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DIOPHANTINE APPROXIMATION IN POSITIVE CHARACTERISTIC

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1 Introduction and notations

1.1 The field $\mathbb{F}(q)$

Let p be a prime number and $q = p^s$ with a positive integer s. We consider the finite field $\mathbb{F}_q$ with q elements. Then we introduce with an indeterminate T, the ring of polynomials $\mathbb{F}_q[T]$ and the field of rational functions $\mathbb{F}_q(T)$. We also consider the absolute value defined on $\mathbb{F}_q(T)$ by $|P/Q| = |T|^{\deg P - \deg Q}$ for $P, Q \in \mathbb{F}_q[T]$, where $|T|$ is a fixed real number greater than one. By completing $\mathbb{F}_q(T)$ with this absolute value we obtain a field denoted by $\mathbb{F}(q)$ which is the field of formal power series with coefficients in $\mathbb{F}_q$. Thus if α is a non-zero element of $\mathbb{F}(q)$ we have

$$\alpha = \sum_{k \leq k_0} u_k T^k \quad \text{with } u_k \in \mathbb{F}_q, u_{k_0} \neq 0 \quad \text{and} \quad |\alpha| = |T|^{k_0}.$$

Observe the analogy between the classical construction of the field of real numbers and the field of power series which we are considering here. The rôles of $\{\pm 1\}$, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ are played by $\mathbb{F}_q^*$, $\mathbb{F}_q[T]$, $\mathbb{F}_q(T)$ and $\mathbb{F}(q)$ respectively. Clearly the same construction as above can be made from an arbitrary base field K instead of $\mathbb{F}_q$, then the resulting field is called the field of power series over K and will be denoted by $\mathbb{F}(K)$. We study here rational approximation to elements of $\mathbb{F}(K)$ which are algebraic over $K(T)$. We are concerned with the case of K having positive characteristic and mainly $K = \mathbb{F}_q$. For a presentation in a larger context and for more references see [13]. Indeed the finiteness of the base field plays an essential rôle in many results and this makes the field $\mathbb{F}(q)$ particularly interesting.

1.2 Continued fractions in $\mathbb{F}(q)$

As in the classical context of the real numbers, we have a continued fraction algorithm in $\mathbb{F}(q)$. For a general study on this subject and more references see [10]. If $\alpha \in \mathbb{F}(q)$ we can write

$$\alpha = a_0 + 1/(a_1 + 1/(a_2 + ... = [a_0, a_1, a_2, ...] \quad \text{where} \quad a_i \in \mathbb{F}_q[T].$$

The a_i are called the partial quotients and we have $\deg a_i > 0$ for $i > 0$. This continued fraction expansion is finite if and only if $\alpha \in \mathbb{F}_q(T)$. As in the classical theory we define recursively the two sequences of polynomials $(x_n)_{n\geq 0}$ and $(y_n)_{n\geq 0}$ by

$$x_n = a_n x_{n-1} + x_{n-2} \quad \text{and} \quad y_n = a_n y_{n-1} + y_{n-2},$$

with the initial conditions $x_0 = a_0$, $x_1 = a_0 a_1 + 1$, $y_0 = 1$ and $y_1 = a_1$. We have $x_{n+1}y_n - y_{n+1}x_n = (-1)^n$, whence x_n and y_n are coprime polynomials. The rational x_n/y_n is called a convergent to α and we have $x_n/y_n = [a_0, a_1, a_2, ..., a_n]$. Because of the ultrametric absolute value we have

$$|\alpha - x_n/y_n| = |x_{n+1}/y_{n+1} - x_n/y_n| = |y_n y_{n+1}|^{-1} = |a_{n+1}|^{-1}|y_n|^{-2}.$$

1.3 A subset of algebraic elements in $\mathbb{F}(q)$

Here we are concerned by elements in $\mathbb{F}(q)$ which are algebraic over $\mathbb{F}_q(T)$ and we introduce a special subset of algebraic elements. Let $r = p^t$ where $t \geq 0$ is an integer. We denote by $H_t(q)$ the subset of irrational elements Θ in $\mathbb{F}(q)$ such that there exist $A, B, C, D \in \mathbb{F}_q[T]$ with

