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Generalized Time Operators and Decay of Quantum Dynamics

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1 Introduction

This is a short review of the results obtained in the paper [4]. In this introduction we explain motivations and ideas behind the work in some detail.

As is well-known, the physical quantity (observable) which describes the total energy of a quantum system S is called the Hamiltonian of S and represented as a self-adjoint operator H acting in the Hilbert space $\mathcal{H}_{\mathrm{S}}$ of quantum states of S. The state $\psi(t)\in\mathcal{H}_{\mathrm{S}}$ at time $t\in\mathbb{R}$ is given by

$$\psi(t)=e^{-itH}\psi$$

with $\psi\in\mathcal{H}_{\mathrm{S}}\setminus\{0\}$ being the intial state (the state at $t=0$) of S, where we use the unit system such that $\hbar=1$ ($\hbar=h/(2\pi)$ with h being the Planck constant). The *transition probability amplitude* of ψ to $\phi\in\mathcal{H}_{\mathrm{S}}\setminus\{0\}$ at time t is given by

$$A_{\psi,\phi}(t):=\frac{\langle\phi,\psi(t)\rangle_{\mathcal{H}_{\mathrm{S}}}}{\|\phi\|_{\mathcal{H}_{\mathrm{S}}}\|\psi(t)\|_{\mathcal{H}_{\mathrm{S}}}}=\frac{\left\langle\phi,e^{-itH}\psi\right\rangle_{\mathcal{H}_{\mathrm{S}}}}{\|\phi\|_{\mathcal{H}_{\mathrm{S}}}\|\psi\|_{\mathcal{H}_{\mathrm{S}}}},$$

where $\langle\cdot,\cdot\rangle_{\mathcal{H}_{\mathrm{S}}}$ and $\|\cdot\|_{\mathcal{H}_{\mathrm{S}}}$ denote the inner product and the norm of $\mathcal{H}_{\mathrm{S}}$ respectively. The square $|A_{\psi,\phi}(t)|^2$ of the modulus of $A_{\psi,\phi}(t)$ is called the *transition probability* of ψ to ϕ at time t. In particular, $|A_{\psi,\psi}(t)|^2$ is called the *survival probability* of ψ at time t. Physically the asymptotic behavior of the transition probability $|A_{\psi,\phi}(t)|^2$ as $t\to\pm\infty$ is very important. It is well-known that, if ψ or ϕ is in the subspace of absolute continuity with respect to H, then $\lim_{t\to\pm\infty} A_{\psi,\phi}(t) = 0$ [8, Proposition 2.2]. In this case, a natural question arises: How fast does $A_{\psi,\phi}(t)$ tend to 0 as $t \to \pm\infty$? In other words, with what order does $A_{\psi,\phi}(t)$ decay in time t going to $\pm\infty$? This question is one of the basic motivations for the present work. Of course one may give an answer to the question at various levels of studies, including analyses of concrete models. But the approach we take here may be a most general one in the sense that we try to find a general mathematical structure governing the order of decay (in time) of transitions probabilities in a way independent of models H. Indeed, as is shown below, such a strucure exists, in which one sees that a class of symmetric operators associated with H, called the *generalized time operators* with respect to H, plays a cetral role.

Our approach is on a line of developments of the representation theory of the canonical commutation relations (CCR). To explain this aspect, we first recall some of the basic facts on the representation theory of the CCR.

A representation of the CCR with one degree of freedom is defined to be a triple $(\mathcal{H}, \mathcal{D}, (Q, P))$ consisting of a complex Hilbert space $\mathcal{H}$, a dense subspace $\mathcal{D}$ of $\mathcal{H}$ and the pair (Q, P) of symmetric operators on $\mathcal{H}$ such that $\mathcal{D} \subset D(QP) \cap D(PQ)$ ($D(\cdot)$ denotes operator domain) and the canonical commutation relation

$$QP - PQ = iI \tag{1.1}$$

holds on $\mathcal{D}$, where $i := \sqrt{-1}$ and I denotes the identity on $\mathcal{H}$[1] . If both Q and P are self-adjoint, then we say that the representation $(\mathcal{H}, \mathcal{D}, (Q, P))$ is self-adjoint.

A typical example of self-adjoint representations of the CCR is the Schrödinger representation $(L^2(\mathbb{R}), C_0^\infty(\mathbb{R}), (Q_S, P_S))$ with Q_S being the multiplication operator by the function $x \in \mathbb{R}$ acting in $L^2(\mathbb{R})$ and $P_S := -iD_x$ the generalized differential operator in the variable x acting in $L^2(\mathbb{R})$.

We remark that there are representations of the CCR which cannot be self-adjoint[2] .

There is a stronger form of representation of the CCR: A double $(\mathcal{H}, (Q, P))$ consisting of a complex Hilbert space $\mathcal{H}$ and a pair (Q, P) of self-adjoint operators on $\mathcal{H}$ is called a Weyl representation of the CCR with one degree of freedom if

$$e^{itQ}e^{isP} = e^{-its}e^{isP}e^{itQ}, \quad \forall t, s \in \mathbb{R}.$$

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[1] One can generalize the concept of the representaion of the CCR by taking the commutation relation (1.1) in the sense of sesquilinear form, i.e., $\mathcal{D} \subset D(Q) \cap D(P)$ and $\langle Q\psi, P\phi\rangle - \langle P\psi, Q\phi\rangle = i\langle\psi, \phi\rangle$, $\psi, \phi \in \mathcal{D}$, where $\langle\cdot\,,\,\cdot\rangle$ denotes the inner product of $\mathcal{H}$.

