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A scaling limit for quantum field models

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Abstract

We study a scaling limit for the generarized spin-boson model and a generalization of the Nelson model. Applying it to a model for the field of the nuclear force with isospin, we obtain an effective potential of the interaction between nucleons. Also, we get some applications to condensed matter physics.

1 Introduction

We consider a scaling limit of abstract quantum field theoritical Hamiltonians for interaction models between particles and a Bose field. The purpose of this paper is to derive a quantum mechanical Hamiltonian in a scaling limit of such a quantum field theoritical Hamiltonian in a general framework.

A typical example is a scaling limit for an interaction model, called the *Nelson model* [8], of norelativistic quantum particles coupled to a Bose field whose Hamiltonian is given by

$$H=-\frac{1}{2M}\Delta\otimes I+I\otimes H_{\mathrm{b}}+gH_{\mathrm{I}},$$

where $M>0$ denotes the mass of the particles, Δ the generalized Laplacian, H_{b} the free Hamiltonian of the Bose field, H_{I} an interaction between the particles and the Bose field, $g\in\mathbb{R}$ a coupling constant which represents the strength of the interaction. A scaled Hamiltonian of H is introduced by

$$H(\Lambda)=-\frac{1}{2M}\Delta\otimes I+\Lambda^{2}I\otimes H_{\mathrm{b}}+g\Lambda H_{\mathrm{I}},\ \Lambda>0.$$

Hiroshima [4, 5] showed that, under suitable conditions, there exists a symmetric operator V_{eff}, called an *effective potential*, such that

$$\mathrm{s}\text{-}\lim_{\Lambda\to\infty}(H(\Lambda)-z)^{-1}=\left(-\frac{1}{2M}\Delta+V_{\mathrm{eff}}-z\right)^{-1}\otimes P_{0}, \tag{1.1}$$

for all $z\in\mathbb{C}\setminus\mathbb{R}$, where P_0 denotes the orthogonal projection onto $\ker H_{\mathrm{b}}$. Physically, a vector belonging to the subspace $\ker H_{\mathrm{b}}$ represents the vacuum of the free Bose field. Therefore one obtains a quantum mechanical Hamiltonian, called a *Schrödinger Hamiltonian*, in the vacuum of the free Bose field in the resolvent sense. Indeed, the limit (1.1) implies that, for all $t \in \mathbb{R}$,

$$\text{s-}\lim_{\Lambda \to \infty} e^{-itH(\Lambda)}(I \otimes P_0) = e^{-it(-\frac{1}{2M}\Delta + V_{\mathrm{eff}})} \otimes P_0. \tag{1.2}$$

According to Davies [3], the limit (1.2) is the weak coupling limit at the same time as the mass of the particles becomes infinity, since we can write

$$H(\Lambda) = \Lambda^2 \left(-\frac{1}{2M\Lambda^2}\Delta \otimes I + I \otimes H_{\mathrm{b}} + \frac{g}{\Lambda} H_{\mathrm{I}} \right),$$

where the factor Λ^2 on the whole Hamiltonian is interpreted as a time scaling.

On the other hand, Arai [1] studied scaling limits for a spin-boson interaction model, called the *spin boson model*, and a model in nonrelativistic quantum electrodynamics, called the *Pauli-Fierz model*, in the dipole approximation without the self-interaction of photons. The methods in [1] have been extended to the *generalized spin boson (GSB) model* [2] and the Pauli-Fierz model with the self-interaction of photons ([6] and the refernces therein).

In this paper, we study a scaling limit for the GSB model and a generalization of the Nelson model. Various branches of physics, such as nuclear physics and condensed matter physics, have many examples of these models, and the interaction H_{I} depends on models (see [2, 12]). From this point of view, it seems natural to consider scaling limits of these general models under conditions as weak as possible.

This paper is organized as follows. In Sec. 2, we introduce some notions, and discuss an abstract scaling limit theorem. In Sec. 3, we introduce the Boson Fock space and define the GSB model. We state a scaling limit for the GSB model under weaker conditions than those in [2]. A scaling limit for the generalization of the Nelson model is treated in Sec. 4. This model describes nonrelativistic quantum particles coupled to a Bose field with some internal degrees of freedom. As a result, we are now able to derive an effective potential that is an operator valued potential in the weak coupling limit. Note that, since the Nelson model has no internal degrees of freedom, the effective potential is a scalar potential. However, in nuclear physics, matrix valued potentials appear as effective potentials. A new feature of our work is in that a quantum mechanical Hamiltonian with such a potential is derived. In the last section, we discuss some examples. The first two examples are concrete realizations of the GSB model in condensed matter physics; the last one the generalization of the Nelson model in nuclear physics.

2 Preliminaries

In this section, we describe an abstract scaling limit theorem ([1, 4, 12]) in convenient form to establish scaling limits for our models. We denote the inner product and the associated norm of a Hilbert space $\mathcal{L}$ by $\langle\cdot,\cdot\rangle_{\mathcal{L}}$ and $\|\cdot\|_{\mathcal{L}}$, respectively. If there is no danger of confusion, we omit the subscript $\mathcal{L}$ in $\langle\cdot,\cdot\rangle_{\mathcal{L}}$ and $\|\cdot\|_{\mathcal{L}}$. Moreover, the domain and range of an operator T is denoted by $D(T)$ and $\mathrm{Ran}(T)$.

To begin with, we introduce the following notions which are useful for describing a condition of a scaling limit theorem.