$$\alpha = \frac{A\alpha^r + B}{C\alpha^r + D}. \tag{1}$$

We put $H(q) = \bigcup_{t\geq 0} H_t(q)$. If t is the smallest non-negative integer such that $\alpha \in \mathbb{F}(q)$ satisfies an equation of type (1) we will say that α is an hyperquadratic element of order t. With our definition an hyperquadratic element of order zero is simply a quadratic element. We observe that elements of $\mathbb{F}(q)$ which are quadratic or cubic over $\mathbb{F}_q(T)$ belong to $H_1(q)$ since then $1, \alpha, \alpha^p, \alpha^{p+1}$ are linked over $\mathbb{F}_q(T)$. Note that if $\alpha \in H_t(q)$, then by iteration in equation (1), $\alpha \in H_{kt}(q)$ for all positive integer k. Moreover $H(q)$ contains also elements of arbitrarily large degree over $\mathbb{F}_q(T)$. Note that if K is a field of positive characteristic p but not necessarily finite we will use the notation $H(K)$ for the corresponding subset of algebraic elements in $\mathbb{F}(K)$. We recall that the analogue of Lagrange's theorem on quadratic real numbers holds in the context of power series over a finite field.

THEOREM 1. *Let $\alpha \in \mathbb{F}(q)$ be irrational. Then the sequence of partial quotients in the continued fraction expansion of α is ultimately periodic if and only if $\alpha \in H_0(q)$.*

As we will see the elements of the subset $H(q)$ have special properties of rational approximation due to the form of equation (1). For these elements the sequence of the degrees of the partial quotients can be bounded (as for instance in the case of quadratic elements) or unbounded. To illustrate this, given a prime p let us introduce the element of $\mathbb{F}(p)$ defined by the following infinite expansion

$$\alpha_1 = [T, T^r, T^{r^2}, \ldots, T^{r^n}, \ldots] \quad \text{where} \quad r = p^t \quad \text{whith } t \geq 0.$$

Because of the property of the Frobenius isomorphism, this element is indeed algebraic (quadratic if r=1) satisfying the equation $\alpha_1 = T + 1/\alpha_1^r$ and it belongs to $H(p)$.

2 General results

2.1 Mahler's Theorem

Diophantine appoximation in the function field case was initiated by K. Mahler [1]. The starting point in the study of rational approximation to algebraic real numbers is a famous theorem of Liouville established in 1850. This theorem has been adapted by Mahler in the fields of power series with an arbitrary base field.

THEOREM 2 (K. Mahler,1949). *Let K be a field and $\alpha \in \mathbb{F}(K)$ be an algebraic element over $K(T)$ of degree $n > 1$. Then there is a positive real number C such that*

$$|\alpha - P/Q| \geq C|Q|^{-n}$$

for all $P, Q \in K[T]$, with $Q \neq 0$.

In the case of real numbers, it is well known that Liouville's theorem has been improved until Roth's theorem which was established in 1955. This improvement can be transposed in fields of power series if the base field has characteristic zero as it was proved by Uchiyama in 1960. In this case the exponent n in the right hand side of the inequality in the above theorem can be replaced by $2 + \epsilon$ for all $\epsilon > 0$. But this is not the case in positive characteristic and consequently the study of rational approximation to algebraic elements becomes more complex.

2.2 The approximation exponent

Let $\alpha \in \mathbb{F}(K)$ be an irrational element. We define the *approximation exponent* of α by

$$\nu(\alpha) = \limsup_{|Q| \to \infty} \left(-\frac{\log|\alpha - P/Q|}{\log|Q|}\right)$$

where P and Q run over polynomials in $K[T]$ with $Q \neq 0$. Considering the continued fraction expansion $\alpha = [a_0, a_1, \ldots, a_n, \ldots]$, since the convergents are the best rational approximations to α, it is clear, from $|\alpha - x_n/y_n| = |a_{n+1}|^{-1}|y_n|^{-2}$, that the approximation exponent can also be defined directly by

$$\nu(\alpha) = 2 + \limsup_k (\deg a_{k+1}/\deg y_k).$$

Observe that $\deg y_k = \sum_{1 \leq i \leq k} \deg a_i$ and therefore $\nu(\alpha)$ is directly connected to the growth of the sequence $(\deg a_i)_{i \geq 1}$. Particularly if the sequence $(\deg a_i)_{i \geq 1}$ is bounded then $\nu(\alpha) = 2$, but this is clearly not a necessary condition. Observe that, because of Mahler's theorem, for all $\alpha \in \mathbb{F}(q)$ algebraic over $\mathbb{F}_q(T)$ and of degree $n > 1$, we have

$$\nu(\alpha) \in [2, n].$$

In the same paper [1], K. Mahler introduced for a prime number p the element of $\mathbb{F}(p)$ defined by

$$\alpha_2 = \sum_{k \geq 0} T^{-r^k} \qquad \text{with } r = p^t \quad \text{and } t > 0.$$

