数理解析研究所講究録
1511巻 2006年 179-187

Hankel determinants and substitutions — some results and problems

Jun-ichi TAMURA

3-3-7-307 Azamino Aoba-ku Yokohama 225-0011 Japan

1. Introduction. Let A^* be a free monoid generated by a non-empty set A, i.e., A^* is the set of finite words over A with the empty word λ. We put $A^\#:=A^*\cup A^{\mathbf{N}}$, where $\mathbf{N}$ is the set of non-negative integers, so that $A^{\mathbf{N}}$ is the set of infinite words over A. Any monoid morphism $\sigma:A^*\to A^*$ can be extended to a map $\sigma:A^\#\to A^\#$ by $\sigma(a_0a_1a_2\cdots):=\sigma(a_0)\sigma(a_1)\sigma(a_2)\cdots$ $(a_n\in A)$, which is a so-called substitution (over A). We say that σ is of constant length k iff $\sigma(x)$ is a finite word of length k for all $x\in A$. A fixed point of a substitution σ is an infinite word $\varphi\in A^{\mathbf{N}}$ satisfying $\sigma(\varphi)=\varphi$.

The fixed point of a substitution σ over $\{a,b\}$ defined by

$$\sigma(a)=ab,\ \sigma(b)=ba\ (\text{resp.},\ \sigma(a)=ab,\ \sigma(b)=a)$$

prefixed by a is referred to as the Thue-Morse word (resp., the Fibnacci word).

Let $q>1$ be an integer. We denote by $\mathrm{ord}_q(n)$ the largest integer $e\geq 0$ such that n is divisile by q^e. We say a word $w=w_1w_2w_3\ldots$ is a q-adic Toeplitz word iff $w_m=w_n$ holds for any positive integers m, n satisfying $\mathrm{ord}_q(m)=\mathrm{ord}_q(n)$. Let σ be a substitution over an infinite alphabet $A_\infty:=\{a_0,a_1,a_2,\ldots\}$ defined by

$$\sigma(a_n)=a_0a_{n+1}\ (n=0,\ 1,\ 2,\ \ldots).$$

For some of the symbols a_0, a_1, a_2, ..., we also write $a_0=a$, $a_1=b$, $a_2=c$, etc. The substitution σ has a unique fixed point

$$\omega=abacabadabacabaeabacabadabacabaf\ldots,$$

which is a 2-adic Toeplitz word. Any 2-adic Toeplitz word over a finite or an infinite alphabet B can be written by

$$\tau(\omega):=\tau(\omega_1)\tau(\omega_2)\tau(\omega_3)\ldots,$$

where τ is a map from A_∞ to B, and $\omega=:\omega_1\omega_2\omega_3\ldots(\omega_i\in A_\infty)$. In this sense the word ω

is a universal 2-adic Toeplitz word.

Our objective is to get something interesting related to determinants

$$H_n^{(m)}=H_n^{(m)}[\varphi]:=\det(\varphi_{m+i+j})_{0\le i\le n-1,\ 0\le j\le n-1},$$

$$H_n=H_n[\varphi]:=H_n^{(0)}[\varphi]$$

for a given infinite word $\varphi=\varphi_0\varphi_1\varphi_2\ldots$ ($\varphi_i\in A$) over a finite, or an infinite alphabet A, where $H_n^{(m)}$ is considered to be an element of $\mathbf{Z}[A]$, i.e., a polynomial in independent variables$\in A$ with integer coefficients. $H_n^{(m)}[\varphi]$ can be extended to $(m,n)\in\mathbf{Z}\times\mathbf{N}$ by setting $H_n^{(0)}:=1$, $\varphi_m:=0$ $(m<0)$, where $\mathbf{N}$ denotes the set of non-negative integers. In the following two sections, we give a very rough survey on the results related to $H_n^{(m)}$. In Section 2, we give some results on $H_n^{(m)}[\varphi]$ for a general word φ, cf. [K-T-W], [T2]. In Section 3, we give some of the results related to $H_n^{(m)}[\varphi]$ when φ is the Thue-Morse word, the Fibonacci word, and a Fibonaci-type word, cf. [A-P-ZXW-ZYW], [K-T-W], [T2]. In Section 4, we give a new characterization of the 2-adic Toeplitz words φ by an algebraic property (completely reducibility) of $H_n[\varphi]$, cf. [M-T-Tn]. We shall give no proofs, but state only results with minimum definition.