Definition 2.1 *Let $\mathcal{L}$ be a Hilbert space, a point t_0 in an interval $I_0 \subset [-\infty,+\infty]$, and $L(t)$ and $M(t)$ $(t \in I_0)$ operators on $\mathcal{L}$ satisfying*

$$\bigcap_{t\in I_0} D(L(t)) \neq \emptyset.$$

(1) We say that $M(t)$ is $L(t)$-bounded uniformly near t_0 if there exist a neighborhood $I \subset I_0$ of t_0 and constants $a, b \geq 0$ such that for any $t \in I \setminus \{t_0\}$, $D(M(t)) \supset D(L(t))$ and

$$\|M(t)\Psi\| \leq a\|L(t)\Psi\| + b\|\Psi\|, \qquad \Psi \in D(L(t)).$$

(2) We say that $M(t)$ is $L(t)$-infinitesimally small uniformly near t_0 if for any $\varepsilon > 0$, there exist an interval $I(\varepsilon) \subset I_0$ and a constant $b(\varepsilon)$ such that for any $t \in I(\varepsilon) \setminus \{t_0\}$, $D(M(t)) \supset D(L(t))$ and

$$\|M(t)\Psi\| \leq \varepsilon\|L(t)\Psi\| + b(\varepsilon)\|\Psi\|, \qquad \Psi \in D(L(t)).$$

Note that, from the Kato-Rellich Theorem, if $M(t)$ is $L(t)$-infinitesimally small uniformly near t_0, then $L(t) + M(t)$ is self-adjoint on $D(L(t))$ for all $t \in I \setminus t_0$ with some neighborhood I of t_0, and moreover, if $L(t)$ is bounded from below, then so is $L(t) + M(t)$.

Let A be a non-negative self-adjoint operator on a Hilbert space $\mathcal{H}$ and B a non-negative self-adjoint operator on a Hilbert space $\mathcal{K}$ with

$$\ker B \neq \{0\}.$$

We denote by P_{B} the orthogonal projection onto $\ker B$ from $\mathcal{K}$. Let $\{C_\Lambda\}_{\Lambda>0}$ be symmetric operators on $\mathcal{X} := \mathcal{H} \otimes \mathcal{K}$. Put

$$K(\Lambda) := K_0(\Lambda) + C_\Lambda,$$

where

$$K_0(\Lambda) := A \otimes I + \Lambda I \otimes B.$$

We consider the scaling limit of $K(\Lambda)$.

Theorem 2.1 *(scaling limit [1, 4, 13]) Suppose that C_Λ is $K_0(\Lambda)$-infinitesimally small uniformly near ∞ and there exists a symmetric operator C on $\mathcal{X}$ such that $D(C) \supset D(A) \otimes \ker B$ and*

$$\text{s-}\lim_{\Lambda\to\infty} C_\Lambda (A-z)^{-1} \otimes P_B = C(A-z)^{-1} \otimes P_B. \tag{2.1}$$

Then, the following (1)-(3) hold.
(1)For any $\Lambda > \Lambda_0$ with some Λ_0, $K(\Lambda)$ is self-adjoint on $D(K_0)$ and bounded from below uniformly in $\Lambda > \Lambda_0$. Moreover, it is essentially self-adjoint on any core for K_0.
(2)The operator

$$K_\infty := A \otimes I + (I \otimes P_B) C (I \otimes P_B)$$

is self-adjoint on $D(A \otimes I)$ and bounded from below. Moreover, it is essentially self-adjoint on any core for $A \otimes I$.
(3)For any $z \in \mathbb{C} \setminus [0,\infty)$ or $z < 0$ with $|z|$ sufficiently large

$$\text{s-}\lim_{\Lambda\to\infty} (K(\Lambda) - z)^{-1} = (K_\infty - z)^{-1} (I \otimes P_B). \tag{2.2}$$

Proof. See [13]. □

If $\ker H_b = \{\alpha\Omega_B \mid \alpha \in \mathbb{C}\}$ with some $\Omega_B \in \mathcal{K}$ ($\|\Omega_B\| = 1$), there exists a symmetric operator $E_B(C)$ such that

$$\langle f, E_B(C) g\rangle = \langle f \otimes \Omega_B, C\,(g \otimes \Omega_B)\rangle, \quad f \in \mathcal{H}, \; g \in D(A),$$

and

$$(I \otimes P_B) C (I \otimes P_B) = E_B(C) \otimes P_B.$$

Hence, we have

$$(K_\infty - z)^{-1} (I \otimes P_B) = (K_{\text{eff}} - z)^{-1} \otimes P_B,$$

where

$$K_{\text{eff}} = A + E_B(C).$$

We note the following fact.

Proposition 2.2 *Let H_n be self-adjoint operators acting on the tensor product of two Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$. Suppose that, there exists a self-adjoint operator H_∞ acting on $\mathcal{H}_1$ such that, for some $z_0 \in \mathbb{C} \setminus \mathbb{R}$,*

$$s\text{-}\lim_{n\to\infty} (H_n - z_0)^{-1} = (H_\infty - z_0)^{-1} \otimes P,$$

where P is an orthogonal projection from $\mathcal{H}_2$ onto $\mathrm{Ran}P$. *Then, for all* $t \in \mathbb{R}$,

$$s\text{-}\lim_{n\to\infty} e^{-itH_n}(I \otimes P) = e^{-itH_\infty} \otimes P.$$

Proof. We need only to prove taht, for all $t \in \mathbb{R}$,

$$\text{s-}\lim_{n\to\infty} \left(e^{-itH_n} - e^{-itH_\infty} \otimes P\right) (H_\infty - z_0)^{-1} \otimes P = 0.$$

We can write

$$\begin{aligned}
&\left(e^{-itH_n} - e^{-itH_\infty} \otimes P\right) (H_\infty - z_0)^{-1} \otimes P \\
&\quad = e^{-itH_n} \left[(H_\infty - z_0)^{-1} \otimes P - (H_n - z_0)^{-1}\right] (I \otimes P) \\
&\qquad + (H_n - z_0)^{-1} \left[e^{-itH_n} - e^{-itH_\infty} \otimes P\right] (I \otimes P) \\
&\qquad\quad + \left[(H_n - z_0)^{-1} - (H_\infty - z_0)^{-1} \otimes P\right] (e^{-itH_\infty} \otimes P).
\end{aligned}$$

In the same way as in [7, p.503, Theorem 2.14], we can prove that

$$\text{s-}\lim_{n\to\infty} (H_n - z_0)^{-1} \left[e^{-itH_n} - e^{-itH_\infty} \otimes P\right] (H_0 - z_0)^{-1} \otimes P = 0.$$

Hence, we obtain the desired result. □

By Proposition 2.2, we obtain the following fact.