[2] For example, consider the Hilbert space $L^2(\mathbb{R}_+)$ with $\mathbb{R}_+ := (0, \infty)$ and define operators q, p on $L^2(\mathbb{R}_+)$ as follows:

$$\begin{aligned} D(q) &:= \left\{ f \in L^2(\mathbb{R}_+) \,\middle|\, \int_{\mathbb{R}_+} |rf(r)|^2 dr < \infty \right\}, \ (qf)(r) := rf(r), \ f \in D(q), \ \text{a.e.} r \in \mathbb{R}_+, \\ D(p) &:= C_0^\infty(\mathbb{R}_+), \quad (pf)(r) := -i\frac{df(r)}{dr}, \ f \in D(p), \ \text{a.e.} r \in \mathbb{R}_+. \end{aligned}$$

Then q is self-adjoint, p is symmetric and $(L^2(\mathbb{R}_+), C_0^\infty(\mathbb{R}_+), (q, p))$ is a representation of the CCR with one degree of freedom. It is not so difficult to prove that p has no self-adjoint extensions (e.g., see [5, Chapter 2, Example D.1]). Therefore (q, p) cannot be extended to a self-adjoint representation of the CCR on $L^2(\mathbb{R}_+)$.

This relation is called the Weyl relation (e.g., [5, §3.3], [17, pp.274–275]). It is easy to see that the Schrödinger representation $(L^2(\mathbb{R}), (Q_S, P_S))$ is a Weyl representation. Von Neumann [14] proved that each Weyl representation on a separable Hilbert space is unitarily equivalent to a direct sum of the Schrödinger representation. This theorem—the von Neumann uniqueness theorem— implies that a Weyl representation of the CCR is a self-adjoint representation of the CCR (for details, see, e.g., [5, §3.5], [16]). But *a self-adjoint representation of the CCR is not necessarily a Weyl representation of the CCR* , namely there are self-adjoint representations of the CCR that are *not* Weyl representations. For example, see [7]. Physically interesting examples of self-adjoint representations of the CCR's with two degrees of freedom which are not necessarily unitarily equivalent to the Schrödinger representation of the CCR's appear in two-dimensional gauge quantum mechanics with singular gauge potentials. These representations, which are closely related to the so-called Aharonov-Bohm effect [1], have been studied by the present author in a series of papers, see [3] and references therein (a textbook description is given in [5, §3.6]).

Schmüdgen [19] presented and studied a weaker version of the Weyl relation with one degree of freedom: Let T be a symmetric operator and H be a self-adjoint operator on a Hilbert space $\mathcal{H}$. We say that (T, H) obeys the *weak Weyl relation* (WWR) if $e^{-itH}D(T) \subset D(T)$ for all $t \in \mathbb{R}$ and

$$Te^{-itH}\psi = e^{-itH}(T+t)\psi, \quad \forall\psi \in D(T), \forall t \in \mathbb{R},$$

where, for later convenience, we use the symbols (T, H) instead of (Q, P). We call $(\mathcal{H}, (T, H))$ a *weak Weyl representation* of the CCR with one degree of freedom. It is easy to see that every Weyl representation of the CCR is a weak Weyl representation of the CCR. But *the converse is not true* [19]. It should be remarked also that the WWR implies the CCR, but *a representation of the CCR is not necessarily a weak Weyl representation of the CCR* . In this sense the WWR is between the CCR and the Weyl relation (cf. [5, §3.7]).

Since the WWR is a relation for e^{-itH}, one may derive from it properties of H such as spectral properties and decay properties of transition probabilities. Indeed, this is true: The WWR was used to study a *time operator* with application to survival probabilities in quantum dynamics [9, 10] (in the article [9], the WWR is called the T-weak Weyl relation), where H is taken to be the Hamiltonian of a quantum system. It was proven in [9] that, if (T, H) obeys the WWR, then H has no point spectrum and its spectrum is purely absolutely continuous [9, Corollary 4.3, Theorem 4.4]. This kind class of H, however, is somewhat restrictive. From this point of view, it would be natural to investigate a general version of the WWR (if any) such that H is not necessarily purely absolutely continuous.

The general version of the WWR we take is defined as follows:

Definition 1.1 Let T be a symmetric operator on a Hilbert space $\mathcal{H}$, H be a self-adjoint operator on $\mathcal{H}$ and $K(t)$ ($t \in \mathbb{R}$) be a bounded self-adjoint operator on $\mathcal{H}$ with $D(K(t)) = \mathcal{H}$, $\forall t \in \mathbb{R}$. We say that (T, H, K) obeys the *generalized weak Weyl relation* (GWWR) in $\mathcal{H}$ if $e^{-itH}D(T) \subset D(T)$ for all $t \in \mathbb{R}$ and

$$Te^{-itH}\psi = e^{-itH}(T+K(t))\psi, \quad \forall\psi \in D(T), \forall t \in \mathbb{R}. \tag{1.2}$$

We call the operator-valued function K the commutation factor in the GWWR. Also we sometimes say that (T, H, K) is a representation of the GWWR.

Obviously the case $K(t) = t$ in the GWWR gives the WWR. Hence the GWWR is certainly a generalization of the WWR. Since the (G)WWR is a weaker version of the Weyl relation, the strong properties arising from the Weyl relation (e.g., spectral properties) may be weakend by the (G)WWR. It is very interesting to investigate this aspect. Thus triples (T, H, K) obeying the GWWR become the main objects of our investigation.

As suggested above, in applications to quantum mechanics and quantum field theory, we have in mind the case where H is the Hamiltonian of a quantum system. In this realization of H, we call T a *generalized time operator*. We show that the GWWR implies a "time-energy uncertainty relation" between H and T (for physical discussions related to this aspect, see [13] and references therein). Mathematically rigorous studies for time-energy uncertainty relations, which, however, do not use time operators, are given in [15]. One can construct generalized time operators for Hamiltonians in both relativistic and nonrelativistic quantum mechanics including Dirac type operators as well as in quantum field theory.

2 Fundamental Properties of the GWWR

Throughout this section, we assume that (T, H, K) obeys the GWWR in a Hilbert space $\mathcal{H}$ (Definition 1.1).

