数理解析研究所講究録
1510 巻 2006 年 33-39

Boundedness of imaginary part of spectrum for second order ordinary differential operators

東北大学理学部 下田 泰史 (Taishi Shimoda)
Mathematical Institute, Tohoku University.

1 Introduction

Let L be a second order ordinary linear differential operator on $\boldsymbol{R}$ of the form:

$$L = -\frac{d^2}{dx^2} + a(x)\frac{d}{dx} + b(x).$$

We regard it as a closed operator on $L^2(\boldsymbol{R})$ with the domain $H^2(\boldsymbol{R})$. We assume that $a(x) \in \mathscr{B}^1(\boldsymbol{R})$ and $b(x) \in \mathscr{B}^0(\boldsymbol{R})$. Here $\mathscr{B}^k(\boldsymbol{R})$ stands for the set of all continuous and complex-valued functions on $\boldsymbol{R}$ which are bounded with their derivatives up to k-th order. We denote by $\sigma(L)$ the spectrum of L in $L^2(\boldsymbol{R})$ and by $\operatorname{Im}\sigma(L)$ the set of all imaginary parts of $\sigma(L)$. We are interested in the relation between the boundedness of $\operatorname{Im}\sigma(L)$ and the well-posedness of the Cauchy problem:

$$\left(-i\frac{d}{dt} - L\right)u(t) = f(t) \in \boldsymbol{C}^0(\boldsymbol{R}; L^2(\boldsymbol{R})), \quad u(0) = u_0 \in L^2(\boldsymbol{R}). \tag{1.1}$$

The sufficient and necessary condition for (1.1) to be L^2-well-posed was given by S. Mizohata [2], which is the following condition:

$$\sup_{x,y\in\boldsymbol{R}}\left|\int_y^x \operatorname{Re} a(s)\,ds\right| < +\infty. \tag{1.2}$$

We remark that $\operatorname{Im}\sigma(L)$ is bounded provided that the condition (1.2) holds. This is because, the gauge transformation

$$u(x) \mapsto \exp\left(\frac{1}{2}\int_0^x a(s)\,ds\right)u(x)$$

is automorphic in $L^2(\boldsymbol{R})$ under (1.2), so we reduce the study of the spectrum of L to that of the operator of Schrödinger type with a complex-valued potential:

$$S = -\frac{d^2}{dx^2} + V(x), \quad \text{where } V(x) = -\frac{1}{2}a'(x) + \frac{1}{4}a(x)^2 + b(x).$$

Since the potential $V(x)$ is bounded, $\operatorname{Im}\sigma(S)$ is bounded, so $\operatorname{Im}\sigma(L)$ is. Therefore we conclude that the L^2-well-posedness implies the boundedness of $\operatorname{Im}\sigma(L)$. In order to make the relation between these notions clear, we give a sufficient and necessary condition for $\operatorname{Im}\sigma(L)$ to be bounded.

Theorem 1.1 *The set* $\operatorname{Im}\sigma(L)$ *is bounded if and only if* $a(x)$ *satisfies the condition:*

$$\limsup_{|x-y|\to+\infty} \frac{\left|\int_y^x \operatorname{Re} a(s)\, ds\right|}{|x-y|} = 0. \tag{1.3}$$

This indicates that the notion of the L^2-well-posedness is more restrictive than that of the boundedness of $\operatorname{Im}\sigma(L)$. Indeed, (1.3) is strictly weaker than (1.2), because, for example, the following function

$$a(x) = \frac{x}{x^2+1}$$

satisfies (1.3), but not (1.2).

We sketch the proof of Theorem 1.1. For a complex number λ, let $\mathscr{P}_\lambda(L)$ be the set of all non-trivial and classical solutions to the equation $(L-\lambda)u = 0$, that is,

$$\mathscr{P}_\lambda(L) = \left\{ u \in \boldsymbol{C}^2(\boldsymbol{R}) \setminus \{0\} \;\middle|\; (L-\lambda)u = 0 \right\}.$$

We note that $\mathscr{P}_\lambda(L)$ is represented as

$$\mathscr{P}_\lambda(L) = \left\{ \exp\left(\frac{1}{2}\int_0^x a(s)\, ds\right) v \;\middle|\; v \in \mathscr{P}_\lambda(S) \right\}.$$

The proof of Theorem 1.1 relies on the result established by Y. Oshime [3], which is a characterization of the resolvent set of L by global behavior of the basis of $\mathscr{P}_\lambda(L)$. We analyze $\mathscr{P}_\lambda(S)$ for the sake of the study of $\mathscr{P}_\lambda(L)$ and prove Theorem 1.1. We introduce Oshime's result in the next section. We present a more precise explanation of the proof of Theorem 1.1 in section 3.

