数理解析研究所講究録
1511 巻 2006 年 197-204

Recent results on analogue in Nevanlinna theory and Diophantine approximation

Junjiro Noguchi

The present note is based on a talk given at Workshop "Analytic Number Theory and Related Topics" on 22 October 2004. We will discuss some recent results obtained on the analogues between Nevanlinna theory in higher dimensional algebraic varieties and Diophantine approximation theory.

1 A basic observation

The unit equation with variables a, b, c is given by

(1.1) $$a+b=c.$$

Why is this equation interesting? There might be several answers, but it is one of them that (1.1) gives a hyperbolic space. In fact, equation (1.1) defines a subvariety X in $\mathbf{P}^2$ with homogeneous coordinates $[a, b, c]$. Since the variables are assumed to be units, X is isomorphic to $\mathbf{P}^1$ minus three distinct points, to say, 0,1, and ∞.

In complex function theory (1.1) was studied by E. Picard for units of entire functions and we konw the famous Picard's Theorem (1879) that a meromorphic function on $\mathbf{C}$ omitting three values of $\mathbf{P}^1$, $0, 1$ and ∞ is necessarily constant. A quantitative theory to measure the frequences to take those three values by a non-constant meromorphic function was established by R. Nevanlinna (1925), in which the second main theorem is viewed in turn as an analogue of *abc*-Conjecture of Masser and Oesterlé.

It is also an interesting subject to study a unit equation in several variables,

(1.2) $$x_1+x_2+\cdots+x_n=0 \quad (n\geq 3).$$

Equation (1.2) defines a variety isomorphic to $\mathbf{P}^{n-2}$ minus n hyperplanes in general position. In complex function theory (1.2) was studied by E. Borel for units of entire functions and the Subsum Theorem for units of entire functions was proved (1897). The corresponding quantitative theory was established by H. Cartan (1933) also by Weyls and Ahlfors (1941), which generalized Nevanlinna's theory. Cartan's second main theorem is viewed as an analogue of a sort of $abc\cdots$-Conjecture.

2 Lang's Conjecture for projective hypersurfaces

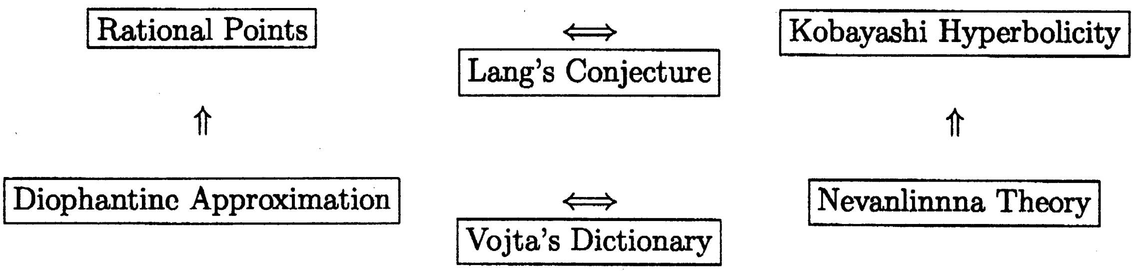
Let k be an algebraic number field, that is, a finite extension of $\mathbf{Q}$. Let X be an algebraic variety defiend over k and let $X(k)$ denote the set of k-rational points of X.

Lang's Conjecture. If there is an embedding $k \hookrightarrow \mathbf{C}$ such that the obtained complex space $X_{\mathbf{C}}$ is Kobayashi hyperbolic, then the cardinality $|X(k)| < \infty$.

An analogue over function fields was dealt with by [7] and [8], and the following finiteness theorem was a result in a special case:

Theorem 2.1 ([8]) *Let X be a Kobayashi hyperbolic compact complex space. Let Y be another compact complex space. Then there is only a finite number of surjective meromorphic mappings from Y onto X.*

So far Nevanlinna theory offers a most effective tool to the Kobayashi hyperbolicity problem for complex algebraic varieties. Analogously Diophantine approximation theroy provides a powerful method to the finiteness problem or distributions of rational points. These relations are described by the following diagram:



We recall

Kobayashi Conjecture. A "generic" hypersurface $X \subset \mathbf{P}^n(\mathbf{C})$ of high degree ($\geq 2n+1$) is Kobayashi hyperbolic.

