数理解析研究所講究録
1510 巻 2006 年 26-32

Strichartz estimates for Schrödinger equations with variable coefficients

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The purpose of this work is to provide a proof of the full (local in time) Strichartz estimates for the Schrödinger operator related to a non trapping asymptotically flat perturbation of the usual Laplacian in $\mathbb{R}^n$.

To be more precise let us first introduce the following space. Let σ_0 be in $]0,1[$. We set

$$\mathcal{B}_{\sigma_0}=\left\{a\in C^\infty(\mathbb{R}^n):\forall\alpha\in\mathbb{N}^n,\ \exists C_\alpha>0:|\partial^\alpha a(x)|\leq\frac{C_\alpha}{\langle x\rangle^{1+|\alpha|+\sigma_0}},\forall x\in\mathbb{R}^n\right\}\tag{1}$$

Here we have set $\langle x\rangle=(1+|x|^2)^{1/2}$.

Let P be a second order differential operator,

$$P=\sum_{j,k=1}^n D_j\big(g^{jk}(x)\,D_k\big)+\sum_{j=1}^n(D_j\,b_j(x)+b_j(x)\,D_j)+V(x)\,,D_j=\frac{1}{i}\frac{\partial}{\partial x_j}\,,\tag{2}$$

with principal symbol $p(x,\xi)=\sum_{j,k=1}^n g^{jk}(x)\,\xi_j\,\xi_k$. (Here $g^{jk}=g^{kj}$).

We shall make the following assumptions.

$$\begin{cases}\text{(i)} & \text{The coefficients } g^{jk},\ b_j,\ V \text{ are real valued, } 1\leq j\leq k\leq n.\\ \text{(ii)} & \text{There exists } \sigma_0>0 \text{ such that } g^{jk}-\delta_{jk}\in\mathcal{B}_{\sigma_0},\ b_j\in\mathcal{B}_{\sigma_0}.\\ & \text{Here } \delta_{jk} \text{ is the Kronecker symbol.}\\ \text{(iii)} & V\in L^\infty(\mathbb{R}^n).\end{cases}\tag{3}$$

$$\text{There exists } \nu>0 \text{ such that for every } (x,\xi) \text{ in } \mathbb{R}^n\times\mathbb{R}^n,\ p(x,\xi)\geq\nu\,|\xi|^2.\tag{4}$$

Then P has a self-adjoint extension with domain $H^2(\mathbb{R}^n)$.

Now we associate to the symbol p the bicharacteristic flow given by the following equations for $j=1,\ldots,n$,

$$\begin{cases}\dot{x}_j(t)=\frac{\partial p}{\partial\xi_j}(x(t),\xi(t)), & x_j(0)=x_j\,,\\ \dot{\xi}_j(t)=-\frac{\partial p}{\partial x_j}(x(t),\xi(t)), & \xi_j(0)=\xi_j\,.\end{cases}\tag{5}$$

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We shall denote by $(x(t,x,\xi),\xi(t,x,\xi))$ the solution, whenever it exists, of the system (5). In fact it is an easy consequence of (3) and (4) that this flow exists for all t in $\mathbb{R}$. Indeed by (4) we have

$$\nu\,|\xi(t)|^2 \leq p(x(t),\,\xi(t)) = p(x,\xi)\,,$$

and it follows from (4) that

$$|\dot{x}_j(t)| \leq 2\sum_{k=1}^{n} |g^{jk}(x)\,\xi_k(t)| \leq C\,|\xi(t)| \leq C\,\nu^{-1/2}\,p(x,\xi)^{1/2}\,.$$

Our last assumption will be the following.

(6) For all (x,ξ) in $T^*\mathbb{R}^n\setminus\{0\}$ we have $\lim\limits_{t\to\pm\infty} |x(t,x,\xi)| = +\infty\,.$

This means that the flow is not trapped backward nor forward. Now let us denote by e^{-itP} the solution of the following initial value problem

$$(7)\qquad \begin{cases} i\,\frac{\partial u}{\partial t} - Pu = 0 \\ u(0,\cdot) = u_0\,. \end{cases}$$

Then the main result of this work is the following.