He observes that this element satisfies $\alpha_2 = 1/T + \alpha_2^r$ and has $\nu(\alpha_2) = r$. Thus it is an algebraic element of degree r. With our notations α_2 belongs to $H(p)$. Note that, according to its continued fraction expansion, the element $\alpha_1 \in H(p)$ introduced above has $\nu(\alpha_1) = r + 1$ and is algebraic of degree $r + 1$.

2.3 Osgood's Theorem

At the begining of the years 1970's, C. Osgood [2] used Differential Algebra to study diophantine approximation in the function field case. We introduce on $\mathbb{F}(K)$ the ordinary formal differentiation where $(aT^n)' = anT^{n-1}$ if $a \in K$ and $n \in \mathbb{Z}$. Observe that if K has positive characteristic p then the subfied of constants for this differentiation is the field of power series in T^p over K. If $\alpha \in \mathbb{F}(K)$ is algebraic of degree n over $K(T)$ then we have $P(\alpha) = 0$ where $P \in K[T][X]$ and differentiating this equation we obtain $\alpha' P'_X(\alpha) + P'_T(\alpha) = 0$. Therefore $\alpha' \in K(\alpha, T)$ and consequently there is an integer d with $0 \leq d \leq n-1$ such that $\alpha' = Q(\alpha)$ where $Q \in K(T)[X]$ and $\deg_X(Q) = d$. If $d \leq 2$ then we say that α satisfies a Riccati differential equation. Then C. Osgood was able to prove :

THEOREM 3 (C. Osgood,1974). *Let K be a field of positive characteristic. Let $\alpha \in \mathbb{F}(K)$ be an algebraic element over $K(T)$, of degree $n > 1$. Then if α does not satisfy a Riccati differential equation there is a positive real number C such that*

$$|\alpha - P/Q| \geq C|Q|^{-([n/2]+1)}$$

for all $P, Q \in K[T]$, with $Q \neq 0$.

In the same paper C. Osgood also introduced a new family of algebraic elements in $\mathbb{F}(p)$ with a maximal approximation exponent. If $n > 1$ is an integer coprime with p then $\alpha_3 \in \mathbb{F}(p)$ defined by $\alpha_3^n = 1 + 1/T$ is algebraic of degree n over $\mathbb{F}_p(T)$ and has $\nu(\alpha_3) = n$. Observe that if t is the order of p in $(\mathbb{Z}/n\mathbb{Z})^*$ and $r = p^t$ then we have $\alpha_3 = (1 + 1/T)^{-(r-1)/n}\alpha_3^r$ and consequently $\alpha_3 \in H(p)$.

2.4 Continued fractions in H(q)

Very shortly afterwards, L. Baum and M. Sweet [3] have studied diophantine approximation in $\mathbb{F}(2)$ by mean of the continued fraction expansion. They gave different examples of algebraic continued fractions with bounded or unbounded partial quotients. Particularly they could prove the following :

THEOREM 4 (L. Baum and M.Sweet,1976). *Let α be the unique root in $\mathbb{F}(2)$ of the algebraic irreducible equation*

$$T\alpha^3 + \alpha + T = 0$$

then the partial quotients of α have degree one or two.

Several years later (1986), this important work by L. Baum and M. Sweet on algebraic continued fractions in $\mathbb{F}(2)$ was deeply extended by W. Mills and D. Robbins [4]. They first pointed at the rôle played by the shape of the equation. In arbitrary positive characteristic they introduced the elements of the subset we have denoted above by $H(q)$. Moreover they developped an algorithm to obtain the continued fraction expansion for such elements. Thus they could describe precisely the expansion for the cubic example introduced by L. Baum and M. Sweet. Also they could produce algebraic and nonquadratic examples in $\mathbb{F}(p)$ for all $p \geq 3$ with all partial quotients of degree one. Finally they proved: If $\alpha \in H(q)$ and if in equation (1) we have $\deg(AD - BC) < r - 1$ then the sequence of partial quotients is unbounded.