The following proposition shows that the vector equation (1.2) can be extended to an operator equality:

Proposition 2.1 *For all $t \in \mathbb{R}$, $e^{-itH}D(T) = D(T)$ and the operator equality*

$$Te^{-itH} = e^{-itH}(T + K(t)) \tag{2.1}$$

holds. Moreover

$$K(0) = 0. \tag{2.2}$$

In Definition 1.1, T is not necessarily closed. But the following proposition holds:

Proposition 2.2 *Let $\bar{T}$ be the closure of T. Then $(\bar{T}, H, K)$ obeys the GWWR.*

For a linear operator A, we denote by $\sigma(A)$ (resp. $\sigma_{\mathrm{p}}(A)$) the spectrum (resp. the point spectrum) of A.

As for the spectrum and the point spectrum of T, the following facts are found:

Corollary 2.3 *For all $t \in \mathbb{R}$, $\sigma(T + K(t)) = \sigma(T)$ and $\sigma_{\mathrm{p}}(T + K(t)) = \sigma_{\mathrm{p}}(T)$, where the multiplicity of each $\lambda \in \sigma_{\mathrm{p}}(T)$ is equal to that of $\lambda \in \sigma_{\mathrm{p}}(T + K(t))$.*

We introduce a stronger notion of commutativity between a linear operator and a self-adjoint operator:

Definition 2.4 We say that a linear operator L on $\mathcal{H}$ strongly commutes with H if $e^{-itH}D(L) \subset D(L)$ for all $t \in \mathbb{R}$ and $e^{-itH}L \subset Le^{-itH}$.

Remark 2.1 One can show that L strongly commutes with H if and only if operator equality $e^{-itH}L = Le^{-itH}$ holds for all $t \in \mathbb{R}$.

The following proposition shows the non-uniqueness of generalized time operators for a given pair (H, K):

Proposition 2.5 *Let S be a symmetric operator on $\mathcal{H}$ strongly commuting with H such that $D(S) \cap D(T)$ is dense (hence $T+S$ is a symmetric operator with $D(T+S) := D(T) \cap D(S)$). Then $(T+S, H, K)$ obeys the GWWR.*

We denote by $\mathsf{B}(\mathcal{H})$ the Banach space of all bounded linear operators on $\mathcal{H}$ with domains equal to $\mathcal{H}$.

The following proposition shows a relation between H and K:

Proposition 2.6 *For all $t \in \mathbb{R}$,*

$$e^{itH}K(-t) + K(t)e^{itH} = 0. \tag{2.3}$$

In particular

$$\sigma(K(t)) = \sigma(-K(-t)), \quad \sigma_{\mathrm{p}}(K(t)) = \sigma_{\mathrm{p}}(-K(-t)), \quad \forall t \in \mathbb{R}. \tag{2.4}$$

The following theorem is concerned with *non-self-adjointness of generalized time operators*:

Theorem 2.7 *Assume that $K : \mathbb{R} \to \mathsf{B}(\mathcal{H})$ is strongly differentiable on $\mathbb{R}$ and let*

$$K'(t) := \text{s-}\frac{dK(t)}{dt}, \tag{2.5}$$

the strong derivative of K in $t \in \mathbb{R}$. Suppose that $K'(0) \neq 0$, H is semi-bounded (i.e., bounded from below or bounded from above) and

$$K(t)T \subset TK(t) \tag{2.6}$$

for all $t \in \mathbb{R}$. Then T is not self-adjoint.

Remark 2.2 In the simple case $K(t) = t$, the fact stated in Theorem 2.7 has been pointed out in [9].

We next describe a method to construct triples obeying the GWWR in direct sums of Hilbert spaces.

Let $\mathcal{H}_1$ be a Hilbert space and $\mathcal{F} := \mathcal{H} \oplus \mathcal{H}_1$. Let (T_1, H_1, K_1) be a triple obeying the GWWR in $\mathcal{H}_1$. We define

$$\widetilde{H} := H \oplus H_1 = \begin{pmatrix} H & 0 \\ 0 & H_1 \end{pmatrix}. \tag{2.7}$$

Proposition 2.8 *Let A be a bounded linear operator from $\mathcal{H}$ to $\mathcal{H}_1$ with $D(A) = \mathcal{H}$ and*

$$\widetilde{T} := \begin{pmatrix} T & A^* \\ A & T_1 \end{pmatrix}, \quad \widetilde{K}(t) := \begin{pmatrix} K(t) & e^{itH}A^*e^{-itH_1} - A^* \\ e^{itH_1}Ae^{-itH} - A & K_1(t) \end{pmatrix}. \tag{2.8}$$

Then $(\widetilde{T}, \widetilde{H}, \widetilde{K})$ obeys the GWWR in $\mathcal{F}$.

Proof: By the functional calculus, we have $e^{-it\widetilde{H}} = e^{-itH} \oplus e^{-itH_1}$ for all $t \in \mathbb{R}$. Then direct computations yield the desired result. ∎

Note that, in Proposition 2.8, $\widetilde{T}$ is not diagonal if $A \neq 0$. This procedure of construction of a new triple obeying the GWWR obviously yields an algorithm to obtain a triple obeying the GWWR in the N direct sum $\oplus_{n=1}^{N} \mathcal{H}_n$ of Hilbert spaces $\mathcal{H}_n$ ($N \geq 2$), provided that, for each n, a triple (T_n, H_n, K_n) obeying the GWWR in $\mathcal{H}_n$ is given.

In concluding this section, we report a result on the problem if the operator H perturbed by a symmetric operator has a generalized time operator.

Let V be a symmetric operator on $\mathcal{H}$ and assume that

$$H(V) := H + V \tag{2.9}$$

is essentially self-adjoint.