Therefore such X defined over k should satisfy $|X(k)| < \infty$ according to Lang's Conjecture. For the existence we have

Theorem 2.2 ([3]) *For every $n \in \mathbf{N}$ there is a number $d(n)$ such that for an arbitrary $d \geq d(n)$ there is a Kobayashi hyperbolic projective hypersurface $X \subset \mathbf{P}^n(\mathbf{C})$ of degree d.*

The following is an example: In $\mathbf{P}^3(\mathbf{C})$ we set

$$X_d = \{x_0^{4d} + \cdots + x_3^{4d} + t(x_0 \cdots x_3)^d = 0\}, \quad t \neq 0. \tag{2.3}$$

Then X_d with $d \geq 7$ is Kobayashi hyperbolic. It is noted that $abc \cdots$-Conjecture would imply $|X_d(k)| < \infty$ if $t \in k^*$. It is also noted that X_1 is a Kummer K3 surface and there is a natural ramified covering $X_d \to X_1$.

Definition. Let X be an algebraic variety defined over k. We say that X satisfies the *arithmetic finiteness property* if $|X(k')| < \infty$ for all finite extensions k' of k.

Let $S \subset M_k$ be an arbitrarily fixed finite subset of places of k containing all infinite places. Let $X_d(U_S)$ denote the subset of all points of $X_d(k)$ whose coordinates in (2.3) are S-units. Then by making use of Schmidt's Subspace Theorem we deduce the following.

Proposition 2.4 ([10]) *Let X_d be as above. Then $|X_d(U_S)| < \infty$.*

By Masuda-Noguchi [3] there exist such examples in $\mathbf{P}^n(\mathbf{C})$ of arbitrary dimension. It is observed that $abc\cdots$-Conjecture implies the arithmetic finiteness property of such projective hypersurfaces. Therefore it is natural and interesting to ask if there is a projective hypersurface satisfying the arithmetic finiteness property. In fact we have

Theorem 2.5 ([14]) *There exists a hypersurface $X \subset \mathbf{P}^n_{\mathbf{Q}}$ satisfying the arithmetic finiteness property.*

We follow Shirosaki's construction of a Kobayashi hyperbolic projective hypersurface ([16]). Let $d, n \in \mathbf{N}$ be co-prime, and assume $d \geq 2e + 8$. Set

$$P(w_0, w_1) = w_0^d + w_1^d + w_0^e w_1^{d-e}.$$

We define inductively

$$P_1(w_0, w_1) = P(w_1, w_1),$$
$$P_n(w_0, \ldots, w_n) = P_{n-1}(P(w_0, w_1), \ldots, P(w_{n-1}, w_n)), \quad n = 2, 3, \ldots$$

We set $X_{e,d} = \{P_n = 0\} \subset \mathbf{P}^n(\mathbf{C})$.

Theorem 2.6 (Shirosaki [16]) *If $e \geq 2$, then $X_{e,d}$ is Kobayashi hyperbolic.*

The proofs of Theorems 2.5 and 2.6 are quite similar by virtue of Nevanlinna's Second Main Theorem and Faltings' Theorem for curves of higher genus (Mordell's Conjecture).

Key Lemma (Yi [22], [16], [14]) (i) *Let $\alpha, \beta \in \mathbf{C}$ and $\alpha \neq 0$. Then the curve*

$$C_{\alpha\beta} = \{[w_0, w_1, w_2] \in \mathbf{P}^2; P(w_0, w_1) = \alpha P(\beta w_1, w_2)\}$$

is hyperbolic for $e \geq 2$, so that if $\alpha, \beta \in \mathbf{Q}$, then $C_{\alpha\beta}$ satisfies the arithmetic finiteness property.

(ii) *Let $f_j = [f_{j0}, f_{j1}] : \mathbf{C} \to \mathbf{P}^1$ be two meromorphic functions satisfying*

$$P(f_{10}, f_{11}) = \exp(g) P(f_{20}, f_{21})$$

with an entire function g. Then $f_0 \equiv f_1$.

Then the proof of Theorem 2.5 is done by the induction on n (cf. [14] for the details).

3 abc-Conjecture for semi-abelian varieties

(a) Analogue over algebraic function fields. It is interesting to consider the problem over algebraic function fields. The case of algberaic function fields is situated in the middle of the Nevnalinna theory and the number theory.