2. General propeties of $H_n[w]$. The set A^* becomes a complete metric space with respect to the metric defined by

$$d(\xi,\eta):=\exp(-\inf\{n;\ \xi_n\neq\eta_n\})\quad(\xi=\xi_0\xi_1\xi_2\cdots,\ \eta=\eta_0\eta_1\eta_2\cdots\in A^*\ (\xi_n,\eta_n\in A)).$$

As usual, $\mathbf{K}((Z))$ denotes the set of formal Laurent series of one variable Z over a field $\mathbf{K}$. We put

$$\mathbf{K}:=\mathbf{Q}(A)\ (\supset A).$$

The set $\mathbf{K}((z^{-1}))$ becomes a metric space induced by a non-Archimedean norm defined by

$$\|\varphi^{(h)}\|:=\exp(-n_0+h),\ n_0:=\inf\{n\in\mathbf{N};\ \varphi_n\neq 0\}\ (\|0\|:=0)$$

for

$$\varphi^{(h)}=\sum_{n\ge 0}\varphi_n z^{-n+h}\in\mathbf{K}((z^{-1}))\tag{1}$$

with $h\in\mathbf{Z}:=\{0,\pm1,\pm2,\ldots\}$. Note that $\|\varphi^{(h)}\|=\exp h$ holds if $\varphi=\varphi_0\varphi_1\varphi_2\cdots\in A^{\mathbf{N}}(\subset$

K^{*}. If φ is a finite word of length k, then $\varphi_n:=0$ for $n\geq k$.) For any given $\varphi=\varphi_0\varphi_1\varphi_2\cdots\in K^{*}$, we say that $(P,Q)\in K[z]^2$ is an h-Padé pair of order m for φ iff

$$\| Q\varphi^{(h)}-P \| < \exp(-m), \quad Q\neq 0, \quad \deg Q:=\deg_z Q\leq m \tag{2}$$

holds. The usual Padé pair (for a formal Laurent series) agrees with the h-Padé pair with h=-1 (for a word), cf. [N-S]. It is known that an h-Padé pair (P,Q) of order m for φ always exists for any $h\in\mathbf{Z}$, $m\geq 0$, $\varphi\in K^{*}$, cf. Lemma 1, [T2]. For h-Padé pairs (P,Q) of order m for φ, a rational function $P/Q\in K(z)$ is uniquely determined for any given $h\in\mathbf{Z}$, $m\geq 0$, $\varphi\in K^{*}$. The element $P/Q\in K(z)$ for an h-Padé pair (P,Q) of order m for $\varphi\in K^{*}$ is referred to as the h-Padé approximant of order m for φ. A number $m\in\mathbf{N}$ is called a normal h-index for $\varphi\in K^{*}$ if (2) implies deg Q=m. A normal h-Padé pair, i.e., deg Q is a normal h-index, is said to be normalized if the leading coefficient of Q equals one. Normal (-1)-indices (resp. (-1)-Padé pairs, (-1)-Padé approximants) will be simply referred to as normal indices (resp. Padé pairs, Padé approximants). The set of all the normal h-indices for φ will be denoted by

$$\Lambda_h(\varphi):=\{m\in\mathbf{N}\,;\ m \text{ is normal h-indices for } \varphi\},$$

$$\Lambda(\varphi):=\Lambda_{-1}(\varphi).$$

We can consider the series (1) over $K=\mathbf{Q}(a,b,\ldots)$ with $a, b, \ldots\in\mathbf{C}$. In such a case, $\varphi^{(h)}$ defined by (1) turns out to be not only an element of $\mathbf{C}((z^{-1}))$, but also an analytic function on $\{z\in\mathbf{C}\,;\ |z|>1\}$, and the h-Padé approximant of order m for $\varphi^{(h)}$ pointwise converges to $\varphi^{(h)}$ with respect to the usual topology on $\mathbf{C}$ for each $z\in\mathbf{C}$, $|z|>1$ as m tends to infinity.