Corollary 2.3 *Let A, B, C and C_Λ be as above. Suppose that* $\ker H_{\mathrm{b}} = \{\alpha\Omega_B \mid \alpha \in \mathbb{C}\}$ *with* $\|\Omega_B\| = 1$. *Then, for all* $t \in \mathbb{R}$,

$$s\text{-}\lim_{\Lambda\to\infty} e^{-itK_\Lambda}(I \otimes P_B) = e^{-itK_{\mathrm{eff}}} \otimes P_B.$$

3 Scaling Limit for the GSB Model

3.1 Boson Fock space

To describe a Bose field, one uses the Boson Fock space over a complex Hilbert space $\mathcal{K}$:

$$\begin{aligned}
\mathcal{F}_{\mathrm{b}}(\mathcal{K}) :&= \bigoplus_{n=0}^{\infty} \bigotimes_{\mathrm{s}}^{n} \mathcal{K} \\
&= \left\{ \psi = \{\Psi^{(n)}\}_{n=0}^{\infty} \;\middle|\; n \geq 0,\ \ \Psi^{(n)} \in \bigotimes_{\mathrm{s}}^{n'} \mathcal{K},\ \ \sum_{n=0}^{\infty} \|\psi^{(n)}\|^2 < \infty \right\},
\end{aligned}$$

where $\bigotimes_s^n \mathcal{K}$ denotes the n-fold symmetric tensor product of $\mathcal{K}$ with $\bigotimes_s^0 \mathcal{K} := \mathbb{C}$.

The *annihilation operator* $a(f)(f \in \mathcal{K})$ is a densely defined closed linear operator on $\mathcal{F}_b(\mathcal{K})$ such that, for all $\psi = \{\Psi^{(n)}\}_{n=0}^{\infty} \in D(a(f)^*)$, $(a(f)^*\psi)^{(0)} = 0$ and

$$(a(f)^*\psi)^{(n)} = \sqrt{n} S_n(f \otimes \Psi^{(n-1)}), \quad n \geq 1,$$

where S_n is the symmetrization operator on $\bigotimes^n \mathcal{K}$ ($S_n{}^* = S_n$, $S_n{}^2 = S_n$, $\bigotimes_s^n \mathcal{K} = S_n(\bigotimes^n \mathcal{K})$). The adjoint $a(f)^*$, called the *creation operator*, and the annihilation operator $a(g)(g \in \mathcal{K})$ obey the *canonical commutation relations*

$$[a(f), a(g)^*] = \langle f, g \rangle, \quad [a(f), a(g)] = 0, \quad [a(f)^*, a(g)^*] = 0$$

for all $f, g \in \mathcal{K}$ on some dense subspace, where $[X, Y] = XY - YX$.

Let

$$\phi(f) := \frac{a(f) + a(f)^*}{\sqrt{2}}, \quad f \in \mathcal{K},$$

which is called the *Segal field operator*. It is shown that $\phi(f)$ is essentially self-adjoint on $\mathcal{F}_0(\mathcal{K})$ [10, §X.7]. We denote its closure by the same symbol $\phi(f)$.

For every self-adjoint operator T on $\mathcal{K}$, one can define a self-adjoint operator $d\Gamma(T)$, called the *second quantization* of T [9, p.302], by

$$d\Gamma(T) := \bigoplus_{n=0}^{\infty} T^{(n)},$$

with $T^{(0)} = 0$ and $T^{(n)}$ is the closure of

$$\left(\sum_{j=1}^{n} I \otimes \cdots \otimes \overset{j\text{th}}{\breve{T}} \otimes \cdots \otimes I \right) \Bigg| \bigotimes_{\text{alg}}^{n} D(T).$$

If T is non-negative, then so is $d\Gamma(T)$.

3.2 Definition of the GSB model

We consider a model of a quantum system $\mathcal{S}$ coupled to a Bose field. We denote the Hilbert space of the system $\mathcal{S}$ by $\mathcal{H}$ which is taken to be an arbitrary separable complex Hilbert space. In concrete realizations, $\mathcal{S}$ may be a system of quantum particles. We denote the one-boson Hilbert space by $\mathcal{K}$ which is taken to be an arbitrary separable complex Hilbert space. The

Hilbert space of the coupled system of $\mathcal{S}$ and the Bose field is given by the tensor product

$$\mathcal{F} := \mathcal{H} \otimes \mathcal{F}_{\mathrm{b}}(\mathcal{K}).$$

We assume that T is a non-negative, injective and self-adjoint operator on $\mathcal{K}$. Then, the free Hamiltonian of the Bose field is defined by

$$H_{\mathrm{b}} := d\Gamma(T)$$

acting on $\mathcal{F}_{\mathrm{b}}(\mathcal{K})$.

Suppose that A is a self-adjoint operator on $\mathcal{H}$ and bounded from below, which denotes physically the Hamiltonian of the quantum system $\mathcal{S}$. Let B_j ($j = 1, \ldots, J, J \in \mathbb{N}$) be bounded self-adjoint operators on $\mathcal{H}$ and $g_j \in \mathcal{K}$ ($j = 1, \ldots, J$). As a total Hamiltonian of the coupled system, we take the following operator:

$$H_{\mathrm{GSB}} := A \otimes I + I \otimes H_{\mathrm{b}} + g \sum_{j=1}^{J} B_j \otimes \phi(g_j), \tag{3.1}$$

where $g \in \mathbb{R}$ denotes a coupling constant of the system $\mathcal{S}$ and the Bose field. Such a Hamiltonian is called the *generalized spin-boson (GSB)* Hamiltonian introduced by Arai and Hirokawa [2]. Altough a scaling limit of the GSB model has been studied in [2], some assumptions are made. One of them is the commutativity of $\{B_j\}_{j=1}^{J}$:

$$[B_j, B_k] = 0, \quad j, k = 1, \cdots, J.$$

We study a scaling limit of the GSB model without this condition.