2 Oshime's result

Here we introduce a part of the result [3] briefly which is concerned with the proof of Theorem 1.1. First we define a notion of global behavior of function.

Definition 2.1 A non-vanishing function $f(x)$ on $\boldsymbol{R}$ is said to *decay uniformly exponentially* if there exist positive constants K and ε such that

$$\frac{|f(x)|}{|f(y)|} \le K e^{-\varepsilon|x-y|} \quad \text{for all } x, y \in \boldsymbol{R} \text{ satisfying } x \ge y.$$

Similarly, we say that $f(x)$ *grows uniformly exponentially* if there exist positive constants K and ε

$$\frac{|f(x)|}{|f(y)|} \le K e^{-\varepsilon|x-y|} \quad \text{for all } x, y \in \boldsymbol{R} \text{ satisfying } x \le y.$$

The notion of uniformly exponential decaying or growing is stronger than that of exponential decaying or growing respectively.

Example 2.2 Let k be a number greater than one, and set

$$f_k(x) = \exp\left(-kx + x \sin\log\sqrt{x^2+1}\right).$$

Obviously f_k decays exponentially. However one can see that f_k does not decay uniformly exponentially in the case where $k \le \sqrt{2}$. On the other hand, in the case where $k > \sqrt{2}$, f_k decays uniformly exponentially.

We remark that for each $u \in \bigcup_{\lambda \in C} \mathscr{P}_\lambda(L)$, $|u|+|u'|$ does not vanish thanks to a uniqueness of solutions.

The following theorem is main result of [3] which characterizes the resolvent set of L by above properties. This is main tool for the proof of Theorem 1.1.

Theorem 2.3 ([3], Theorem 21, 22, 23) *A complex number λ is an element of $\rho(L)$ if and only if one of the following statements holds.*

(i) *For any $u \in \mathscr{P}_\lambda(L)$, $|u|+|u'|$ grows uniformly exponentially.*

(ii) *For any $u \in \mathscr{P}_\lambda(L)$, $|u|+|u'|$ decays uniformly exponentially.*

(iii) *There exist two functions $u_+, u_- \in \mathscr{P}_\lambda(L)$ such that $|u_+|+|u'_+|$ decays uniformly exponentially and $|u_-|+|u'_-|$ grows uniformly exponentially.*

The resolvent set $\rho(L)$ are classified into three subsets, according to **(i)**, **(ii)** and **(iii)**. They are denoted by $\rho_-(L)$, $\rho_+(L)$ and $\rho_0(L)$. The next proposition describes topological properties of these subsets.

Proposition 2.4 ([3], Proposition 11) *$\rho_+(L)$, $\rho_-(L)$ and $\rho_0(L)$ are mutually disjoint open sets.*

This plays a crucial role in the proof of the necessary part of Theorem 1.1, that is, (1.3) follows from the boundedness of $\mathrm{Im}\,\sigma(L)$.

The following theorem is the characterization of the resolvent set for Schrödinger operators with complex-valued potentials.

Theorem 2.5 ([3], Theorem 24) *If (1.2) holds, then $\rho(L) = \rho_0(L)$.*

This theorem gives us an information about the location of $\sigma(L)$ under the condition (1.3).

3 Proof of Theorem 1.1

In this section, we mention the sketch of the proof of Theorem 1.1. The main difficulty is to show the necessity of (1.3) for $\operatorname{Im}\sigma(L)$ to be bounded. The sufficiency is obtained as a direct consequence of Theorem 2.3. Before going into the detail, we define a notation. For an $f \in \boldsymbol{C}^1(\boldsymbol{R})$ satisfying $|f(x)| + |f'(x)| \neq 0$, we set

$$Q[f](x,y) = \frac{|f(x)| + |f'(x)|}{|f(y)| + |f'(y)|}.$$

If we set $u(x) = v(x)\exp\left(\frac{1}{2}\int_0^x a(s)\,ds\right)$, then we have by simple calculation

$$\frac{1}{K} \leq \frac{Q[u](x,y)}{Q[v](x,y)} \exp\left(\frac{1}{2}\int_x^y \operatorname{Re} a(s)\,ds\right) \leq K \quad \text{for all } x, y \in \boldsymbol{R}, \tag{3.1}$$

where K is a positive constant independent of x and y.