Proposition 2.9 *Assume that the following conditions (i)–(iii) hold:*

(i) *The operators T, H and $K(t)$ ($t \in \mathbb{R}$) are reduced by a closed subspace $\mathcal{M}$ of $\mathcal{H}$. We denote their reduced part by $T_{\mathcal{M}}, H_{\mathcal{M}}$ and $K_{\mathcal{M}}(t)$ respectively.*

(ii) *The operator $\overline{H(V)}$ is reduced by a closed subspace $\mathcal{N}$ of $\mathcal{H}$.*

(iii) *There exists a unitary operator $U : \mathcal{M} \to \mathcal{N}$ such that $UH_{\mathcal{M}}U^{-1} = \overline{H(V)}_{\mathcal{N}}$.*

Let

$$T_V := (UT_{\mathcal{M}}U^{-1}) \oplus 0, \quad K_V(t) := (UK_{\mathcal{M}}(t)U^{-1}) \oplus 0 \tag{2.10}$$

relative to the orthogonal decomposition $\mathcal{H} = \mathcal{N} \oplus \mathcal{N}^{\perp}$. Then $(T_V, \overline{H(V)}, K_V)$ obeys the GWWR.

Proof: It is obvious that T_V is symmetric and $K_V(t)$ is a bounded self-adjoint operator. By direct computations, one sees that $(T_V, \overline{H(V)}, K_V)$ obeys the GWWR. ∎

A method to find the unitary operator U in Proposition 2.9 is to use the method of wave operators with respect to the pair $(H, \overline{H(V)})$. In that case, U would be one of the wave operators

$$W_{\pm} := \text{s-}\lim_{t \to \pm\infty} e^{it\overline{H(V)}} J e^{-itH} P_{\mathrm{ac}}(H)$$

(if they exist) ($P_{\mathrm{ac}}(H)$ is the orthogonal projection onto the absolutely continuous space of H and J is a linear operator),

$$\mathcal{M} = (\ker W_{\pm})^{\perp}$$

and

$$\mathcal{N} = \overline{\mathrm{Ran}(W_\pm)}$$

(e.g., [8, §4.2], [18, p.34, Proposition 4]). This method was taken in [9, 10] in the case where H is the 1-dimensional Laplacian and V is a real-valued function on $\mathbb{R}$ (hence $H(V)$ is a one-dimensional Schrödinger operator).

3 Transition Probability Amplitudes and the Point Spectra of Hamiltonians

Let (T, H, K) be a triple obeying the GWWR in a Hilbert space $\mathcal{H}$. The following proposition is concerned with upper bounds of the modulus of a transition probabilty amplitude in time t.

Proposition 3.1 *Suppose that there is a constant $\alpha > 0$ such that the strong limit*

$$L_\alpha := \text{s-}\lim_{t\to\infty} \frac{K(t)}{t^\alpha} \in \mathsf{B}(\mathcal{H}) \tag{3.1}$$

exists. Let S be a symmetric operator strongly commuting with H. Then, for all $\psi, \phi \in D(T) \cap D(S)$ and $t > 0$,

$$\left|\left\langle \psi, e^{-itH} L_\alpha \phi \right\rangle\right| \le \frac{\|(T+S)\psi\|\|\phi\| + \|\psi\|\|(T+S)\phi\|}{t^\alpha} + \|\psi\| \left\| \left(L_\alpha - \frac{K(t)}{t^\alpha} \right) \phi \right\|. \tag{3.2}$$

Remark 3.1 Proposition 3.1 is a generalization of [9, Theorem 4.1] where the special case $K(t) = t$ is considered.

The following corollary is a generalized version of [9, Corollary 4.3]:

Corollary 3.2 *Suppose that the assumption of Proposition 3.1 holds. Then, for all $\psi, \phi \in \mathcal{H}$,*

$$\lim_{t\to\infty} \left\langle \psi, e^{-itH} L_\alpha \phi \right\rangle = 0. \tag{3.3}$$

This corollary implies an interesting structure of the point spectrum of H:

Corollary 3.3 *Suppose that the assumption of Proposition 3.1 holds. Then, for all $E \in \mathbb{R}$, $\ker(H - E) \subset \ker L_\alpha$. In particular, if $\ker L_\alpha = \{0\}$, then $\sigma_{\mathrm{p}}(H) = \emptyset$.*

Proof: Let $\psi_E \in \ker(H - E)$. Then $e^{itH}\psi_E = e^{itE}\psi_E$. Taking $\psi = \psi_E$ in (3.3), we obtain $\langle \psi_E, L_\alpha \phi \rangle = 0$ for all $\phi \in \mathcal{H}$. This implies that $L_\alpha \psi_E = 0$, i.e., $\psi_E \in \ker L_\alpha$. ∎

Remark 3.2 Corollary 3.3 is a generalization of [9, Corollary 4.3] where the case $K(t) = t$ is considered.

4 Generalized Weak CCR and Time-Energy Uncertainty Relations

Let A, B be symmetric operators on a Hilbert space $\mathcal{H}$ and $C \in \mathsf{B}(\mathcal{H})$ be a self-adjoint operator. We say that (A, B, C) obeys the *generalized weak CCR* (GWCCR) if

$$\langle A\psi, B\phi\rangle - \langle B\psi, A\phi\rangle = \langle \psi, iC\phi\rangle\,, \quad \forall \psi, \phi \in D(A) \cap D(B). \tag{4.1}$$

The case $C = I$ (the identity on $\mathcal{H}$) is the usual CCR with one degree of freedom in the sense of sesquilinear form.

For a symmetric operator A on a Hilbert space, a constant $a \in \mathbb{R}$ and a unit vector $\psi \in D(A)$, we define

$$(\Delta A)_\psi(a) := \|(A - a)\psi\|, \tag{4.2}$$

an *uncertainty of A in the state vector ψ*. The quantity $(\Delta A)_\psi(a)$ with $a = \langle \psi, A\psi\rangle$ is the usual uncertainty of A in the state vector ψ. We set

$$(\Delta A)_\psi := (\Delta A)_\psi(\langle \psi, A\psi\rangle). \tag{4.3}$$

We also introduce

$$\delta_C := \inf_{\psi \in (\ker C)^\perp, \|\psi\|=1} |\langle \psi, C\psi\rangle|. \tag{4.4}$$