3.3 Scaling limit for the GSB model

To state main results of this section, we need some assumptions.

(A.1) The vectors $g_j (j = 1, \ldots, J)$ satisfy the following conditions:

$$g_j \in D(T^{-3/2}), \quad j = 1, \ldots, J, \tag{3.2}$$

and

$$\langle g_j, g_k \rangle \ , \ \langle g_j, T^{-1} g_k \rangle \ , \ \langle T^{-1} g_j, T^{-1} g_k \rangle \ \in \mathbb{R}, \quad j, k = 1, \ldots, J. \tag{3.3}$$

(A.2) There exists a dense subspace $\mathcal{D} \subset D(A)$ such that

$$B_j \mathcal{D} \subset \mathcal{D}, \quad j = 1, \cdots, J. \tag{3.4}$$

(A.3) $[B_j, A]|\mathcal{D}(j = 1, \cdots, J)$ are bounded.

We introduce a scaled Hamiltonian by

$$H_{\text{GSB}}(\Lambda) := A \otimes I + \Lambda^2 I \otimes H_{\text{b}} + g\Lambda H_{\text{I}}, \quad \Lambda > 0, \tag{3.5}$$

where

$$H_{\text{I}} := \sum_{j=1}^{J} B_j \otimes \phi(g_j).$$

Let P_0 be the orthogonal projection onto $\ker H_{\text{b}}$ which is the one-dimensional subspace generated by the Fock vacuum $\Omega := \{1, 0, 0, \cdots\} \in \mathcal{F}_{\text{b}}(\mathcal{K})$:

$$\ker H_{\text{b}} = \{\alpha\Omega \mid \alpha \in \mathbb{C}\}.$$

Then, we obtain the following result which is one of the main theorems in this paper.

Theorem 3.1 *Assume (A.1)-(A.3). Let $z \in \mathbb{C} \setminus \mathbb{R}$ or $z < 0$ with $|z|$ surfficiently large. Then,*

$$\text{s-}\lim_{\Lambda \to \infty} (H_{\text{GSB}}(\Lambda) - z)^{-1} = (A + V_{\text{eff}} - z)^{-1} \otimes P_0, \tag{3.6}$$

where

$$V_{\text{eff}} = -\frac{g^2}{2} \sum_{j,k} \langle T^{-1} g_j, g_k \rangle B_k B_j. \tag{3.7}$$

Proof. See [12]. □

If A is a bounded self-adjoint operator on $\mathcal{H}$, then (A.2) and (A.3) hold. Therefore, Theorem 3.1 implies the following corollary:

Corollary 3.2 *Suppose that (A.1) holds and that A is bounded. Then, (3.6) holds for all $z \in \mathbb{C} \setminus \mathbb{R}$ or $z < 0$ with $|z|$ surfficiently large.*

We can state a result of another type without the condition (A.3). To do this, we introduce some objects. We denote by $[B_j, A]$ the closure of $[B_j, A]|\mathcal{D}(j = 1, \cdots, J)$. Put $\mu_0 := \inf \sigma(A)$ and

$$\tilde{A} := A - \mu_0,$$

which is a non-negative self-adjoint operator on $\mathcal{H}$. We need the following assumption.

(A.4) $\mathcal{D}$ is a core for A and $[B_j, A]$ $(j = 1, \cdots, J)$ are $\tilde{A}^{1/2}$-bounded, i.e. $D(\tilde{A}^{1/2}) \subset D([B_j, A])$ and there exist constants $a_j, b_j \geq 0$ such that, for all $u \in D(\tilde{A}^{1/2})$,

$$\|[B_j, A]u\| \leq a_j \left\|\tilde{A}^{1/2}u\right\| + b_j\|u\|. \tag{3.8}$$

Moreover, $[B_j, A]|\mathcal{D}$ $(j = 1, \cdots, J)$ are commuting with $B_k (k = 1, \cdots, J)$ on $\mathcal{D}$.

Then, we obtain the following theorem:

Theorem 3.3 *Assume (A.1), (A.2) and (A.4). Then, (3.6) holds for all* $z \in \mathbb{C} \setminus \mathbb{R}$ *or* $z < 0$ *with* $|z|$ *surfficiently large.*

Proof. See [12]. □

4 Scaling limit for a generalization of the Nelson model

4.1 Definition of the model

In this section, we study a model of a quantum system $\mathcal{S}$ coupled to a Bose field with some internal degrees of freedom. We denote the Hilbert space of the system $\mathcal{S}$ by $L^2(\mathbb{R}^d; \mathcal{H})$. Here $d \in \mathbb{N}$ and $\mathcal{H}$ is taken to be an arbitary separable complex Hilbert space. In concrete realizations, $\mathcal{S}$ may be a system of quantum particles with some internal degrees of freedom such as spin and isospin. The Hilbert space of the coupled system of $\mathcal{S}$ and the Bose field is given by the tensor product

$$\mathcal{F} := L^2(\mathbb{R}^d; \mathcal{H}) \otimes \mathcal{F}_{\mathrm{b}}(\mathcal{K}) \simeq \mathcal{H} \otimes L^2(\mathbb{R}^d; \mathcal{F}_{\mathrm{b}}(\mathcal{K})). \tag{4.1}$$

We defined the *quantized scalar field* by

$$\Phi(g) := \int_{\mathbb{R}^d}^{\oplus} \phi(g(x))dx$$

on $L^2(\mathbb{R}^d; \mathcal{F}_{\mathrm{b}}(\mathcal{K}))$, where $g : x \in \mathbb{R}^d \mapsto g(x) \in \mathcal{K}$ denotes a strongly continuous function. Then, $\Phi(g)$ is self-adjoint (see [11, Theorem XIII.85 (b)]).