3.1 Sufficiency

In this subsection, we prove that $\operatorname{Im}\sigma(L)$ is bounded under the condition (1.3). Since $\operatorname{Im}\sigma(S)$ is bounded, it suffices for the proof to show that $\rho(S) \subset \rho(L)$. From the hypothesis (1.3), for any $\varepsilon > 0$, there exists a positive constant $M = M(\varepsilon, a)$ such that

$$\left|\int_y^x \operatorname{Re} a(s)\,ds\right| \leq \varepsilon\,|x-y| + M.$$

Given $\lambda \in \rho(S)$, set $u = v\exp\left(\frac{1}{2}\int_0^x a(s)\,ds\right)$ for a $v \in \mathscr{P}_\lambda(S)$. Applying the above inequality to (3.1), we obtain for every $\varepsilon > 0$

$$\frac{e^{-M/2}}{K}\,e^{-(\varepsilon/2)|x-y|} \leq \frac{Q[u](x,y)}{Q[v](x,y)} \leq K\,e^{M/2}\,e^{(\varepsilon/2)|x-y|} \quad \text{for all } x, y \in \boldsymbol{R}.$$

This reveals that $|u| + |u'|$ grows or decays uniformly exponentially if $|v| + |v'|$ does. In view of Theorem 2.3, we obtain $\lambda \in \rho(L)$ and the sufficiency follows.

Remark 3.1 We can prove that $\rho(S) = \rho_0(S) = \rho_0(L) = \rho(L)$ with the aid of Theorem 2.5. By this relation, the study of the location of the spectrum of L reduces to that of Schrödinger type operators, which has been analyzed by many authors (see [1]).

3.2 Necessity

We shall prove that (1.3) holds if $\operatorname{Im}\sigma(L)$ is bounded. For this, we show the contraposition of this statement, that is, $\operatorname{Im}\sigma(L)$ is unbounded if (1.3) fails to hold. In what follows, we assume that (1.3) does not hold. First we define some notations. Let $I(\lambda)$ be a positive function of λ defined in $\{\lambda \in \boldsymbol{C} | \operatorname{Re}\lambda > 0\}$ such that

$$I(\lambda) = \sup_{x\in\boldsymbol{R}} |V(x) - i \operatorname{Im}\lambda| \Big/ \sqrt{\operatorname{Re}\lambda}.$$

This function represents the growth order of functions of $\mathscr{P}_\lambda(S)$. For simplicity, we set

$$\delta = \limsup_{|x-y|\to\infty} \frac{\left|\displaystyle\int_y^x \operatorname{Re} a(s)\, ds\right|}{|x-y|}.$$

This is a positive number under the assumption. Finally, let Ω be a subset of $\boldsymbol{C}$ defined as

$$\Omega = \{\lambda \in \boldsymbol{C} \mid \operatorname{Re}\lambda > 0,\ I(\lambda) < \delta/2\}.$$

We note that Ω is open and $\operatorname{Im}\Omega$ is unbounded, moreover, any line parallel to the real axis intersects with Ω.

Now we introduce two propositions which lead us to the conclusion of the necessity part of Theorem 1.1.

Proposition 3.2 *There exists a negative number R such that*

$$\{\lambda \in \boldsymbol{C} \mid \operatorname{Re}\lambda < R\} \subset \rho_0(L).$$

Proposition 3.3 *If (1.3) is not satisfied, then $\rho_0(L) \cap \Omega = \emptyset$ holds.*

Remark 3.4 Proposition 3.2 holds even though (1.3) does not hold.

We will sketch the proof of them later. If we admit them, the proof of the necessity part proceeds as follows. Given an arbitrary $q \in \boldsymbol{R}$, let ℓ be the line $\boldsymbol{R} + i\,q$ which is parallel to the real axis. By Proposition 3.2, we see that ℓ intersects with $\rho_0(L)$. Set $p = \sup_{\lambda\in\ell\cap\rho_0(L)} \operatorname{Re}\lambda$. This is a finite number thanks to Proposition 3.3. Since $p + i\,q$ is a boundary point of $\rho_0(L)$, it does not belong to $\rho_0(L)$. By Proposition 2.4, we have $p + i\,q \notin \rho_+(L) \cup \rho_-(L)$. This implies $p + i\,q \in \sigma(L)$ due to Theorem 2.3. Consequently we obtain that $\operatorname{Im}\sigma(L)$ is unbounded since q is arbitrary. Now the proof of Theorem 1.1 is completed.

In the rest of this paper, we sketch the proof of Propositions 3.2 and 3.3.