Proposition 4.1 *Suppose that (A, B, C) obeys the GWCCR. Then, for all $\psi \in D(A) \cap D(B) \cap (\ker C)^\perp$ with $\|\psi\| = 1$ and all $a, b \in \mathbb{R}$,*

$$(\Delta A)_\psi(a)(\Delta B)_\psi(b) \geq \frac{\delta_C}{2}. \tag{4.5}$$

Proposition 4.2 *Suppose that (A, B, C) obeys the GWCCR with $C \geq 0$. Then, for all $\lambda \in \mathbb{R}$, $\ker(B - \lambda) \cap D(A) \subset \ker C$ and $\ker(A - \lambda) \cap D(B) \subset \ker C$.*

Proof: Let $\psi \in \ker(B - \lambda) \cap D(A)$. Then, taking $\phi = \psi$ in (4.1), we have $\langle \psi, C\psi\rangle = 0$. Since C is nonnegative, it follows that $C\psi = 0$, i.e., $\psi \in \ker C$. ∎

The following proposition gives a connection of the GWWR with the GWCCR:

Proposition 4.3 *Let (T, H, K) be a triple obeying the GWWR in $\mathcal{H}$. Assume that K is strongly differentiable on $\mathbb{R}$. Then $(T, H, K'(0))$ obeys the GWCCR:*

$$\langle T\psi, H\phi\rangle - \langle H\psi, T\phi\rangle = \langle \psi, iK'(0)\phi\rangle\,, \quad \psi, \phi \in D(T) \cap D(H). \tag{4.6}$$

Propositions 4.3 and 4.1 yield the following result:

Corollary 4.4 *Suppose that the same assumption as in Proposition 4.3 holds. Then, for all $\psi \in D(T) \cap D(H) \cap (\ker K'(0))^\perp$ with $\|\psi\| = 1$ and all $t, E \in \mathbb{R}$,*

$$(\Delta T)_\psi(t)(\Delta H)_\psi(E) \geq \frac{\delta_{K'(0)}}{2}. \tag{4.7}$$

In applications to quantum theory, (4.7) gives a time-energy uncertainty relation if H is the Hamiltonian of a quantum system.

5 The Point Spectra of Generalized Time Operators

For a linear opeartro L on a Hilbert apce $\mathcal{H}$, we introduce a subset of $\mathcal{H}$:

$$N_0(L) := \{\psi \in D(L) | \langle \psi, L\psi \rangle = 0\}. \tag{5.1}$$

It is obvious that $\ker L \subset N_0(L)$.

Remark 5.1 If L is a non-negative self-adjoint operator, then $N_0(L) = \ker L$.

Proposition 5.1 *Assume that (T, H, K) obeys the GWWR and K is strongly differentiable on $\mathbb{R}$. Then, for all $E \in \mathbb{R}$,*

$$\ker(T - E) \subset N_0(K'(0)). \tag{5.2}$$

Corollary 5.2 *Assume that (T, H, K) obeys the GWWR and K is strongly differentiable on $\mathbb{R}$. Then:*

(i) *If $N_0(K'(0)) = \{0\}$, then $\sigma_{\mathrm{p}}(T) = \emptyset$.*

(ii) *If $K'(0) \geq 0$ or $K'(0) \leq 0$, then $\sigma_{\mathrm{p}}(T|[D(T) \cap (\ker K'(0))^{\perp}]) = \emptyset$.*

Remark 5.2 Corollary (5.2) is a generalization of [9, Corollary 4.2] where the case $K(t) = t$ is considered.

It may be interesting to note that the behavior of $K(t)$ at $t = 0$ and $t = \infty$ is respectively related to $\sigma_{\mathrm{p}}(T)$ (Corollary 5.2) and $\sigma_{\mathrm{p}}(H)$ (Corollary 3.3).

6 Commutation Formulas and Absolute Continuity

In this section we show commutation relations derived from the GWWR. Moreover, in the special case where the commutation factor $K(t)$ is of the form tC with C a bounded self-adjoint operator, we show that H is reduced by $\overline{\mathrm{Ran}(C)}$ ($\mathrm{Ran}(C)$ denotes the range of C) and its reduced part is absolutely continuous.

6.1 General cases

For $p \geq 0$, we introduce a class of Borel measurable functions on $\mathbb{R}$:

$$L^1_p(\mathbb{R}) := \left\{F : \mathbb{R} \to \mathbb{C}, \text{Borel measurable} | \int_{\mathbb{R}} |F(t)|(1 + |t|^p)dt < \infty\right\}. \tag{6.1}$$

It is easy to see that $L^1_p(\mathbb{R})$ includes the space $\mathcal{S}(\mathbb{R})$ of rapidly decreasing C^∞-functions on $\mathbb{R}$.

We say that a Borel measurable function f is in the set $\mathcal{M}_p$ if it is the Fourier transform of an element $F_f \in L^1_p(\mathbb{R})$:

$$f(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} F_f(t) e^{-it\lambda} dt, \quad \lambda \in \mathbb{R}. \tag{6.2}$$

Note that, for each $f \in \mathcal{M}_p$, F_f is uniquely determined. We have

$$\mathcal{S}(\mathbb{R}) \subset \mathcal{M}_p. \tag{6.3}$$

Moreover, every element f of $\mathcal{M}_p$ is bounded, $[p]$ times continuously differentiable ($[p]$ denotes the largest integer not exceeding p) and, for $j = 1, \cdots, [p]$, $d^j f/d\lambda^j$ is bounded.

Let H be a self-adjoint operator on a Hilbert space $\mathcal{H}$ and $S : \mathbb{R} \to \mathsf{B}(\mathcal{H})$ be Borel measurable such that, for all $\psi \in \mathcal{H}$,

$$\|S(t)\psi\| \le c(1 + |t|^p)\|\psi\|, \quad t \in \mathbb{R}$$

with constants $c > 0$ and $p \ge 0$ independent of ψ. Then, for all $\psi \in \mathcal{H}$ and $f \in \mathcal{M}_p$, the strong integral

$$f(H, S)\psi := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} F_f(t) e^{-itH} S(t)\psi dt \tag{6.4}$$

exists and $f(H, S) \in \mathsf{B}(\mathcal{H})$.