Now we define a total Hamiltonian H acting on $\mathcal{F}$ by

$$H_{\mathrm{GN}} := -\Delta \otimes I + I \otimes H_{\mathrm{b}} + gH_{\mathrm{I}}, \tag{4.2}$$

where $g \in \mathbb{R}$ denotes a coupling constant, Δ the generalized Laplacian and

$$H_{\mathrm{I}} := \sum_{j=1}^{J} B_j \otimes \Phi(g_j). \tag{4.3}$$

Here, $B_j (j = 1, \cdots, J)$ are bounded self-adjont operators on $\mathcal{H}$. The functions $g_j (j = 1, \cdots, J, J \in \mathbb{N})$ from $\mathbb{R}^d$ to $\mathcal{K}$ are strongly continuous.

Example 4.1 *(the Nelson model)* If $\dim \mathcal{H} = 1$ and $B_j = 1$, then the Hamiltonian H_{NR} is written as

$$H_{\mathrm{Nelson}} := -\Delta \otimes I + I \otimes H_{\mathrm{b}} + g \sum_{j=1}^{J} \Phi(g_j)$$

on $L^2(\mathbb{R}^d; \mathcal{F}_{\mathrm{b}}(\mathcal{K}))$, which is called the *Nelson Hamiltonian.* The weak coupling limit of this Hamiltonian is studied by Hiroshima [4, 5].

4.2 Scaling limit for the model

To begin with, we introduce a scaled Hamiltonian $H(\Lambda)$ $(\Lambda > 0)$ by

$$H_{\mathrm{GN}}(\Lambda) := -\Delta \otimes I + \Lambda^2 I \otimes H_{\mathrm{b}} + g\Lambda H_{\mathrm{I}}. \tag{4.4}$$

In order to describe our result, we introduce some notations, and formulate our assumption.

We denote by $L^\infty(\mathbb{R}^d; \mathcal{K})$ the set of mesurable functions $f : \mathbb{R}^d \mapsto \mathcal{K}$ for which

$$\|f\|_\infty := \operatorname{ess.} \sup_{x \in \mathbb{R}^d} \|f(x)\|_{\mathcal{K}} < \infty.$$

For $\alpha \in \mathbb{R}$, we define a $\mathcal{K}$-valued function $T^\alpha f$ on $\mathbb{R}^d$ as follows: if $f(x) \in D(T^\alpha)$ a.e.$x \in \mathbb{R}^d$ with respect to Lebesuge mesure,

$$(T^\alpha f)(x) := T^\alpha f(x).$$

Definition 4.1 *Let $\alpha \in \mathbb{R}$. $L^\infty_\alpha(\mathbb{R}^d; \mathcal{K})$ denotes the set of $\mathcal{K}$-valued functions f on $\mathbb{R}^d$ satisfing the following conditions:*
(i) f is strongly continuous with $f \in L^\infty(\mathbb{R}^d; \mathcal{K})$.
(ii) $f(x) \in D(T^\alpha)$ $(x \in \mathbb{R})$ and $T^\alpha f \in L^\infty(\mathbb{R}^d; \mathcal{K})$.

A $\mathcal{K}$-valued function f on $\mathbb{R}^d$ is said to be diferentiable with respect to x_μ if the net

$$\frac{f(x_1, \cdots, x_\mu + \varepsilon, \cdots, x_d) - f(x)}{\varepsilon} \tag{4.5}$$

converges as $\varepsilon \to 0$ for any $x = (x_1, \cdots, x_d) \in \mathbb{R}^d$. Then, we denote the limit of (4.5) by $\partial_\mu f$. One can define the n times diferentiability ($n \in \mathbb{N}$), inductively:

$$\partial_\mu^n f := \partial_\mu(\partial_\mu^{(n-1)} f), \ n \geq 1.$$

(A.5) *The functions $g_j (j = 1, \cdots, J)$ are twice diferensiable and satisfy the following conditions:*
(i) $g_j \in L^\infty_{-3/2}(\mathbb{R}^d; \mathcal{K})$.
(ii) $\partial_\mu(T^{-1}g_j) \in L^\infty_{-1/2}(\mathbb{R}^d; \mathcal{K}) \cap L^\infty_{1/2}(\mathbb{R}^d; \mathcal{K})$ for $\mu = 1, \cdots, d$.
(iii) $\partial_\mu^2(T^{-1}g_j) \in L^\infty_{-1/2}(\mathbb{R}^d; \mathcal{K})$ for $\mu = 1, \cdots, d$.
Moreover, we assume that for any $j, k = 1, \cdots, J$ and $x \in \mathbb{R}^d$

$$\langle g_j(x), g_k(x)\rangle, \ \ \langle g_j(x), T^{-1}g_k(x)\rangle, \ \ \langle T^{-1}g_j(x), T^{-1}g_k(x)\rangle \in \mathbb{R}. \tag{4.6}$$

We are now ready to describe our result. Let

$$V_{\text{eff}} = -\frac{g^2}{2} \sum_{1 \leq j,k \leq J} B_k B_j V_{j,k}, \tag{4.7}$$

on $L^2(\mathbb{R}^d; \mathcal{H})$, where

$$V_{j,k}(x) = \left\langle g_j(x), T^{-1}g_k(x)\right\rangle, \ \ \text{a.e.} x \in \mathbb{R}. \tag{4.8}$$

Theorem 4.1 *Assume (A.5). Let $z \in \mathbb{C} \setminus \mathbb{R}$ or $z < 0$ with $|z|$ sufficiently large. Then,*

$$\text{s-}\lim_{\Lambda \to \infty} (H_{\text{GN}}(\Lambda) - z)^{-1} = (H_{\text{eff}} - z)^{-1} \otimes P_0, \tag{4.9}$$

where

$$H_{\text{eff}} = -\Delta + V_{\text{eff}} \tag{4.10}$$

on $L^2(\mathbb{R}^d; \mathcal{H})$.