Theorem 1 *Assume that the operator P satisfies the conditions (3), (4), (6). Let $T > 0$ and (q,r) be a couple of real numbers such that $q > 2$ and $\frac{2}{q} = \frac{n}{2} - \frac{n}{r}$. Then there exists a positive constant C such that*

$$(8)\qquad \|e^{-itP}u_0\|_{L^q([-T,T],L^r(\mathbb{R}^n))} \leq C\,\|u_0\|_{L^2(\mathbb{R}^n)}\,,$$

for all u_0 in $L^2(\mathbb{R}^n)$.

Such estimates are known in the litterature under the name of Strichartz estimates. They have been proved for the flat Laplacian by Strichartz [St] when $p = q = \frac{2n+4}{n}$ and extended to the full range of (p,q) given by the scaling by Ginibre-Velo [GV] and Yajima [Y]. The limit case $q = 2$ (the end point) when $n \geq 3$ is due to Keel-Tao [KT] . These estimates have been a key tool in the study of non linear equations. Very recently several works appeared showing a new interest for such estimates in the case of variable coefficients. Steffilani-Tataru [ST] proved Theorem 1 under conditions (4) and (6) for compactly supported perturbations of the flat Laplacian. In [B] Burq gave an alternative proof of this result using the work of Burq-Gérard-Tzvetkov [BGT]. In the same work Burq announced without proof that if you accept to replace in the right hand side of (8) the L^2 norm by an H^ε norm, for any small $\varepsilon > 0$, then you can weaken the decay hypotheses on the coefficients of P in the sense that you may replace in the definition (1) of $\mathcal{B}_{\sigma_0}$ the power $|\alpha| + 1 + \sigma_0$ by $|\alpha| + \sigma_0$. We have also to mention a recent work of Hassel-Tao-Wunsch [HTW1] who proved in dimension $n = 3$ a weaker form of our result corresponding to the case where $q = 4$, $r = 3$, under conditions similar to ours. Still more recently these three authors announced the same result as ours under hypotheses on the coefficients similar to ours (see [HTW2]).

It is also worthwhile to mention the work of Burq-Gérard-Tzvetkov who investigate the Strichartz estimates on compact Riemannian manifolds. In that case they show that such estimates hold with the L^2 norm replaced by the $H^{1/q}$ norm. In the same paper these authors show that the same result holds on $\mathbb{R}^n$ when the coefficients of their Laplacian (and its derivatives) are merely bounded. Let us note also that these estimates concern also the wave equation and many works have been devoted to this case. However we would like to emphasize that, due to the finite speed of propagation, the extension to the variable coefficients case appear to be much less technical (see [SS]).

Let us now give some ideas on the proof. It is by now well known that a proof of the Strichartz estimates can be done using a dispersion result, duality arguments and the Hardy-Littlewood-Sobolev lemma. This has been formulated as an abstract result in the paper [KT] as follows. Assume that for every $t \in \mathbb{R}$ we have an operator $U(t)$ which maps $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$ and satisfies,

$$\begin{cases} \text{(i)} & \|U(t)\,f\|_{L^2(\mathbb{R}^n)} \leq C\,\|f\|_{L^2(\mathbb{R}^n)}\,, \quad \forall t \in \mathbb{R}\,, \ \ C \text{ independent of } t\,. \\ \text{(ii)} & \|U(s)(U(t))^*\,g\|_{L^\infty(\mathbb{R}^n)} \leq C\,|t-s|^{-\frac{n}{2}}\,\|g\|_{L^1(\mathbb{R}^n)}\,, \quad t \neq s\,, \end{cases}$$

then the Strichartz estimates (5) hold for $U(t)$. It is not difficult to see that the serious estimate to be proved is (ii). In the case when $U(t) = e^{it\Delta_0}$ (the flat Laplacian) this estimate is obtained by the explicit formula giving the solution in term of the data u_0. In the variable coefficients case such a formula is of course out of hope and the better we can have is a parametrix. However due to strong technical difficulties (which we try to explain below) which seem to be serious we are not able to write a parametrix for e^{-itP} so we have to explain what we do instead. First of all let $\varphi_0 \in C_0^\infty(\mathbb{R}^n)$ be such that $\varphi_0(x) = 1$ if $|x| \leq \frac{3}{2}$ and $\operatorname{supp}\varphi_0 \subset [-1,1]$. With a large $R > 0$ we write

$$e^{-itP}\,u_0(x) = \varphi_0\Big(\frac{x}{R}\Big)\,e^{-itP}\,u_0(x) + \Big(1 - \varphi_0\Big(\frac{x}{R}\Big)\Big)\,e^{-itP}\,u_0(x) = v + w\,.$$