2.5 Diophantine approximation in H(K)

Independently and at about the same time, J-F. Voloch [5], inspired by C. Osgood's works on diophantine approximation in positive characteristic, pointed at the importance of the algebraic equation stated above. He first observed that if $\alpha \in H(K)$ then α satisfies a Riccati differential equation. He could prove the following

THEOREM 5 (J-F. Voloch,1988). *Let K be a field of positive characteristic. If $\alpha \in H(K)$ and has approximation exponent $\nu(\alpha)$, then there is a positive real* *number C such that*

$$|\alpha - P/Q| \geq C|Q|^{-\nu(\alpha)}$$

for all $P, Q \in K[T]$, with $Q \neq 0$.

COROLLARY:*Let $\alpha \in H(K)$ then $\nu(\alpha) = 2$ if and only if the sequence of partial quotients for α is bounded.*

Later B. de Mathan [6], by studying more deeply rational approximation of elements in $H(K)$, obtained the following theorem which contains Theorem 5.

THEOREM 6 (B. de Mathan,1992).*Let K be a field of positive characteristic. If $\alpha \in H(K)$ and has approximation exponent $\nu(\alpha)$, then*

$$\liminf_{|Q|\to\infty} |Q|^{\nu(\alpha)}|\alpha - P/Q| \neq 0, \infty \quad \textit{and} \quad \nu(\alpha) \in \mathbb{Q}.$$

2.6 Singularity of H(K)

Later by adapting the method used by A. Thue on rational approximation to algebraic real numbers, we could prove the following [7] :

THEOREM 7 (B. de Mathan,A.L.,1996). *Let K be a field of positive characteristic. Let $\alpha \in \mathbb{F}(K)$ be an algebraic element over $K(T)$ of degree $n > 1$. Assume that $\alpha \notin H(K)$. Then for every $\epsilon > 0$ there is a positive real number C such that*

$$|\alpha - P/Q| \geq C|Q|^{-([n/2]+1+\epsilon)}$$

for all $P, Q \in K[T]$, with $Q \neq 0$.

Using Osgood's theorem and the hypothesis of a finite base field, we could prove almost the same result [8]:

THEOREM 8 (B. de Mathan,A.L.,1998). *Let $\alpha \in \mathbb{F}(q)$ be an algebraic element over $\mathbb{F}_q(T)$ of degree $n > 1$. Assume that $\alpha \notin H(q)$. Then there is a positive real number C such that*

$$|\alpha - P/Q| \geq C|Q|^{-([n/2]+1)}$$

for all $P, Q \in K[T]$, with $Q \neq 0$.

3 Two subclasses in H(q)

In spite of the attempt made by Mills and Robbins [5], the possibility of describing the continued fraction expansion for all the elements in $H(q)$ is yet out of reach. Nevertheless an explicit description is possible for many examples and also for large subclasses.

3.1 Elements of class IA

We will say that an element in $H(q)$ is of class IA if we have $AD - BC \in \mathbb{F}_q^*$ in equation (1). The number α_1 introduced above belongs to this subclass. Observe that, according to the property first stated by Mills and Robbins, if α is of class IA and $r > 1$ then $\deg(AD - BC) = 0 < r - 1$ and the sequence of partial quotients is unbounded. Such algebraic elements have been studied by Schmidt [10] and also by Thakur [9]. They proved independently the following theorem and its corollary.

THEOREM 9 (W. Schmidt, D. Thakur, 1999-2000). *$\alpha \in \mathbb{F}(q)$ is algebraic of class IA if and only if there exist $k \geq 0$, $a_j, c_i \in \mathbb{F}_q[T]$ with $1 \leq j \leq k$ and $i \geq 1$, an integer $t \geq 1$ and $\epsilon \in \mathbb{F}_q^*$ such that*

$$\alpha = [a_1, a_2, \ldots, a_k, c_1, c_2, \ldots, c_n, \ldots]$$

where for $l \geq 1$ we have

$$c_{l+t} = \begin{cases} \epsilon c_l^r & \text{if } l \text{ is odd,} \\ \epsilon^{-1} c_l^r & \text{if } l \text{ is even .} \end{cases}$$

Observe that the expansion for such an element is determined by the first $t + k$ partial quotients. In the case $r = 1$ we obtain the classical periodic expansion for quadratic power series. The fact that the continued fraction expansion can be given explicitly for these elements, by chosing the integer t and the polynomials c_i for $1 \leq i \leq t$, implies the following important corollary.