5 Examples

5.1 Lattice spin system interacting with a Bose field

Let Λ be a finite set of the ν-dimentional lattice $\mathbb{Z}^\nu$ and consider the case where an N component spin $\mathbb{S} = (S^{(1)}, S^{(2)}, \cdots S^{(N)})$ sits on each site $i \in \Lambda$ and each component $S^{(n)}$ on $\mathbb{C}^s$ ($s \in \mathbb{N}$) obeys the following anticommuting relations:

$$\{S^{(n)}, S^{(m)}\} = 2\delta_{nm}, \quad i = 1, \cdots, N.$$

The Hilbert space of this spin system is given by $\mathcal{H}_\Lambda = \bigotimes_{i\in\Lambda}\mathcal{H}_i$ with $\mathcal{H}_i = \mathbb{C}^s$, $i \in \Lambda$. The spin at site i is defined by $\mathbb{S}_i = (S_i^{(1)}, S_i^{(2)}, \cdots S_i^{(N)})$, $S_i{}^{(n)} = I \otimes \cdots \otimes S^{(n)} \otimes \cdots \otimes I$ with $S^{(n)}$ acting on $\mathcal{H}_i$. A Hamiltonian of the spin system interacting with a Bose field is given by

$$H_\Lambda := \left(- \sum_{(i,j)\subset\Lambda} J_{ij}\mathbb{S}_i \cdot \mathbb{S}_j \right) \otimes I + I \otimes H_{\mathrm{b}} + \alpha \sum_{i\in\Lambda}\sum_{n=1}^{N} S_i{}^{(n)} \otimes \phi(g_i{}^{(n)}),$$

acting in $\mathcal{H}_\Lambda \otimes \mathcal{F}_{\mathrm{b}}(L^2(\mathbb{R}^\nu))$, where $J_{ij} \in \mathbb{R}$, i, $j \in \Lambda$, are constants and $g_i{}^{(n)} \in L^2(\mathbb{R}^\nu)$, $i \in \Lambda$, $n = 1, \cdots, N$. Here, $\alpha \in \mathbb{R}$ is a coupling constant. This model is a general type of a lattice spin system interacting with a Bose field (see [2]), which is a concrete realization of the abstract model H in (3.1) with the following choice:

$$\mathcal{H} = \mathcal{H}_\Lambda, \quad \mathcal{K} = L^2(\mathbb{R}^\nu), \quad g = \alpha$$
$$A = - \sum_{(i,j)\subset\Lambda} J_{ij}\mathbb{S}_i \cdot \mathbb{S}_j, \quad T = \omega, \quad B_j = S_i{}^{(n)}, \quad g_j = g_i{}^{(n)},$$

where $\omega : \mathbb{R}^\nu \to [0, \infty)$ is a Borel measurable function, almost everywhere finite with respect to the Lebesgue measure on $\mathbb{R}^\nu$, physically denoting the dispersion relation of a free boson in momentum representation. Let

$$H(\lambda) := \left(- \sum_{(i,j)\subset\Lambda} J_{ij}\mathbb{S}_i \cdot \mathbb{S}_j \right) \otimes I + \lambda^2 I \otimes H_{\mathrm{b}} + \alpha\lambda \sum_{i\in\Lambda}\sum_{n=1}^{N} S_i{}^{(n)} \otimes \phi(g_i{}^{(n)}).$$

By applying Corollary 3.2, we obtain the following theorem:

Theorem 5.1 *Suppose that*

$$\omega^{-3/2} g_i{}^{(n)} \in L^2(\mathbb{R}^\nu), \quad i \in \Lambda, \quad n = 1, \cdots, N.$$

Then, for all $z \in \mathbb{C} \setminus \mathbb{R}$ or $z < 0$ with $|z|$ surfficiently large,

$$\operatorname*{s-lim}_{\lambda\to\infty} (H(\lambda) - z)^{-1} = (H_{\mathrm{eff}} - z)^{-1} \otimes P_0,$$

where

$$H_{\mathrm{eff}} = - \sum_{(i,j)\subset\Lambda} J_{ij}\mathbb{S}_i \cdot \mathbb{S}_j + \sum_{i\in\Lambda} E_i + \sum_{i\neq j} V_{i,j}$$

and

$$E_i = -\frac{\alpha^2}{2}\sum_{n=1}^{N}\left\|\frac{g_i^{(n)}}{\sqrt{\omega}}\right\|^2, \quad V_{i,j} = -\frac{\alpha^2}{2}\sum_{n,m}\left\langle\frac{g_i^{(n)}}{\sqrt{\omega}}, \frac{g_j^{(m)}}{\sqrt{\omega}}\right\rangle S_i^{(n)}S_j^{(m)}.$$

Proof. Note that the following anticommutation relations:

$$\{S_i^{(n)}, S_i^{(m)}\} = 2\delta_{nm}, \quad i = 1, \cdots, N.$$

□

Remark 5.1 *Physically, E_i and $V_{i,j}$ above are considered respectively as the self-energy of each spin and an effective interaction between two spins. In particular, the case where*

$$\nu = 3, \quad N = 3, \quad s = 2, \quad \omega(k) = |k|, \quad g_i^{(n)} = \widehat{\rho(\cdot - x_i)}/\sqrt{|k|}$$

is interesting. Here, x_i denots the coordinate of a lattic point and ρ a real distribution sutisfying $\hat{\rho}/|k|^{1/2}, \hat{\rho}/|k|^2 \in L^2(\mathbb{R}^3)$. This case is considered as a lattice spin system interacting with phonons.