Proposition 1 (cf. [T1]). Let $\varphi\in K^{*}$ be a word over $K=\mathbf{Q}(A)$ with an alphabet A possibly consisting of infinite letters.

$$H_{h+1}^{(m)}[\varphi]=(-1)^{\lfloor m/2\rfloor}\prod_{Q(z)=0} P(z) \quad (h,m\in\mathbf{Z},\ m\geq 0),$$

where (P,Q) is a normalized h-Padé pair of degree m for φ, $\lfloor x\rfloor$ denotes the largest integer not exceeding a real number x, and $\prod_{Q(z)=0}$ indicates a product taken over all the zeros z of Q with their multiplicity in any field $\widetilde{K}$ containing an algebraic closure of K.

Remark 1. We can take $\widetilde{K}=\mathbf{C}$ in Proposition 1 in the case where A is a subset (possibly empty) of $\mathbf{C}$.

Remark 2. If m is not a normal h-index of φ, then $P, Q\in K[z]$ have common zeros. Hence, it follows from Proposition 1 that $m\notin\Lambda_h(\varphi)$ implies $H_{h+1}^{(m)}=0$. The converse of this fact is valid, cf. Lemma 2, [T2].

In particular, Proposition 1 holds for all the fixed point $\varphi\in A^{\mathbf{N}}(\subset K^*)$ of a substitution over any alphabet A. The following remark is usefull, while it is valid only for a word φ consising of at most two symbols.

Remark 3. Let M be a matrix of size n×n with entries consisting of two variables a, b (symbols). Then

$$\det M=(a-b)^{n-1}(pa+qb)\in\mathbf{Z}[a,b],$$

where p, q are integers defined by

$$p=\det M\,|_{(a,b)=(1,0)},\quad q=\det M\,|_{(a,b)=(0,1)}.$$

3. Thue-Morse, and Fibonacci cases.

J.-P. Allouche, J. Peyrière, Z.-X. Wen and Z.-Y. Wen considered $H_{n,m}(\zeta)$ for the Thue-Morse sequence ζ=abbabaab··· with (a,b)=(1,0), and showed that the 2-dimensional word $H_n^{(m)}(\zeta)$ (mod 2) of $(n,m)\in\mathbf{N}^2$ is 2-dimensionally automatic; it is remarkable that $\Lambda(\zeta)=\mathbf{N}$ is known, cf. [A-P-ZXW-ZYW].

In general, it is very difficult to give an explicit formula of $H_n^{(m)}(\varphi)$ for a given infinite word φ that is not periodic, while explcit formulae of $H_n^{(m)}(\eta)$ are completely given for the Fibonacci word η=abaabab..., cf. Theorems 1-5 in [K-T-W]. By f_n we denote the n-th Fibonacci number ($f_{-1}=f_0=1$, $f_n=f_{n-1}+f_{n-2}$). Let

$$n = \sum_{i \geq 0} \delta_i(n) f_i \quad (\delta_i(n) \in \{0,1\}, \ \delta_{i+1}(n)\delta_i(n)=0 \text{ for all } i \geqq 0)$$

be the reperesentation of n in the Fibonacci base due to Zeckendorf. We write

$$m \equiv_k n$$

iff $\delta_i(m)=\delta_i(n)$ holds for all $0 \leqq i < k$. We put

$$\chi(k,S) := -1 \text{ if } k \equiv s \pmod 6 \text{ for an } s \in S,$$
$$1 \text{ otherwise,}$$