COROLLARY.*Let μ be a rational real number with $\mu \geq 2$. Then there is an element α in $H(q)$ such that $\nu(\alpha) = \mu$.*

3.2 Elements in H(q) with linear partial quotients

Now our goal is to present a second subclass of $H(q)$. This one contains non-quadratic elements with all partial quotients of degree one. The existence of such elements was first pointed at in [4]. The references for what is presented here are [11] and [12]. Here r is as above, l is an integer with $l \geq r$ and ϵ is a given element in $\mathbb{F}_q^*$. We consider a sequence $(a_i)_{i \geq 1}$ of polynomials with $a_i = \lambda_i T$ and $\lambda_i \in \mathbb{F}_q^*$ for $i \geq 1$. Then, for $i \geq 0$ and $k \geq -1$, we introduce the polynomials $x_{i,k} \in \mathbb{F}_q[T]$ defined by

$$x_{0,k} = 0 \qquad x_{1,k} = 1 \quad \text{and} \quad x_{i,k} = a_{k+i} x_{i-1,k} + x_{i-2,k}.$$

Observe that for $i \geq 2$, $x_{i,k}$ is a polynomial of degree $i - 1$ depending on the $i - 1$ elements of $\mathbb{F}_q^*$ from λ_{k+2} to λ_{k+i}. We have the following result :

THEOREM 10 (A.L.,J-J. Ruch, 2002). *Let $\epsilon \in \mathbb{F}_q^*$ and $l \geq r$ be given. Let $\alpha = [\lambda_1 T, \lambda_2 T, \ldots, \lambda_n T, \ldots] \in \mathbb{F}(q)$. Then α satisfies the algebraic equation*

$$(E) \quad \alpha = \frac{\epsilon x_{l+1,-1}\alpha^r + x_{l-r+1,-1}}{\epsilon x_{l,0}\alpha^r + x_{l-r,0}}$$

if and only if the sequence $(\lambda_n)_{n\geq 1}$ satisfies for $n \geq 1$ the system

$$(S) \quad \begin{cases} x_{2r,(n-2)r+l} = \epsilon^{(-1)^{n-1}} \lambda_n^r T^r x_{r,l-r} \\ x_{r,(n-1)r+l} = x_{r,l-r} \end{cases}$$

Observe that the coefficients in equation (E) depend upon the l elements $\lambda_1, \ldots, \lambda_l$ and ϵ in $\mathbb{F}_q^*$. Moreover, with the above notations, one can remark that here we have $\deg(AD - BC) = r - 1$. The existence of sequences $(\lambda_n)_{n\geq 1}$ solutions of the system (S) will eventually depend on the choice of ϵ and of the first l terms $\lambda_1, \lambda_2, \ldots, \lambda_l$. We are now going to describe three families of solutions of (S).

3.2.1 Solutions in characteristic two

THEOREM 11 (A.L.,J-J. Ruch, 2004). *We assume that $q = 2^s$ and $r = 2^t$ with two positive integers s and t. Let $\lambda_1, \lambda_2, \ldots, \lambda_l$ be given in $\mathbb{F}_q^*$ such that*

$$(H_r) \qquad x_{r,l-r}T^{-r+1} \in \mathbb{F}_q^*.$$

Then there is a sequence $(\lambda_n)_{n\geq 1}$ in $\mathbb{F}_q^*$ satisfying the system (S). This sequence $(\lambda_n)_{n\geq 1}$ is defined for $n \geq 0$ by the following formulas

$$\lambda_{rn+l+1} = \epsilon^{(-1)^n} \lambda_{n+1}^r \prod_{l-r+2 \leq i \leq l} \lambda_i^{-1}$$

$$\lambda_{rn+l+k} = \begin{cases} \lambda_{l+2-k} & \text{if} \quad n \equiv 0 \mod 2 \\ \lambda_{l-r+k} & \text{if} \quad n \equiv 1 \mod 2 \end{cases} \qquad 2 \leq k \leq r.$$

We can notice that condition (H_r) only involves the elements $\lambda_{l-r+2}, \ldots, \lambda_l$. Moreover (H_2) is empty and (H_4) reduces to $\lambda_l = \lambda_{l-2}$.