Theorem 6.1 *Assume that (T, H, K) obeys the GWWR. Suppose that K is strongly continuous and, for all $\psi \in \mathcal{H}$,*

$$\|K(t)\psi\| \le c(1 + |t|^p)\|\psi\|, \tag{6.5}$$

where $c > 0$ and $p \ge 0$ are constants independent of ψ. Let $f \in \mathcal{M}_p$. Then, for all $\psi \in D(\bar{T})$, we have $f(H)\psi \in D(\bar{T})$ and

$$\bar{T} f(H)\psi = f(H)\bar{T}\psi + f(H, K)\psi, \tag{6.6}$$

where $f(H) := \int_{\mathbb{R}} f(\lambda) dE_H(\lambda)$.

6.2 A special case

In this subsection we consider a special case of a triple (T, H, K) obeying the GWWR in a Hilbert space $\mathcal{H}$: We assume that K is of the form

$$K_C(t) := tC, \quad t \in \mathbb{R} \tag{6.7}$$

with C being a bounded self-adjoint operator on $\mathcal{H}$. In this case a more detailed analysis is possible as shown below.

We set

$$\begin{aligned} C^1_{\mathrm{b}}(\mathbb{R}) &:= \{f \in C^1(\mathbb{R}) | f \text{ and } f' \text{ are bounded}\}, & (6.8)\\ C^1_{\mathrm{b},+}(\mathbb{R}) &:= \{f \in C^1(\mathbb{R}) | \text{for some } a \in \mathbb{R}, \sup_{\lambda \ge a} |f(\lambda)| < \infty \text{ and} \\ &\qquad \sup_{\lambda \ge a} |f'(\lambda)| < \infty\}. & (6.9)\end{aligned}$$

Theorem 6.2 *Let C be a bounded self-adjoint operator on $\mathcal{H}$ and suppose that (T, H, K_C) obeys the GWWR.*

(i) *Let $f \in C_{\mathrm{b}}^1(\mathbb{R})$. Then $f(H)D(\bar{T}) \subset D(\bar{T})$ and*

$$\bar{T}f(H)\psi - f(H)\bar{T}\psi = if'(H)C\psi \tag{6.10}$$

for all $\psi \in D(\bar{T})$.

(ii) *Suppose that H is bounded from below. Then, for all $f \in C_{\mathrm{b},+}^1(\mathbb{R})$, the same conclusion as that of part (i) holds. In particular, for all $z \in \mathbb{C}$ with $\Re z > 0$, $e^{-zH}D(\bar{T}) \subset D(\bar{T})$ and, for all $\psi \in D(\bar{T})$*

$$\bar{T}e^{-zH}\psi - e^{-zH}\bar{T}\psi = -ize^{-zH}C\psi. \tag{6.11}$$

Corollary 6.3 *Let C be a bounded self-adjoint operator on $\mathcal{H}$ and suppose that (T, H, K_C) obeys the GWWR. Then H is reduced by $\overline{Ran(C)}$.*

As in the case of [19, 3.2 Corollary 2], we have from Proposition 6.2 and Corollary 6.3 the following theorem. For a self-adjoint operator H, we set

$$E_H(\lambda) := E_H((-\infty, \lambda]), \quad \lambda \in \mathbb{R}.$$

Theorem 6.4 *Suppose that (T, H, K_C) obeys the GWWR. Then H is reduced by $\overline{\mathrm{Ran}(C)}$ and the reduced part $H|\overline{\mathrm{Ran}(C)}$ is absolutely continuous. Moreover, for all $\psi, \phi \in D(\bar{T})$, the Radon-Nikodym derivative $d\langle\psi, E_H(\lambda)C\phi\rangle/d\lambda$ is given by*

$$\frac{d\langle\psi, E_H(\lambda)C\phi\rangle}{d\lambda} = i\left(\left\langle\bar{T}\psi, E_H(\lambda)\phi\right\rangle - \left\langle E_H(\lambda)\psi, \bar{T}\phi\right\rangle\right). \tag{6.12}$$

7 Absence of Minimum-Uncertainty States

Let (A, B, C) be a triple obeying the GWCCR. A vector $\psi_0 \in D(A) \cap D(B) \cap (\ker C)^\perp$ with $\|\psi_0\| = 1$ which attains the equality $(\Delta A)_{\psi_0}(\Delta B)_{\psi_0} = \delta_C/2 > 0$ in the uncertainty relation (4.5) with $a = \langle\psi_0, A\psi_0\rangle$ and $b = \langle\psi_0, B\psi_0\rangle$ is called a *minimum-unertainty state* for (A, B, C).

Remark 7.1 It is well-known that the Schrödinger representation $(Q_{\mathrm{S}}, P_{\mathrm{S}})$ of the CCR has a minimum-uncertainty state. Indeed, the vector $f_0 \in L^2(\mathbb{R})$ given by $f_0(x) := (2\pi)^{-1/4}\sigma^{-1/2}e^{-(x-a)^2/(4\sigma^2)}$, $x \in \mathbb{R}$ with $a \in \mathbb{R}$ and $\sigma > 0$ being constants is a minimum-uncertainty state for $(Q_{\mathrm{S}}, P_{\mathrm{S}}, I)$: $(\Delta Q_{\mathrm{S}})_{f_0}(\Delta P_{\mathrm{S}})_{f_0} = 1/2$. It follows from this fact that every representation (Q, P) of the CCR unitarily equivalent to the Schrödinger one has a minimum-uncertainty state. In particular, by the von Neumann uniqueness theorem mentioned in Introduction of the present paper, every Weyl representation has a minimum-uncertainty state. Also the Fock representation of the CCR with one degree of freedom has a minimum-uncertainty state.

In this section, in contrast to the facts stated in Remark 7.1, we give a sufficient condition for a triple (T, H, C) to have *no* minimum-unertainty states.