5.2 Model of a Fermi field interacting with a Bose field

Let $\mathcal{F}_{\mathrm{f}}(\mathcal{L})$ be the fermion Fock space over the Hilbert space $\mathcal{L}$ and $\psi(f), f \in \mathcal{L}$, the fermion annihilation operators on $\mathcal{F}_f(\mathcal{L})$, which are bounded. We denote by H_{f} the second quantization operator of a self-adjoint operator T' acting on $\mathcal{L}$. Then, a Hamiltonian of a quantum system of a Fermi field interacting with a Bose field is given by

$$H := H_{\mathrm{f}} \otimes I + I \otimes H_{\mathrm{b}} + \alpha\sum_{j=1}^{J}\psi(f_j)^*\psi(f_j) \otimes \phi(g_j),$$

acting in $\mathcal{F}_{\mathrm{f}}(\mathcal{L}) \otimes \mathcal{F}_{\mathrm{b}}(K)$, where $f_j \in \mathcal{L}, j = 1, \cdots, J$ and $\alpha \in \mathbb{R}$ is a coupling constant. In the case $\mathcal{L} = L^2(\mathbb{R}^3; \mathbb{C}^2)$ and $\mathcal{K} = L^2(\mathbb{R}^3)$, this model may serve as a model of electrons interacting with phonons in a metal. (See [2].)

Let

$$H(\Lambda) = H_{\mathrm{f}} \otimes I + \Lambda^2 I \otimes H_{\mathrm{b}} + \alpha\Lambda\sum_{j=1}^{J}\psi(f_j)^*\psi(f_j) \otimes \phi(g_j).$$

Applying Theorem 3.1, we can prove the following theorem:

Theorem 5.2 *Suppose that (3.2),(3.3) and*

$$f_j \in D(T'), \quad j = 1, \cdots, J.$$

Then, for all $z \in \mathbb{C} \setminus \mathbb{R}$ or $z < 0$ with $|z|$ surfficiently large,

$$\text{s-}\lim_{\Lambda \to \infty} (H(\Lambda) - z)^{-1} = (H_{\text{eff}} - z)^{-1} \otimes P_0,$$

where

$$H_{\text{eff}} = H_{\text{f}} - \frac{\alpha^2}{2} \sum_{j,k} \langle g_j, T^{-1} g_k \rangle \psi(f_j)^* \psi(f_j) \psi(f_k)^* \psi(f_k).$$

Proof. It is well known or easy to see that, for $f \in D(T')$,

$$\psi(f) D(H_f) \subset D(H_f), \quad \psi(f)^* D(H_f) \subset D(H_f),$$

and

$$[H_f, \psi(f)^*] = \psi(T'f)^*, \quad [H_f, \psi(f)] = -\psi(T'f).$$

This implies (3.4) and $[\psi(f_j)^* \psi(f_j), H_f]$ are bounded. Thus, we obtain the desired result. □

5.3 Interaction between nucleons and pions with isospin

In this section, we give a concrete realization of Theorem 4.1, which is an interaction model between nucleons and pions with isospin (see [12, Section 5.1]).

Let $\sigma_j, \tau_j (j = 1, 2, 3)$ be the Pauli matrices:

$$\sigma_1 = \tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and

$$\sigma_j{}^{(i)} = 1_2 \otimes \cdots \otimes \overset{i\text{th}}{\overbrace{\sigma_j}} \otimes \cdots \otimes 1_2, \quad j = 1, 2, 3,$$

$$\tau_\alpha{}^{(i)} = 1_2 \otimes \cdots \otimes \overset{i\text{th}}{\overbrace{\tau_\alpha}} \otimes \cdots \otimes 1_2, \quad \alpha = 1, 2, 3,$$

where 1_2 is the 2×2 identity matrix. Physically, $\sigma^{(i)} = (\sigma_1{}^{(i)}, \sigma_2{}^{(i)}, \sigma_3{}^{(i)})$ and $\tau^{(i)} = (\tau_1{}^{(i)}, \tau_2{}^{(i)}, \tau_3{}^{(i)})$ denote the spin and the isospin of the ith particle, respectively. Set

$$\mathcal{H}_N := \left[\bigotimes^N \mathbb{C}^2 \right] \bigotimes \left[\bigotimes^N \mathbb{C}^2 \right].$$

If there is no danger of confusion, we denote the operators ${\sigma_j}^{(i)} \otimes (\bigotimes^N 1_2)$ and $(\bigotimes^N 1_2) \otimes {\tau_\alpha}^{(i)}$ acting on $\mathcal{H}_N$ by the same symbol ${\sigma_j}^{(i)}$ and ${\tau_\alpha}^{(i)}$, respectively. We denote by $\hbar$ the Planck constant divided by 2π. Put

$$B_{j,\alpha}^{(i)} := \frac{\hbar}{2}{\sigma_j}^{(i)}{\tau_\alpha}^{(i)}, \quad i = 1, \cdots, N, \; j, \alpha = 1, 2, 3,$$

which act on $\mathcal{H}_N$. It is straightforward to see that

$$\left[B_{j,\alpha}^{(i)}, B_{k,\beta}^{(l)}\right] = 0, \quad i \neq l, \; j, k, \alpha, \beta = 1, 2, 3. \tag{5.1}$$

By the anticommutativity of the Pauli matrices, it follows that, for $i = 1, \cdots, N$, $j, k, \alpha, \beta = 1, 2, 3$,

$$\left\{B_{j,\alpha}^{(i)}, B_{k,\beta}^{(i)}\right\} = \frac{\hbar^2}{4}\delta_{jk}\delta_{\alpha\beta}, \tag{5.2}$$

where $\{X, Y\} = XY + YX$ and δ_{ij} is Kronecker's delta.