Nevanlinna theory ——— Number theory

\ Theory over function field /

There are a number of works on this subject for $\mathbf{P}^n$ ($n \geq 1$) over algebraic function fields (Voloch, Mason, Brownawell-Masser, J. T.-Y. Wang, myself,...; cf. [9], [10] and their references). The problem for abelian varieties was first dealt with by A. Buium.

Theorem 3.1 (Buium [2]) *Let A be an abelian variety. Let D be a reduced divisor on A which is Kobayashi hyperbolic. Let C be a smooth compact curve. Then there exists a number N depending on C, A and D such that for every morphism $f : C \to A$, either*

$$f(C) \subset D \quad \textit{or} \quad \mathrm{mult}_x f^* D \leq N \quad (\forall x \in C).$$

Corollary 3.2 *Let the notation be as in Theorem 3.1. If $f(C) \not\subset D$, then*

$$\text{“height}\,(f)\text{”} = \deg(f) \leq N|f^{-1}(D)|.$$

This is an estimate of type of *abc*-Conjecture. His proof based on Kolchin's theory of differential algebra and he posed two problems:

- Find a proof by complex geometry.
- The Kobayashi hyperbolicity assumption for D is too strong, and the ampleness should suffice.

Theorem 3.3 (Noguchi-Winkelmann [12]) *Let A be a semi-abelian variety with a smooth equivariant algebraic compactification $A \hookrightarrow \bar{A}$. Let $\bar{D}$ be an effective reduced ample divisor on $\bar{A}$, and $D = \bar{D} \cap A$. Let C be a smooth algebraic curve with smooth compactification $C \hookrightarrow \bar{C}$. Then there exists a number $N \in \mathbf{N}$ such that for every morphism $f : C \to A$ either*

$$f(C) \subset D \quad \textit{or} \quad \mathrm{mult}_x f^* D \leq N \quad (\forall x \in C).$$

Furthermore, the number N depends only on the numerical data involved as follows:

(i) *The genus of $\bar{C}$ and the number $\#(\bar{C} \setminus C)$ of the boundary points of C,*

(ii) *the dimension of A,*

(iii) *the toric variety (or, equivalently, the associated "fan") which occurs as closure of the orbit in $\bar{A}$ of the maximal connected linear algebraic subgroup $T \cong (\mathbf{C}^*)^t$ of A,*

(iv) *all intersection numbers of the form $D^h \cdot B_{i_1} \cdots B_{i_k}$, where the B_{i_j} are closures of A-orbits in $\bar{A}$ of dimension n_j and $h + \sum_j n_j = \dim A$.*

In particular, if we let A, $\bar{A}$, C and D vary within a flat connected family, then we can find a uniform bound for N. For abelian varicties this specializes to the following result:

Theorem 3.4 (Noguchi-Winkelmann [12]). *There is a function $N : \mathbf{N} \times \mathbf{N} \times \mathbf{N} \to \mathbf{N}$ such that the following statement holds.*

Let C be a smooth compact curve of genus g, let A be an abelian variety of dimension n, and let D be an ample effective divisor on A with intersection number $D^n = d$.

Then for an aribitrary morphism $f : C \to A$, either

$$f(C) \subset D \quad or \quad \mathrm{mult}_x f^*D \leq N(g, n, d) \quad (\forall x \in C).$$

As an application a *finiteness theorem* was proved for morphisms from a non-compact curve into an abelian variety omitting an ample divisor.

(b) Nevnalinna thcory. In Ncvanlinna theory for a holomorphic curve $f : \mathbf{C} \to A$ into a semi-abelian variety A we lately proved thc next result.

Theorem 3.5 (Noguchi-Winkelmann-Yamanoi [15]) *Let D be a reduced divisor on a semi-abelian variety A. Then there is an equivariant compactification $\bar{A} \supset A$ of A such that for an arbitrary algebraically non-degenerate holomorphic curve $f : \mathbf{C} \to A$*

$$(1 - \epsilon)T_f(r; L(\bar{D})) \leq N_1(r; f^*D)||_\epsilon, \quad \forall \epsilon > 0, \tag{3.6}$$

where $\bar{D}$ is the closure of D in $\bar{A}$.