It is not difficult to see that the Strichartz estimates for v will be ensured by the result of Staffilani-Tataru [ST] while the same estimate for w leads to consider an operator which is a small perturbation of the Laplacian.

Now it is not a surprise that microlocal analysis is strongly needed in our proof. So let $\xi_0 \in \mathbb{R}^n$, $|\xi_0| = 1$ be a fixed direction. Let $\chi_0 \in C^\infty(\mathbb{R})$, $\chi_0(s) = 1$ if $s \leq \frac{3}{4}$, $\chi_0(s) = 0$ if $s \geq 1$, $0 \leq \chi_0 \leq 1$ and let us set $\chi_+(x) = \chi_0\big(\frac{-x\cdot\xi_0}{\delta_1}\big)$, $\chi_-(x) = \chi_0\big(\frac{x\cdot\xi_0}{\delta_1}\big)$, $\delta_1 > 0$. We set $U_+(t) = \chi_+\,e^{-itP}$, $U_-(t) = \chi_-\,e^{-itP}$. Now since $\chi_+(x)+\chi_-(x) \geq 1$ for all x in $\mathbb{R}^n$ then Strichartz estimates separately for $U_+(t)$ and $U_-(t)$ will give the result. It is therefore sufficient to prove the estimate (ii) above for $U_+(s)\ (U_+(t))^* = \chi_+\,e^{i(s-t)P}\,\chi_+$ (and for $U_-(s)(U_-(t))^*$). In our proof we shall construct a parametrix for these operators.

Our construction relies heavily on the theory of FBI transform (see Sjöstrand [Sj] and Melin-Sjöstrand [MS]) viewed as a Fourier integral operator with complex phase. One of the reason of our choice is that in our former works on the analytic smoothing effect [RZ2] we have already done similar constructions (but only near the outgoing points: see below). Let us explain very roughly the main ideas. The standard FBI transform is given by

$$Tv(\alpha,\lambda) = c_n\,\lambda^{3n/4}\int_{\mathbb{R}^n} e^{i\lambda(y-\alpha_x)\cdot\alpha_\xi - \frac{\lambda}{2}|y-\alpha_x|^2 + \frac{\lambda}{2}|\alpha_\xi|^2}\,v(y)\,dy \tag{9}$$

where $\alpha = (\alpha_x, \alpha_\xi) \in \mathbb{R}^n \times \mathbb{R}^n$ and c_n is a positive constant.

Let us note that the phase can be written $i\lambda\varphi_0$ where $\varphi_0(y,\alpha) = \frac{i}{2}\,(y - (\alpha_x + i\alpha_\xi))^2$. Then T maps $L^2(\mathbb{R}^n)$ into the space $L^2(\mathbb{R}^{2n}, e^{-\lambda|\alpha_\xi|^2}\,d\alpha)$. The adjoint T^* of T is given by a similar formula and we have,

(10) $$T^*T \text{ is the identity operator on } L^2(\mathbb{R}^n)\,.$$

We embed the transform T into a continuous family of FBI transform

(11) $$\begin{cases} T_\theta v(\alpha,\lambda) = \lambda^{3n/4}\int_{\mathbb{R}^n} e^{i\lambda\varphi(\theta,y,\alpha)}a(\theta,y,\alpha)\,v(y)\,dy \text{ with} \\ \varphi(0,y,\alpha) = \frac{1}{2}(y-(\alpha_x+i\alpha_\xi))^2\,, a(0,y,\alpha) = c_n\,. \end{cases}$$

Let us set $U(\theta,t,\alpha,\lambda) = T_\theta[K_\pm(t)\,u_0](\alpha,\lambda)$, where $K_\pm(t) = \chi_\pm\, e^{-itP}\,\chi_\pm$. Then it is shown that if φ satisfies the eikonal equation,