for a subset S of $\{0,1,2,3,4,5\}$. Then we have, for instance,

<u>Proposition 2</u> (Theorem 3, [K-T-W]). For any k, m, $i \geqq 0$ integers satisfying

$$m \equiv_{k+1} i, \quad 0 \leqq i \leqq f_{k+1}-1,$$

the following formulae hold:

$$H_{f_k}^{(m)}[\eta]\,|_{(a,b)=(1,0)} = \chi(k;2)\chi(k;1,4)^i f_{k-1},$$
if either $\delta_{k+1}(m)=0$ and $0 \leqq i < f_{k-1}$,
or $\delta_{k+1}(m)=1$ and $0 \leqq i < f_k$,
$$= \chi(k;1,2,4) f_{k-2},$$
if either $\delta_{k+1}(m)=0$ and $i=f_{k-1}$,
or $i=f_{k+1}-1$,
$$= 0 \text{ otherwise,}$$
$$H_{f_k}^{(m)}[\eta]\,|_{(a,b)=(0,1)} = \chi(k;1,2,4)\chi(k;1,4)^i f_{k-2},$$
if either $\delta_{k+1}(m)=0$ and $0 \leqq i < f_{k-1}$,
or $\delta_{k+1}(m)=1$ and $0 \leqq i < f_k$,
$$= \chi(k;2) f_{k-3},$$
if either $\delta_{k+1}(m)=0$ and $i=f_{k-1}$,
or $i=f_{k+1}-1$,
$$= 0 \text{ otherwise.}$$

Notice that Proposition 2 together with Remark 3 gives a part of the explicit formulae for $H_n^{(m)}[\eta]$. In comparison with the automayicity result for the 2-dimensional word $(H_n^{(m)}(\zeta) \pmod 2))_{(n,m)\in\mathbf{N}^2}$ for the Thue-Morse sequence

ζ=10010110$\cdots$ given in [A-P-ZXW-ZYW], we gave an explicit expression of the 2-dimensional word $H_n^{(m)}(\eta)$ $(n,m)\in\mathbb{N}^2$ for the Fibonacci word η=abaabab..., which is rather complicated, cf. Theorem 5 in [K-T-W].

In [T2], we developped a theory of analysis on words, especially for words of Fibonacci type, i.e. the fixed points of substitutions of the form

$$\sigma(a):=a^k b,\ \ \sigma(b):=a\ \ (k>0). \tag{3}$$

We denote by $\varepsilon=\varepsilon(a,b;k)$ the fixed point of the substitution defined by (3), and by $|w|_x$:=the number of occurences of an identical symbol x appearing in a word w. In Sections 2, 3 in [T2], we gave explicit formulae for the continued fraction expansion with partial denominators$\in K[z]$ $(K=\mathbb{Q}(a,b))$ and normalized Padé pairs for the Laurent series $\varepsilon^{(-1)}(z)=\varepsilon(z;a,b;k)$ defined by (1) with $\varphi=\varepsilon$, cf. Theorems 1-12, [T2]. For instance, we have

<u>Proposition 3</u> (Theorem 8, [T2]). Let $k\geq 2$. The continued fraction expansion of the Laurent series $\varepsilon^{(-1)}(z)\in K((z^{-1}))$ for $\varepsilon=\varepsilon(a,b;k)\in\{a,b\}^{\mathbb{N}}$ is given by

$$\varepsilon^{(-1)}(z)=[0;a^{-1}(z-1),(-1)^m(a-b)^{-1}h_m{}^2 b_{m-1}{}^*,(-1)^m(a-b)h_m{}^{-1}h_{m+1}{}^{-1}(z-1)]_{m=0}^{\infty},$$

where

$$h_n:=|\sigma^n(a)|_a a+|\sigma^n(a)|_b b\ (=g_n a+g_{n-1}b\in\mathbb{Z}[a,b]),\ \ f_n=f_{n,k}:=|\sigma^n(a)|,$$

$$b_n{}^*=b_n{}^*(z;k):=z^{f_n}\sum_{0\leq i\leq f_{n+1}-1}z^i\cdot\sum_{1\leq j\leq k-1}(k-j)z^{(j-1)f_{n+1}}+k\sum_{0\leq i\leq f_n-1}z^i\in\mathbb{Z}[z].$$