We denote by m and c the mass of a pion and the speed of light, respectively. Let

$$\omega(k) = \sqrt{\hbar^2 k^2 c^2 + m^2 c^4} \; (k \in \mathbb{R}^3),$$

where ω denotes a dispersion relation of one free pion. Let

$$\mathcal{K} = \bigoplus^3 L^2(\mathbb{R}^3).$$

The function ω defines a multiplication operator on $\mathcal{K}$. We denote it by the same symbol ω:

$$\omega f := (\omega f_1, \omega f_2, \omega f_3), \quad f = (f_1, f_2, f_3) \in \mathcal{K}$$

with $f_i \in D(\omega)$. Let $H_{\mathrm{b}} = d\Gamma(\omega)$. Then, H_{b} represents the free Hamiltonian of the pion field.

Let

$$\phi_\alpha(f) := \phi(f_\alpha), \qquad f \in L^2(\mathbb{R}^3),$$

where

$$f_\alpha := (\delta_{\alpha 1} f, \delta_{\alpha 2} f, \delta_{\alpha 3} f).$$

We denote by ρ the density of a nucleon, which is a real distribution satisfying $\widehat{\partial_j \rho}/\sqrt{\omega} \in L^2(\mathbb{R}^3)$, where $\widehat{\partial_j \rho}$ denotes the Fourier transform of $\partial_j \rho$. Let

$$\Phi_\alpha(g_j^{(i)}) = \int_{\mathbb{R}^{3N}}^{\oplus} \phi_\alpha(g_j^{(i)}(x)) dx,$$

where

$$g_j^{(i)}(x) = -\frac{\sqrt{\hbar}}{\sqrt{(2\pi)^3\omega}}\widehat{\hbar\partial_j\rho}\; e^{-ik\cdot x_i}$$

for $x := (x_1, \cdots, x_N) \in \mathbb{R}^{3N}$. Here $x_i \in \mathbb{R}^3$ indicates the coordinate of the ith nucleon.

A Hamiltonian of spin-nucleons interacting with pions, acting on $\mathcal{H}_N \otimes L^2(\mathbb{R}^{3N}; \mathcal{F}_{\mathrm{b}}(\mathcal{K}))$, is defined by

$$H(\hbar, c, M) := -\frac{\hbar^2}{2M}\Delta\otimes I + \frac{\hbar}{2}\sum_{i=1}^{N}\sigma_3{}^{(i)}\otimes I + I\otimes H_{\mathrm{b}} + g\sum_{\substack{1\le i\le N\\ 1\le j,\alpha\le 3}} B_{j,\alpha}^{(i)}\otimes\Phi_\alpha(g_j{}^{(i)}),$$

where $g \in \mathbb{R}$ is a coupling constant.

Now, we define the scaled Hamiltonian by

$$H(\Lambda) := \frac{1}{\Lambda^2}H(\Lambda^2\hbar, \Lambda^2 c, \Lambda^2 M).$$

Then, we can write

$$H(\Lambda) = -\frac{\hbar^2}{2M}\Delta\otimes I + \frac{\hbar}{2}\sum_{i=1}^{N}\sigma_3{}^{(i)}\otimes I + \Lambda^2 I\otimes H_{\mathrm{b}} + g\Lambda H_{\mathrm{I}},$$

where

$$H_{\mathrm{I}} = \sum_{i=1}^{N}\sum_{1\le j,\alpha\le 3} B_{j,\alpha}^{(i)}\otimes\Phi_\alpha(g_j{}^{(i)}).$$

We now ready to derive a quantum mechanical Hamiltonian from $H(\Lambda)$. Let

$$H_{\text{eff}} = -\frac{\hbar^2}{2M}\Delta + \frac{\hbar}{2}\sum_{i=1}^{N}\sigma_3{}^{(i)} + \sum_{1\le i<l\le N} E_{i,l} + NE_0,$$

where

$$E_{i,l}(x) = -\frac{g^2\hbar}{(2\pi)^3}\int_{\mathbb{R}^3}\left(\frac{\hbar}{2}\sigma^{(i)}\cdot\hbar k\right)\left(\frac{\hbar}{2}\sigma^{(l)}\cdot\hbar k\right)\frac{|\hat\rho(k)|^2}{\omega(k)^2}e^{-ik\cdot(x_i-x_l)}dk,$$

a.e.$x = (x_1, \cdots, x_N) \in \mathbb{R}^{3N}$ and

$$E_0 = -\frac{3}{2}\frac{g^2\hbar}{(2\pi)^3}\sum_{\alpha=1}^{3}\tau_\alpha{}^{(i)}\tau_\alpha{}^{(l)}\left(\frac{\hbar}{2}\right)^2\int_{\mathbb{R}^3}|\hbar k|^2\frac{|\hat\rho(k)|^2}{\omega(k)^2}dk.$$

Theorem 5.3 *Suppose that*

$$\omega^{-3/2}g_j{}^{(i)} \in L^2(\mathbb{R}^3), \quad i=1,\cdots,N, \quad j=1,2,3.$$

Then, for all $z \in \mathbb{C}\setminus\mathbb{R}$ *or* $z<0$ *with sufficiently large,*

$$\text{s-}\lim_{\Lambda\to\infty}(H(\Lambda)-z)^{-1} = (H_{\text{eff}}-z)^{-1}\otimes P_0.$$

Proof. Applying Theorem 4.1, we have

$$\text{s-}\lim_{\Lambda\to\infty}(H(\Lambda)-R\otimes I-z)^{-1} = (H_{\text{eff}}-R-z)^{-1}\otimes P_0,$$

where

$$R=\frac{\hbar}{2}\sum_{i=1}^{N}\sigma_3{}^{(i)}.$$

(For detail see [12].) Since R is bounded, we can prove the desired result in the same way as in Theorem 2.1. □

Remark 5.2 *Physically,* E_0 *and* $E_{i,l}$ *above are considered respectively as the self-energy of each nucleon and an effective potential of the force between two nucleons caused by the exchange of pions.*

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