(12) $$\Big[\frac{\partial\varphi}{\partial\theta} + p\Big(x,\frac{\partial\varphi}{\partial x}\Big)\Big](\theta,x,\alpha) = 0\,,$$

and if the symbol a satisfies appropriate transport equations then U is a solution of the following equation

$$\Big(\frac{\partial U}{\partial t} + \lambda\,\frac{\partial U}{\partial\theta}\Big)(\theta,t,\alpha,\lambda) \sim 0\,.$$

It follows that essentially we have, $U(\theta,t,\alpha,\lambda) = V(\theta-\lambda t,\alpha,\lambda)$. In particular this shows that $U(0,t,\alpha,\lambda) = U(-\lambda t,0,\alpha,\lambda)$. Written in terms of the transformations T_θ this reads

$$T[K_\pm(t)u_0](\alpha,\lambda) = T_{-\lambda t}[\chi_\pm^2\,u_0](\alpha,\lambda)\,.$$

Applying T^* to both members and using (10) we obtain

$$K_\pm(t)\,u_0(x) = T^*\{T_{-\lambda t}[\chi_\pm^2\,u_0](\cdot,\lambda)\}(t,x)\,.$$

Thus we have expressed the solution in terms of the data through a Fourier integral operator with complex phase.

This short discussion shows that as usual the main point of the proof is to solve the eikonal and transport equations. Let us point out the main difficulties which occur in solving these equations. They are of three types: the bad behavior of the flow from incoming points and for large time, the global (in θ, x) character of all our constructions and the mixing of C^∞ coefficients and complex variables (coming from the non real character of our phase). Let us discuss each of them. First of all whatever the method you use to solve an eikonal equation (symplectic geometry or another one) a precise description of the flow of the symbol p is needed. Let us recall (see (5)) that our flow $(x(t,x,\xi),\xi(t,x,\xi))$, issued from the point $(x,\xi)\in T^*\mathbb{R}^n\setminus 0$, is defined for all $t\in\mathbb{R}$. In the case of the flat Laplacian we have $\xi(t,x,\xi) = \xi$ and $x(t,x,\xi) = x + 2t\xi$. Let now $(x,\xi)\in T^*\mathbb{R}^n\setminus\{0\}$ and assume that $x\cdot\xi\geq 0$. Then it is easy to see that $|x(t,x,\xi)|^2 \geq |x|^2 + 4t^2\,|\xi|^2$ for $t\geq 0$ so that $|x(t,x,\xi)|$ becomes larger and larger while $x(t,x,\xi)$ may vanish for a large $t<0$. Such a point is called "outgoing for $t\geq 0$" and "incoming for $t<0$". In the case of a perturbed Laplacian this distinction between the directions is very important. Indeed although the flow from outgoing points for $t\geq 0$ is very well described for $t\geq 0$ and has very similar properties to the flat case, it has a bad behavior for $t<0$ in what concerns its derivatives with respect to (x,ξ). For instance $\frac{\partial x_j}{\partial\xi_k}(t,x,\xi)$ does not behave at all as $2t\,\delta_{jk}$. This is of great importance and causes some trouble in the proof. However still when $t<0$, the flow behaves correctly as long as the point $(x(t,x,\xi),\xi(t,x,\xi))$ is outgoing for $t\geq 0$. Roughly speaking that is the reason why we are not able to construct a parametrix for e^{-itP} while it is possible for the operator $\chi_\pm\,e^{-itP}\chi_\pm$.