If $(a,b)\in\mathbb{C}^2$, then Proposition 3 is valid under the condition

$$a\neq b,\ \ h_n(=g_n a+g_{n-1}b)\neq 0\ \text{for all}\ n\geq 0. \tag{4}$$

We can give explicit formulae for the continued fraction expansion for $\varepsilon^{(-1)}(z)=\varepsilon(z;a,b;k)\in\mathbb{C}((z^{-1}))$ with $(a,b)\in\mathbb{C}^2$, which does not satisfy (4). For example,

<u>Proposition 4</u> (Theorem 9, [T2]). Let $(a,b)\in\mathbb{C}^2$ with $h_t=0$, $a\neq 0$, $t\geq 0$. Then

$$\varepsilon^{(-1)}(z)=\varepsilon^{(-1)}(z;a,-g_{t-1}{}^{-1}g_t a)$$
$$=[0;a_1,d_{-1},c_0,d_0,\dots,c_{t-2},d_{t-2},e_1,e_2,i_m,j_m]_{m=0}^{\infty}$$

holds with partial denominators$\in\mathbb{C}[z]$ given by

$$a_1=a^{-1}(z-1),$$
$$c_m=(-1)^m(a-b)h_m{}^{-1}h_{m+1}{}^{-1}(z-1),$$

$$d_m=(-1)^{m+1}(a-b)^{-1}h_{m+1}{}^2 b_m{}^*,$$

$$e_1=(-1)^{t-1}(a-b)h_{t-1}{}^{-2}(z-1)b_{t-1},$$

$$e_2=(-1)^{t-1}(a-b)^{-1}h_{t-1}{}^2 b_t{}^*,$$

$$i_m=(-1)^{m+t-1}(a-b)h_{t-1}{}^{-2}g_m{}^{-1}g_{m+1}{}^{-1}(z-1),$$

$$j_m=(-1)^{m+t}(a-b)^{-1}h_{t-1}{}^2 g_{m+1}{}^2 b_{m+t+1}{}^*.$$

Concerning such continued fractions, we studied uniform convergence in Section 5, [T2]. Related to the product formula (Proposition 1), we studied the distribution, and the simplicity of the zero points of $Q(z)$ for the Padé pairs (P,Q) for $\varepsilon(z;a,b;k)$ in Section 4, [T2].

It is an interesting problem which asks whether we can do the same for the Thue-Morse word ζ. The fact $\Lambda(\zeta)=\mathbf{N}$ (cf. [A-P-ZXW-ZYW]), we mentioned, says that all the denominators of the continued fraction for $\zeta^{(-1)}(z)$ are of degree 1. It is of special interest to find the the continued fraction expansion for $\zeta^{(-1)}(z)$ in a closed form.

We could not give a completely explicit formula for $H_n{}^{(m)}[\varepsilon(z;a,b;k)]$ when $k\geqq 2$; we gave the following

Proposition 5 (cf. Corollary 6, [T2]).

$$H_{f_n}{}^{(0)}[\varepsilon(z;a,b;k)])=r_n(g_n a+g_{n-1}b)(a-b)^{f_n-1},$$

$$H_{f_{n+1}-1}{}^{(0)}[\varepsilon(z;a,b;k)]=s_n(g_n a+g_{n-1}b)(a-b)^{f_{n+1}-2} \quad (n\geqq 0);$$

and $H_m{}^{(0)}=0$ for all $m\neq f_n$ and $m\neq f_{n+1}-1$ $(n\geqq 0)$, where $r_n\neq 0$, $s_n\neq 0$ are integers independent of a, b.