Remark. In Noguchi-Winkelmann-Yamanoi [13] we proved (3.6) with a higher level truncated counting function $N_k(r; f^*D)$ for some special compactification of A. (see [17] and [4] for related results). In the case of abelian A (3.6) with truncation level one was obtained by Yamanoi [20] (see [21] for an important result in the transcendental case).

(c) Analogue in Diophantine approximation. Rccall

abc-Conjecture. Let $a, b, c \in \mathbf{Z}$ be co-prime numbers satisfying

$$a + b = c. \tag{3.7}$$

Thcn for an arbitrary $\epsilon > 0$ there is a number $C_\epsilon > 0$ such that

$$\max\{|a|, |b|, |c|\} \leq C_\epsilon \prod_{\text{prime } p|(abc)} p^{1+\epsilon}.$$

Notice that the order of abc at every prime p is counted only by "$1+\epsilon$" when it is positive.

As in §1 we put $x = [a, b] \in \mathbf{P}^1(\mathbf{Q})$. After Vojta's notational dictionary [18], this is equivalent to

$$(3.8) \qquad (1-\epsilon)h(x) \leq N_1(x;0) + N_1(x;\infty) + N_1(x;1) + C_\epsilon$$

for $x \in \mathbf{P}^1(\mathbf{Q})$ (cf. [5], [19]). This is quite analogous to (3.6). Here we follow the notation in [18] for number theory and [6] for the Nevanlinna theory (cf. [5], [19]); in particular,

$$h(x) = \text{ the height of } x.$$
$$N_1(x;*) = \text{ the counting function at } * \text{ truncated to level 1 (see below).}$$

Motivated by the results in (a) and (b), we formulate an analogue of abc-Conjecture for semi-abelian varieties. Let k be an algerbaic number field and let $S \subset M_k$ be an arbitrarily fixed finite subset of places of k containing all infinite places.

Let A be a semi-abelian variety over k and let D be a reduced divisor on A. Let $\bar{A}$ be an equivariant compactification of A such that the closrure $\bar{D}$ of D in $\bar{A}$ contains no A-orbit. Let $\sigma_{\bar{D}}$ denote a regular section of the line bundle $L(\bar{D})$ defining the divisor $\bar{D}$.

abc-Conjecture for semi-abelian variety. For an arbitrary $\epsilon > 0$ there exits a constant $C_\epsilon >$ such that for all $x \in A(k) \setminus D$

$$(3.9) \qquad (1-\epsilon)h_{L(\bar{D})}(x) \leq N_1(x;S,\bar{D}) + C_\epsilon.$$

Here $h_{L(\bar{D})}(x)$ dentoes the height function with respect to $L(\bar{D})$ and $N_1(x,\bar{D};S)$ denotes the S-counting function truncated to level one:

$$N_1(x;S,\bar{D}) = \frac{1}{[k:\mathbf{Q}]} \sum_{v\in M_k\setminus S,\, \mathrm{ord}_{\mathfrak{p}_v}\sigma_{\bar{D}}(x)\geq 1} \log N_{k/\mathbf{Q}}(\mathfrak{p}_v).$$

It may be interesting to specialize the above conjecture in two forms.

abc-Conjecture for S-units. We assume that a and b are S-units in (3.7); that is, x in (3.8) is an S-unit. Then for every $\epsilon > 0$ there is a constant $C_\epsilon > 0$ such that

$$(3.10) \qquad (1-\epsilon)h(x) \leq N_1(x;S,1) + C_\epsilon.$$

abc-Conjecture for elliptic curve. Let C be an elliptic curve defined as a closure of an affine curve,

$$y^2 = x^3 + c_1 x + c_0, \quad c_i \in k^*.$$

In a neighborhood of $\infty \in C$ $\sigma_\infty = \frac{x}{y}$ gives an affine parameter with $\sigma_\infty(\infty) = 0$. For every $\epsilon > 0$ there is a constant $C_\epsilon > 0$ such that for $w \in C(k)$

$$(1-\epsilon)h(w) \leq N_1(w;S,\infty) = \frac{1}{[k:\mathbf{Q}]} \sum_{v\in M_k\setminus S,\, \mathrm{ord}_{\mathfrak{p}_v}\sigma_\infty(w)\geq 1} \log N_{k/\mathbf{Q}}(\mathfrak{p}_v) + C_\epsilon.$$

References

[1] Buium, A., The abc theorem of abelian varieties, Intern. Math. Res. Notices **5** (1994), 219-233.