Let us now describe our method of resolution of the eikonal equation. The classical method uses the ideas of symplectic geometry. Roughly speaking the manifold constructed from the flow is a Lagrangian manifold on which the symbol $\tau + p(x,\xi)$ is constant. If it projects (globally) and clearly on the basis then it is a graph of some function φ which is the desired phase. However this general method leads immediately to a difficulty in our case. Indeed since we want that for $\theta = 0$ the phase φ coincides with the phase φ_0 of the FBI transform (see (9)) which is non real, we should take, in solving the flow, data which are non real, so the flow itself would be non real; but our symbol has merely C^∞ coefficients. To circumvent this difficulty a method has been proposed by Melin-Sjöstrand [MS] which uses the almost analytic machinery. Another method, different in spirit, that the one described above and known under the name of "Lagrangian ideals", has been introduced by Hörmander [H]. Here the initial data in the flow are kept real. Let us set $u_j(x,\xi) = \xi_j - \frac{\partial\varphi_0}{\partial x_j}(x,\xi) = \xi_j - \alpha_\xi^j - i(x_j - \alpha_x^j)$. Then obviously we have $\{u_j, u_k\} = 0$ if $j \neq k$ (where $\{,\}$ denotes the Poisson bracket). Now let us set $v_j(\theta, x, \xi) = u_j(x(-\theta,x,\xi), \xi(-\theta,x,\xi))$, $j = 1, \ldots, n$. Then for every θ in $\mathbb{R}$ the Poisson bracket of v_j and v_k still vanishes if $j \neq k$. Thus the ideal generated by the v_j's is closed under the Poisson bracket. The main step in Hörmander's method is to show that this ideal is generated by functions of the form $\xi_j - \Phi_j(\theta, x, \alpha)$. This will imply that one can find a function $\varphi = \varphi(\theta, x, \alpha)$ such that $\frac{\partial\varphi}{\partial x_j}(\theta, x, \alpha) = \Phi_j(\theta, x, \alpha)$ and it turns out that φ is the desired phase. To achieve its main step, Hörmander uses a precise version of the Malgrange preparation theorem which is discussed in [H], tome 1. This is the way we chosed to use in our case. The proof is made separately for outgoing and incoming points. Since the v_j's are defined by mean of the backward flow, in both cases we encounter the difficulty caused by the bad behavior of the flow from incoming points. As it can be seen many technical difficulties arise in the procedure.

The next step in the proof is the resolution of the transport equations. Here also the cases of outgoing and incoming points have to be separated. We have also to be careful since these are first order equations with non real C^∞ coefficients. The first case is easier. Indeed due to the good behavior of the flow and the decay of the perturbation one can cut the Taylor expansion of the coefficients of the vector field to some order and thus reduce ourselves to the case of polynomial coefficients. Then by classical holomorphic methods one can solve the equations modulo flat terms which will be enough for our purpose. In the second case there is no more such an asymptotic and the situation is much more intricate. So we use the classical idea which consists in straightening the vector field. This forces us to enter in the almost analytic machinery of Melin-Sjöstrand [MS]. Of course all the constructions made above are done microlocally and in a neighborhood of the bicharacteristic. Therefore to define the general FBI transform T_θ (see (11)) as well as to pass from the standard T to $T_{-\lambda t}$ we have to insert many microlocal cut-off. Of course we have to check at each microlocalization that the remainder leads to an acceptable error. At this stage of the proof the operator $K_\pm(t) = \chi_\pm\, e^{-itP}\, \chi_\pm$ is writen as

$$K_\pm(t)\, u_0(x) = \int k_\pm(t,x,y)\, u_0(y)\, dy$$

where

$$k_\pm(t,x,y) = \int e^{i\lambda F(-\lambda t, x, y, \alpha)}\, a(\lambda t, x, y, \alpha)\, d\alpha\,.$$

Thus the dispersion estimate would follow from the bound

$$|k_\pm(t,x,y)| \leq \frac{C}{|t|^{n/2}}$$

for $0 < |t| \leq T$.

Here we have two regimes according to the fact that $|\lambda t| \geq 1$ or $|\lambda t| \leq 1$. In the first case on the support of $a(\lambda t, x, y, \alpha)$ we could be very far from the critical point of F. Fortunately the phase F has enough convexity to produce the desired bound of $k_{\pm}$. In the second regime we are close to the critical point of F so we expect a stationnary phase method to work. However since the phase F is non real and since the determinant of its Hessien in α degenerates in some direction when $|\lambda t| \to 0$ we cannot apply the standard results as they appear in [H]. Instead, after a careful study of the phase F we use merely an integration by part method with an appropriate vector field to conclude. The proof of the main theorem is then obtained by using the Littlewood-Paley theory.

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