4. Toeplitz cases. In this section, we consider Hankel determinants for 2-adic Toeplitz words

$$w=w_1w_2w_3\ldots \quad (w_i\in A).$$

Note that the numbering of the symbols starts from 1 (not from 0), cf. Sections 2, 3. Recall that a word $w=w_1w_2w_3\ldots$ $(w_i\in A)$ is a 2-adic Toeplitz word iff

$$\mathrm{ord}_2(m)=\mathrm{ord}_2(n) \Longrightarrow w_m=w_n.$$

Without loss of generality, we may suppose $A=\{a_0,a_1,a_2,\dots\}$. In some cases, we use symbols a, b, c, ... instead of a_0, a_1, a_2, ... as before. Recall also the universal 2-adic Toeplitz word

$$\omega=abacabadabacabaeabacabadabacabaf\dots$$

and the notation

$$H_n[\omega]:=H_n^{(0)}[\omega]$$

defined in Section 1. For example, by direct calculation, we have

$$H_7[\omega] = \begin{vmatrix} a & b & a & c & a & b & a \\ b & a & c & a & b & a & d \\ a & c & a & b & a & d & a \\ c & a & b & a & d & a & b \\ a & b & a & d & a & b & a \\ b & a & d & a & b & a & c \\ a & d & a & b & a & c & a \end{vmatrix}$$

$$= -4ab^2c^4+abc^5-ac^6+16ab^2c^3d-12abc^4d+2ac^5d-24ab^2c^2d^2$$
$$+8abc^3d^2+ac^4d^2+16ab^2cd^3+8abc^2d^3-4ac^3d^3-4ab^2d^4-12abcd^4$$
$$+ac^2d^4+4abd^5+2acd^5-ad^6$$

$$= -a(2b-c-d)^2(c-d)^4.$$

We say that a form (i.e, a homogeneous polynomial) $P\in\mathbf{Z}[A]$ is <u>completely reducible</u> iff P=0, or P can be factorized into linear forms$\in\mathbf{Z}[A]$, i.e.,

$$P=P_1P_2\cdots P_k \quad (\deg P_i=1 \text{ for all } 1\leq i\leq k).$$

One can check that $H_n[\omega]$ are non-zero completely reducible forms for small n (for $n\leq 30$ or so) by using the soft "Mathematica". This is a curious phenomenon, since, for instance, $H_2[w]$ (resp., $H_3[w]$, etc.) is not completely reducible for any word w having abc (resp., abacd, etc.) as its prefix of w. Related to such a phenomenon, we can show the following

<u>Proposition 6</u> (Main Theorem in [M-T-Tn]). Let w be a fixed point of a substitution of constant length 2. Suppose w is a word strictly over an alphabet

consisting of at least 3 symbols. Then $H_n[w]$ is completely reducible for all $n \geq 1$ if and only if w is a 2-adic Toeplitz word.

The proof of this proposition together with something more interesting (probably) will appear in the forthcoming paper [M-T-Tn].

References

[A-P-ZXW-ZYW] J.-P. Allouche, J. Peyrière, Z.-X. Wen and Z.-Y. Wen, Hankel determinants of the Thue-Morse sequence, prépublications Univ. de Paris-sud Mathématiques No.96-68 (1996), 1-26.

[M-T-Tn] A. Manada, J.-I. Tamura and S. Tanaka, A characterization of Toelitz words by an algebraic property of Hankel determinants, in prepartion.

[N-S] E. M. Nikishin and V. N. Sorokin, Rational Approximations and Orthogonality (in Russian), Nauka, Moskva, 1988.

[K-T-W] T. Kamae, J. Tamura and Z.-Y. Wen, Hankel determinants for the Fibonacci word and Padé approximation, Acta Arith. 89(1999), 123-161.

[T1] J. Tamura, A class of transcendental numbers having explicit g-adic and Jacobi-Perron expansions of arbitrary dimension, Acta Arith. 71(1995), 301-329.

[T2] J. Tamura, Padé approximation for words generated by certain substututions, and Hankel determinants, in Number Theory and its applications, (eds.) S. Kanemitsu and K. Györy, Kluwer Acadmic Publishers, 1999, 309-346.