Theorem 7.1 (Absence of minimum-uncertainty state) *Suppose that (T, H, K_C) obeys the GWWR with T being closed. Assume that H is bounded from below and that $C \geq 0$ with $\delta_C > 0$. Then there exist no vectors $\psi_0 \in D(H) \cap D(T) \cap (\ker C)^{\perp}$ with $\|\psi_0\| = 1$ such that*

$$(\Delta T)_{\psi_0}(\Delta H)_{\psi_0} = \frac{\delta_C}{2} > 0. \tag{7.1}$$

Remark 7.2 An essential condition in this theorem is the boundedness below of H (note that the operators Q_S and P_S in the Schrödinegr representation of the CCR are unbounded both above and below).

Remark 7.3 Theorem 7.1 is an extension of [9, Theorem 5.1], where the case $C = I$ is considered. A new point here is that one does not need to assume the analytic continuation property of the weak Weyl relation (the GWWR with $C = I$) as is done in [9, Theorem 5.1].

8 Power Decays of Transition Probability Amplitudes in Quantum Dynamics

In Section 3 we have derived an estimate for transition probability amplitudes in time t. In this section we consider a triple (T, H, K_C) obeying the GWWR (discussed in Section 6.2) and show that, for state vectors in "smaller" subspaces, transition probability amplitudes decay in powers of t as $|t| \to \infty$. We apply the results to two-point correlation functions of Heisenberg operators.

Let H be a self-adjoint operator on a Hilbert space $\mathcal{H}$ and $C \neq 0$ be a bounded self-adjoint opeartor on $\mathcal{H}$. We introduce a set of generalized time operators:

$$\mathsf{T}(H, C) := \{T | (T, H, K_C) \text{ obeys the GWWR}\}. \tag{8.1}$$

By Proposition 2.5, if $T \in \mathsf{T}(H, C)$, then $T + S \in \mathsf{T}(H, C)$ for all symmetric operators S on $\mathcal{H}$ strongly commuting with H such that $D(T) \cap D(S)$ is dense in $\mathcal{H}$.

8.1 A simple case

Theorem 8.1 *Let $T \in \mathsf{T}(H, C)$ and $\psi, \phi \in D(T)$. Then, for all $t \in \mathbb{R} \setminus \{0\}$,*

$$\left|\left\langle \phi, e^{-itH} C\psi \right\rangle\right| \leq \frac{1}{|t|}(\|T\phi\|\|\psi\| + \|\phi\|\|T\psi\|). \tag{8.2}$$

Proof: In the present case, we have $L_\alpha = C$ with $\alpha = 1$. Hence Proposition 3.1 gives the desired result. ∎

Remark 8.1 For vectors $\phi, \psi \in \mathcal{H}$, we can define a set of operators

$$\mathsf{T}_{\phi,\psi}(H,C) := \{T \in \mathsf{T}(H,C) | \phi, \psi \in D(T)\}$$

and put

$$c_{\phi,\psi} := \inf_{T \in \mathsf{T}_{\phi,\psi}(H,C)} (\|T\phi\|\|\psi\| + \|\phi\|\|T\psi\|),$$

then (8.2) implies that

$$\left|\left\langle \phi, e^{-itH} C\psi \right\rangle\right| \leq \frac{c_{\phi,\psi}}{|t|}. \tag{8.3}$$

Remark 8.2 Let $T \in \mathsf{T}(H,C)$. Then, for all $\psi \in D(T)$ with $\|\psi\| = 1$, $T - \langle \psi, T\psi \rangle$ is in the set $\mathsf{T}(H,C)$. Hence (8.2) implies that

$$\left|\left\langle \psi, e^{-itH} C\psi \right\rangle\right|^2 \leq \frac{4(\Delta T)_\psi^2}{t^2}. \tag{8.4}$$

Hence Theorem 8.1 gives a generalization of [9, Theorem 4.1].

8.2 Higher order dcays in smaller subspaces

As demonstrated in a concrete example [9, Proposition 3.2], the modulus of a transition probability amplitude $\left|\left\langle \phi, e^{-itH}\psi \right\rangle\right|$ may decay faster than $|t|^{-1}$ as $|t| \to \infty$ for a class of vectors ϕ and ψ. In this subsection we investigate this aspect in an abstract framework and show that $\left|\left\langle \phi, e^{-itH}\psi \right\rangle\right|$ decays faster than $|t|^{-1}$ for all ϕ and ψ in smaller subspaces.

Theorem 8.2 *Let $T \in \mathsf{T}(H,C)$. Assume that*

$$CT \subset TC. \tag{8.5}$$

Let $n \in \mathbb{N}$ and $\psi, \phi \in D(T^n)$. We define constants $d_k^T(\phi,\psi), k = 1, \cdots, n$ by the following recursion relation:

$$d_1^T(\phi,\psi) := \|T\phi\|\|\psi\| + \|\phi\|\|T\psi\|, \tag{8.6}$$

$$d_n^T(\phi,\psi) := \|T^n\phi\|\|\psi\| + \|\phi\|\|T^n\psi\| + \sum_{r=1}^{n-1} {}_nC_r d_{n-r}^T(\phi, T^r\psi), \ n \geq 2, \tag{8.7}$$

where ${}_nC_r := n!/[(n-r)!r!]$. Then, for all $t \in \mathbb{R} \setminus \{0\}$,

$$\left|\left\langle \phi, e^{-itH} C^n\psi \right\rangle\right| \leq \frac{d_n^T(\phi,\psi)}{|t|^n}. \tag{8.8}$$

Theorem 8.2 can be generalized.