Sketch of the proof of Proposition 3.2 We set $D_R = \{\lambda \in \boldsymbol{C} | \operatorname{Re}\lambda < R\}$ for $R < 0$. We can see that D_R is contained in $\rho(L)$ for some $R < 0$ in a usual manner of the spectral analysis. Therefore it suffices for the proof of Proposition 3.2 to show that we can choose an R' smaller than R so that $D_{R'} \cap \rho(L) \subset \rho_0(L)$. For this, we need an information about $\mathscr{P}_\lambda(S)$, which is given by the following lemma.

Lemma 3.5 *For any* $\lambda \in \boldsymbol{C}$ *satisfying* $\operatorname{Re}\lambda < -2\sup|\operatorname{Re}V(x)|$, *there exists a* $v \in \mathscr{P}_\lambda(S)$ *such that*

$$|v(x)|^2 \geq \cosh\sqrt{-\operatorname{Re}\lambda}\,x \quad \textit{for all } x \in \boldsymbol{R}.$$

This lemma implies that if we take a sufficiently small R', then for every $\lambda \in D_{R'}$ there exists a $u \in \mathscr{P}_\lambda(L)$ such that

$$|u(x)|^2 \geq \exp\left(\left(\sqrt{-\operatorname{Re}\lambda} - \sup_{s\in\boldsymbol{R}}|a(s)|\right)|x|\right).$$

The right hand side tends to $+\infty$ as $|x|$ goes to $+\infty$, so u neither grows nor decays uniformly exponentially. By means of Theorem 2.3, we have $D_{R'} \notin \rho_+(L) \cup \rho_-(L)$. Now we proved $D_{R'} \cap \rho(L) \subset \rho_0(L)$. And hence Proposition 3.2 applies.

Sketch of the proof of Proposition 3.3 As in the proof of Proposition 3.2, we need an information about the exponential growth order of elements of $\mathscr{P}_\lambda(S)$.

Lemma 3.6 *For any* $\lambda \in \boldsymbol{C}$ *satisfying* $\operatorname{Re}\lambda > 0$ *and any* $v \in \mathscr{P}_\lambda(S)$, *the following estimate holds.*

$$Q[v](x,y) \leq K\exp\left(\frac{I(\lambda)}{2}|x-y|\right) \quad \textit{for all } x,y \in \boldsymbol{R},$$

where K *is a positive constant depending only on* λ.

We admit this lemma and prove Proposition 3.3. Suppose that (1.3) fails to hold. Given an arbitrary $\lambda \in \Omega$, we shall show that $\lambda \notin \rho_0(L)$. From the assumption, we can choose a sequence $\{(x_j, y_j)\}$ in $\boldsymbol{R}^2$ so that $x_j - y_j \to +\infty$ as $j \to +\infty$ and one of the following **(i)** and **(ii)** holds:

$$\text{(i)} \quad \int_{y_j}^{x_j} \operatorname{Re}a(s)\,ds > (\delta - 2\,I(\lambda))(x_j - y_j) \quad (j = 0,1,\ldots).$$

$$\text{(ii)} \quad \int_{y_j}^{x_j} \operatorname{Re}a(s)\,ds < -(\delta - 2\,I(\lambda))(x_j - y_j) \quad (j = 0,1,\ldots).$$

For simplicity, we restrict ourselves to the case **(i)**. As a direct consequence of Lemma 3.6, we have for any $v \in \mathscr{P}_\lambda(S)$

$$Q[v](x,y) \geq \frac{1}{K}\exp\left(\frac{I(\lambda)}{2}|x-y|\right) \quad \text{for all } x,y \in \boldsymbol{R}.$$

This implies, together with (3.1), for every $u \in \mathscr{P}_\lambda(L)$

$$Q[u](x_j,y_j) \geq K'\exp\left(\frac{\delta - 2\,I(\lambda)}{2}|x_j - y_j|\right) \quad (j = 0,1,\ldots).$$

Here K' is a positive constant independent of u and j. The right hand side of the above inequality tends to $+\infty$ as j goes to $+\infty$. This indicates that there is no element of $\mathscr{P}_\lambda(L)$ decaying uniformly exponentially. Therefore Theorem 2.3 gives $\lambda \notin \rho_0(L)$. Now the proof of Proposition 3.3 is completed.

References

[1] E. B. Davies, *Non-self-adjoint differential operators*, Bull. London Math. Soc. **34** (2002), 513–532.

[2] S. Mizohata, *On the Cauchy problem*, Academic Press, Inc., Orlando, FL; Science Press, Beijing, 1985.

[3] Y. Oshime, *On the second order ordinary differential operator with complex coefficients without decay*, preprint