[2] Buium, A., Intersection multiplicities on abelian varieties, Math. Ann. **310** (1998), 653-659.

[3] Masuda, K. and Noguchi, J., A construction of hyperbolic hypersurfaces of $\mathbf{P}^n(\mathbf{C})$, Math. Ann. **304** (1996), 339-362.

[4] McQuillan, M., A toric extension of Faltings' "Diophantine approximation on abelian varieties", J. Diff. Geom. **57** (2001), 195–231.

[5] Noguchi, J., On Nevanlinna's second main theorem, Geometric Complex Analysis, Proc. the Third International Research Institute, Math. Soc. Japan, Hayama, 1995, pp. 489–503, World Scientific, Singapore, 1996.

[6] Noguchi, J. and Ochiai, T., Geometric Function Theory in Several Complex Variables, Japanese edition, Iwanami, Tokyo, 1984; English Translation, Transl. Math. Mono. **80**, Amer. Math. Soc., Providence, Rhode Island, 1990.

[7] Noguchi, J., Hyperbolic fibre spaces and Mordell's conjecture over function fields, Publ. RIMS, Kyoto University **21** (1985), 27-46.

[8] Noguchi, J., Meromorphic mappings into compact hyperbolic complex spaces and geometric Diophantine problems, International J. Math. **3** (1992), 277-289.

[9] Noguchi, J., Value distribution theory over function fields and a Diophantine equation, Analytic Number Theory 1994 (Ed. Y. Motohashi), Surikaisekikenkyujokokyuroku Vol. **958**, pp. 33-42, Research Institute of Mathematics Sciences, Kyoto University, 1996.

[10] Noguchi J., Nevanlinna-Cartan theory over function fields and a Diophantine equation, J. reine angew. Math. **487** (1997), 61-83; Correction to the paper, Nevanlinna-Cartan theory over function fields and a Diophantine equation, J. reine angew. Math. **497** (1998), 235.

[11] Noguchi, J. and Winkelmann, J., A note on jets of entire curves in semi-Abelian varieties, Math. Z. **244** (2003), 705–710.

[12] Noguchi, J. and Winkelmann, J., Bounds for curves in abelian varieties, J. reine angew. Math. **572** (2004), 27–47.

[13] Noguchi, J., Winkelmann, J. and Yamanoi, K., The second main theorem for holomorphic curves into semi-Abelian varieties, Acta Math. **188** no.1 (2002), 129–161.

[14] Noguchi, J., An arithmetic property of Shirosaki's hyperbolic projective hypersurface, Forum Math. **15** (2003), 935–941.

[15] Noguchi, J., Winkelmann, J. and Yamanoi, K., The second main theorem for holomorphic curves into semi-Abelian varieties II, preprint/arXiev 2004. 83-96.

[16] Shirosaki, M., On some hypersurfaces and holomorphic mappings, Kodai Math. **21** (1998), 29-34.

[17] Siu, Y.-T. and Yeung, S.-K., Addendum to "Defects for ample divisors of abelian varieties, Schwarz lemma, and hyperbolic hypersurfaces of low degrees," *American Journal of Mathematics* **119** *(1977), 1139–1172*, Amer. J. Math. **125** (2003), 441–448.

[18] Vojta, P., Diophantine Approximations and Value Distribution Theory, Lecture Notes in Math. vol. **1239**, Springer, Berlin-Heidelberg-New York, 1987.

[19] Vojta, P., A more general abc conjecture, Intern. Math. Res. Notices **21** (1998), 1103-1116.

[20] Yamanoi, K., Holomorphic curves in abelian varieties and intersection with higher codimensional subvarieties, Forum Math. **16** (2004), 749–788.

[21] Yamanoi, K., The second main theorem for small functions and related problems, Acta Math. **192** (2004), 225–294.

[22] Yi, H.-X., A question of Gross and the uniqueness of entire functions, Nagoya Math. J. **138** (1995), 169–177.

J. Noguchi
Graduate School of Mathematical Sciences
University of Tokyo
Komaba, Meguro,Tokyo 153-8914
Japan
e-mail: noguchi@ms.u-tokyo.ac.jp