Theorem 8.3 *Let $T, T_1, \cdots, T_n \in \mathsf{T}(H,C)$ such that $CT \subset TC, CT_j \subset T_jC$, $j = 1, \cdots, n$. Let $\phi \in D(T_n \cdots T_1) \cap D(T^{n-1})$ and $\psi \in \cap_{r=1}^{n-1} \cap_{1 \le i_1 < \cdots < i_r \le n} D(T^{n-r}T_{i_1} \cdots T_{i_r})$. For $k = 1, \cdots, n$, we define a constant*

$$\delta_n^T(\phi, \psi; T_1, \cdots, T_n) := \|T_n \cdots T_1\phi\|\|\psi\| + \|\phi\|\|T_1 \cdots T_n\psi\| \tag{8.9}$$

$$+\sum_{r=1}^{n-1} \sum_{1 \le i_1 < \cdots < i_r \le n} d_{n-r}^T(\phi, T_{i_1} \cdots T_{i_r}\psi). \tag{8.10}$$

Then, for all $t \in \mathbb{R} \setminus \{0\}$,

$$\left|\left\langle \phi, e^{-itH} C^n \psi \right\rangle\right| \le \frac{\delta_n^T(\phi, \psi; T_1, \cdots, T_n)}{|t|^n}. \tag{8.11}$$

Finally we discuss the case where condition (8.5) is not necessarily satisfied. For $n \ge 2$ and $r = 1, \cdots, n-1$, we introduce a set

$$\mathsf{J}_{n,r} := \{j := (j_1, \cdots, j_{r+1}) \in \{0,1\}^{r+1} | j_1 + \cdots + j_{r+1} = n - r\} \tag{8.12}$$

and, for each $j \in \mathsf{J}_{n,r}$, we define

$$K_{n,r}^{(j)} := T^{j_1}CT^{j_2}C \cdots CT^{j_r}CT^{j_{r+1}}. \tag{8.13}$$

Let

$$\begin{aligned} &\mathcal{D}_n(T,C) \\ &:= \left\{\psi \in D(T^n) \cap \left(\cap_{r=1}^{n-1} \cap_{j \in \mathsf{J}_{n,r}} D(K_{n,r}^{(j)})\right) \middle| K_{n,r}^{(j)}\psi \in \mathrm{Ran}(C^r|D(T^r)), \right. \\ &\qquad\qquad \left. j \in \mathsf{J}_{n,r},\ r = 1, \cdots, n-1 \right\} \quad (n \ge 2). \end{aligned} \tag{8.14}$$

We set $\mathcal{D}_1(T,C) := D(T)$.

Remark 8.3 If (8.5) holds, then $\mathcal{D}_n(T,C) = D(T^n)$.

Theorem 8.4 *Let $T \in \mathsf{T}(H,C)$. Then, for all $\phi \in D(T^n)$ and $\psi \in \mathcal{D}_n(T,C)$ and $t \in \mathbb{R} \setminus \{0\}$,*

$$|\left\langle \phi, e^{-itH} C^n \psi \right\rangle| \le \frac{d_n(\phi, \psi)}{|t|^n}, \tag{8.15}$$

where $d_n(\phi, \psi) > 0$ is a constant independent of t.

8.3 Correlation functions

In this subsection, we show that the existence of generalized time-operators gives upper bounds for correlation functions for a class of linear operators. For a linear operator A on $\mathcal{H}$ and a self-adjoint operator H on $\mathcal{H}$, we define

$$A(t) := e^{itH} A e^{-itH}, \quad t \in \mathbb{R}, \tag{8.16}$$

the Heisenberg operator of A with respect to H. Let B be a linear operator on $\mathcal{H}$. Let

$$\psi \in \cap_{t\in\mathbb{R}}[D(Ae^{-itH}) \cap D(Be^{-itH})]$$

with $\|\psi\| = 1$. Then we can define

$$W(t, s; \psi) := \langle A(t)\psi, B(s)\psi\rangle, \quad s, t \in \mathbb{R}. \tag{8.17}$$

We call it the *two-point correlation function* of A and B with repect to the vector ψ.

Theorem 8.5 *Let $T \in \mathsf{T}(H, C)$. Suppose that ψ is an eigenvector of H such that $A\psi \in D(T)$ and $B\psi \in \mathrm{Ran}(C|D(T))$. Then, for all $t, s \in \mathbb{R}$ with $t \neq s$,*

$$|W(t, s; \psi)| \leq \frac{c_{A,B,T}}{|t - s|}, \tag{8.18}$$

where

$$c_{A,B,T} := \inf_{\chi\in D(T), B\psi=C\chi} \|TA\psi\|\|\chi\| + \|A\psi\|\|T\chi\|.$$

Proof: Let E be the eigenvalue of H with eigenvector ψ. Then we have

$$W(t, s) = e^{i(t-s)E}\left\langle A\psi, e^{-i(t-s)H}B\psi\right\rangle. \tag{8.19}$$

There exists a vector $\chi \in D(T)$ such that $B\psi = C\chi$. Hence, applying Theorem 8.1, we obtain

$$|W(t, s; \psi)| \leq \frac{\|TA\psi\|\|\chi\| + \|A\psi\|\|T\chi\|}{|t - s|}.$$

Thus (8.18) follows. ∎

Theorem 8.5 can be strengthened:

Theorem 8.6 *Let $T \in \mathsf{T}(H, C)$ with (8.5). Suppose that ψ is an eigenvector of H such that $\psi \in D(A)$ and $A\psi \in D(T^n)$ and $B\psi \in \mathrm{Ran}(C^n|D(T^n))$. Then, for all $t, s \in \mathbb{R}$ with $t \neq s$,*

$$|W(t, s)| \leq \frac{c^{(n)}_{A,B,T}}{|t - s|^n}, \tag{8.20}$$

where

$$c^{(n)}_{A,B,T} := \inf_{\chi\in D(T^n), B\psi=C^n\chi} d^T_n(A\psi, \chi).$$

Proof: This follow from (8.19) and an application of Theorem 8.2. ∎

In the case where H is bounded below, we can discuss the decay of the heat semi-group $e^{-\beta H}$ in $\beta > 0$. See [4] for the details.

9 Concluding Remarks

In this paper, we have presented some basic aspects of the theory of generalized time operators developed in the paper [4]. There are other interesting results. For example, a formulation of an abstract version of Wigner's time-energy uncertainty relation [20], existence of a structure producing successively triples obeying the GWWR, a method of constructions of generalized time operators of partial differential operators including Schrödinger and Dirac operators, and Fock space representations of the GWWR, which have applications to quantum field theory. For the details we refer the reader to [4].

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