\left(\delta_i^j - y^i y^j\right)\eta_j \quad (i = 1, \ldots, n)\end{aligned} \tag{A.7}$$

$$\xi = -\frac{1}{x^2}\sum_{i=1}^{n} y^i \zeta_i = -|z|\sum_{i=1}^{n} z^i \zeta_i, \tag{A.8}$$

$$\eta_i = \frac{1}{x}\zeta_i - \frac{1}{x}\frac{y^i}{y^n}\zeta_n = |z|\zeta_i - |z|\frac{z^i}{z^n}\zeta_n \quad (i = 1, \ldots, n-1), \tag{A.9}$$

and on the tangent space to the cotangent bundle,

$$\begin{aligned}\partial_{z^i} = &-\frac{z^i}{|z|^3}\partial_x + \sum_{j=1}^{n-1}\left(\frac{\delta_i^j}{|z|} - \frac{z^i z^j}{|z|^3}\right)\partial_{y^j} - |z|\zeta_i\partial_\xi - \frac{1}{|z|}\sum_{j=1}^{n} z^i z^j \partial_\xi \\ &+ \sum_{j=1}^{n-1}\left(\frac{z^i}{|z|}\zeta_j - \frac{z^i z^j}{|z| z^n}\zeta_n - \frac{\delta_j^i |z|}{z^n}\zeta_n + \frac{\delta_n^i\,|z|\,z^j}{(z^n)^2}\zeta_n\right)\partial_{\eta_j},\end{aligned} \tag{A.10}$$

$$\partial_x = -\frac{1}{x^2}\sum_{i=1}^{n} y^i \partial_{z^i} + \sum_{i=1}^{n}\left(-2xy^i\xi + \sum_{j=1}^{n-1}\left(\delta_j^i - y^i y^j\right)\eta_j\right)\partial_{\zeta_i}, \tag{A.11}$$

$$\partial_{y^i} = \frac{1}{x}\partial_{z^i} - \frac{1}{x}\frac{y^i}{y^n}\partial_{z^n} + \sum_{j=1}^{n-1}\left(-\delta_j^i x^2\xi - x\sum_{k=1}^{n-1}\left(\delta_k^i y^j + \delta_j^i y^k\right)\eta_k\right)\partial_{\zeta_j} \tag{A.12}$$

$$+\left(x^2\frac{y^i}{y^n}\xi + x\sum_{j=1}^{n-1}\frac{y^i y^j}{y^n}\eta_j\right)\partial_{\zeta_n}, \tag{A.13}$$

$$\partial_{\zeta_i} = \begin{cases} -\frac{y^i}{x^2}\partial_\xi + \frac{1}{x}\partial_{\eta_i} = -|z|z^i\partial_\xi + |z|\partial_{\eta_i}, & \text{if } i \neq n, \\ -\frac{y^n}{x^2}\partial_\xi - \sum_{j=1}^{n-1}\frac{y^j}{xy^n}\partial_{\eta_j} = |z|z^n\partial_\xi - \sum_{j=1}^{n-1}\frac{|z|z^j}{z^n}\partial_{\eta_j}, & \text{if } i = n,\end{cases} \tag{A.14}$$

$$\partial_\xi = -x^2\sum_{i=1}^{n} y^i\partial_{\zeta_i} = -\sum_{i=1}^{n}\frac{z^i}{|z|^3}\partial_{\zeta_i}, \tag{A.15}$$

$$\partial_{\eta_i} = x\sum_{j=1}^{n}\left(\delta_i^j - y^i y^j\right)\partial_{\zeta_j} = \sum_{j=1}^{n}\left(\frac{\delta_i^j}{|z|} - \frac{z^i z^j}{|z|^3}\right)\partial_{\zeta_j}. \tag{A.16}$$

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