3.2.2 Solutions in odd characteristic

We introduce the following definition. Let $(\lambda_n)_{n\geq 1}$ and $(y_n)_{n\geq -1}$ be two sequences in $\mathbb{F}_q^*$. Given two integers r and l with $l \geq r \geq 3$, we say that the sequence $(\lambda_n)_{n\geq 1}$ is (r, l)-derived from the sequence $(y_n)_{n\geq -1}$ if $\lambda_1, \ldots, \lambda_l$ are given and for $n \geq 0$ we have

$$\lambda_{rn+l+1} = (1/2)(y_n + y_{n-1}^{-1}) \quad \text{and} \quad \lambda_{rn+l+k} = y_n^{(-1)^{k+1}} \quad \text{for } 2 \leq k \leq r.$$

Then we have the following theorem :

THEOREM 12 (idem). *We assume that p is an odd prime number. Let $q = p^s$, $r = p^t$ and $l \geq r$ be given with $s \geq 1$ and $t \geq 1$. Let $\lambda_1, \ldots, \lambda_{l-r+1}$ and ϵ be given in $\mathbb{F}_q^*$ such that*

$$(H) \quad \epsilon = [2\lambda_1, -2\lambda_2, \ldots, (-1)^{l-r} 2\lambda_{l-r+1}, (-1)^{l-r+1}]^{-r}.$$

We put $\lambda_i = 1$ for $l - r + 2 \leq i \leq l$.
Let $(y_n)_{n \geq -1}$ be the sequence in $\mathbb{F}_q^*$ defined by the initial conditions

$$y_{-1} = 1, \quad y_i = 2\epsilon^{(-1)^i} \lambda_{i+1}^r - y_{i-1}^{-1}, \quad 0 \leq i \leq l - r$$

and for $n \geq -1$ by the recursive formula

$$y_{nr+l+k} = y_n^{(-1)^k r} \epsilon^{(-1)^{nr+l+k}}, \quad 0 \leq k \leq r - 1.$$

Then the sequence $(\lambda_n)_{n \geq 1}$ in $\mathbb{F}_q^*$ which is (r, l)-derived from the sequence $(y_n)_{n \geq -1}$ satisfies the system (S).

It is possible to check that for all $l \geq r$ there exist $\epsilon \in \mathbb{F}_q^*$ and tuples in $(\mathbb{F}_q^*)^{l-r+1}$, $(\lambda_1, \ldots, \lambda_{l-r+1})$ satisfying condition (H) in the above theorem.

3.2.3 Singular solutions in certain fields of odd characteristic

THEOREM 13 (idem). *We assume that p is an odd prime number. Let $r = p^t$ and $q = p^{2s+1}$ be given with $s \geq 1$ and $t \geq 1$.*
Let $e \in \mathbb{F}_q^*$ be such that $e \notin \mathbb{F}_p$ and $e^r + e + 1 \neq 0$.
We put $\lambda_1 = (e^r + e + 1)^{r^{2s}} (2e)^{-1}$ and $\lambda_i = 1$ for $2 \leq i \leq r$. We put $\epsilon = e^{r-1}$.
Let $(u_n)_{n \geq -1}$ be the sequence in $\mathbb{F}_q^*$ defined by the initial condition $u_{-1} = e^{-r}$ and for $n \geq -1$ and $0 \leq k \leq r - 1$ by

$$u_{r(n+1)+k} = \Big(\frac{u_n}{(1 + k u_n)(1 + (k+1) u_n)} \Big)^r.$$

Let $(y_n)_{n \geq -1}$ be the sequence in $\mathbb{F}_q^*$ defined by the initial condition $y_{-1} = 1$ and for $n \geq -1$ and $0 \leq k \leq r - 1$ by

$$y_{r(n+1)+k} = y_n^{(-1)^k r} \epsilon^{(-1)^{n+k+1}} \big(1 + u_n (1 + k u_n)^{-1}\big).$$

Then the sequence $(\lambda_n)_{n \geq 1}$ in $\mathbb{F}_q^*$ which is (r, r)-derived from the sequence $(y_n)_{n \geq -1}$ satisfies the system (S) with $l = r$.

Observe that when considering the sequence $(u_n)_{n \geq -1}$ identically zero and $l = r$ the solution in Theorem 13 becomes the one given in Theorem 12. Moreover the cardinality $q = p^{2s+1}$ of the base field is important to ensure the existence of a sequence $(u_n)_{n \geq -1}$ defined as above. In particular and for instance these solutions do not exist in a prime field.

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