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On the defining group of a category of systems of linear inequalities (線形不等式系のなす淡中圏を定める代数群について)

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During today's talk, the speaker gives a partial answer to a question about systems of linear inequalities. It was raised by the speaker himself 3 years ago [1] in this same seires of workshops.
(The speaker had expected 30 minutes would be appropriate to explain his result and its background. In reality, it was too short. He had to abridge an initially prepared draft and is afraid the audience did not get a perspective of the treated subject. The speaker apologizes for his insufficient preparation. An unabridged transcript of the draft of the speech is presented below. He hopes it would help the people understand the way in which our example introduced there has relation to a general theory.)

Notation $L := \mathbb{R} \cap \bar{\mathbb{Q}}$

System of lin. ineq.

- Linear $f_0, f_1, \dots, f_n \in L[T_0, T_1, \dots, T_n]$

$$f_0 \wedge f_1 \wedge \cdots \wedge f_n \neq 0$$

- $c(0), c(1), \dots, c(n) \in \mathbb{Z}$

Fixed $\delta \in \mathbb{R}_{>0}$ and variable $Q \in \mathbb{R}_{>1}$

$$|f_i| < Q^{-c(i)-\delta} \quad (i = 0, 1, \dots, n)$$

Example $(n = 1)$ $\alpha \in L$

$$\begin{aligned} |f_0| = |T_0| &< Q^{1-\delta}, \\ |f_1| = |T_1 - \alpha T_0| &< Q^{-1-\delta} \end{aligned}$$

Let's first recall the problem of our interest.

For brevity, we call L the field of real algebraic numbers.

By a system of linear inequalities, we formally mean a pair of data: One is

- a family of linear forms $f_0, f_1, \dots, f_n$ with coefficients in L and in the indeterminates $T_0, T_1, \dots, T_n$. We assume they are linearly independent. In other words, the volume form they define does not vanish. The other datum is
- a family of rational integers $c(0), c(1), \dots, c(n)$.

More intuitively, with the help of an arbitrarily fixed positive real number δ and a variable real number Q larger than 1, it's a system of linear inequalities ...

A typical example with $n = 1$ is attached to a single real algebraic number α. In this example, (for small δ,) the actual inequalities are ...

When we took up simultaneously all the systems of linear inequalities, we did not directly handle those raw data. We have introduced an equivalence class. Namely, we have associated each system with a filtered vector space in the following way:

Associated filtered vect. sp

$$V := \mathbb{Q}T_0 \oplus \mathbb{Q}T_1 \oplus \cdots \oplus \mathbb{Q}T_n$$

We call V the vector space of rational linear forms in $T_0, T_1, \ldots, T_n$.

$$\{w(1) < \cdots < w(s)\} = \{c(0), c(1), \ldots, c(n)\}$$

The symbols $w(1), \ldots, w(s)$ are strictly increasing rational integers such that as a set, it is identical with the set of $c(0), c(1), \ldots, c(n)$. An L-vector space $V^{w(j)}$ is defined to be the subspace of the scalar extension of V to L, spanned by all f_i's with $c(i)$ at least $w(j)$.

$$\begin{aligned} V \otimes_{\mathbb{Q}} L &\supset V^{w(j)} \\ &:= \langle f_i \mid c(i) \geq w(j) \rangle_L \end{aligned}$$

$$V \otimes_{\mathbb{Q}} L = V^{w(1)} \supsetneq \cdots \supsetneq V^{w(s)} \supsetneq 0$$

Thus we obtain a descending filtration on the scalar extension of V.

Example (continuation)

$$V \otimes_{\mathbb{Q}} L = V^{-1} \supsetneq V^1 = L \cdot (T_1 - \alpha T_0) \supsetneq 0$$

For our example, the whole space is V^{-1}. The only nontrivial filter is V^1, which is generated over L by $T_1 - \alpha T_0$.

Cat. of lin. ineq.

$$\begin{aligned} \mathrm{Obj}(\underline{C}) &= \{\text{filtr. vect. sp as above}\} \\ \mathrm{Hom}(\underline{C}) &= \{\mathbb{Q}\text{-lin. filtr.}\} \\ &= \{\mathbb{Q}\text{-lin., preserving sols.}\} \end{aligned}$$

These filtered vector spaces make up a category $\underline{C}$. The morphisms of $\underline{C}$ are $\mathbb{Q}$-linear filtered maps. Intuitively they can be said to be preserving solutions to the relevant systems of linear inequalities.

$\underline{C}_0^{ss}$: full subcat. of $\underline{C}$

$$\mathrm{Obj}(\underline{C}_0^{ss}) = \{\text{semi-stable of slope } 0\}$$

What we are concerned with is the full subcategory $\underline{C}_0^{ss}$ consisting of semi-stable objects of slope 0. Here the condition of slope 0 is the same as the sum of the numbers $c(0), c(1), \ldots, c(n)$ of the system being 0. Under the condition of slope 0, semi-stability is equivalent, by the subspace theorem of SCHMIDT, to the classical condition that the system of linear inequalities is a general ROTH system. By definition, a general ROTH system has only finitely many solutions in $\mathbb{Z}$.

Thm (former [3][2]) $\exists$ an affine gp scheme $G/\mathbb{Q}$ s.t.

$$\underline{\mathrm{Rep}_{\mathbb{Q}}}(G) \simeq \underline{C}_0^{ss}$$

Our former result [3][2] was the theorem which says that there exists an affine group scheme G over $\mathbb{Q}$ such that the category of finite dimensional representations over $\mathbb{Q}$ of G is equivalent to the category $\underline{C}_0^{ss}$.

<u>Problem</u> $G = ?$

A natural question, which was touched upon at the beginning of the talk, is that what is the group scheme G?

<u>Thm</u> (today's) $\forall$ 1-dim. anisotropic torus $\tilde{G}/\mathbb{Q}$ which splits$/L$ is a quotient gp of G!

And today's main result is that any 1-dimensional anisotropic torus $\tilde{G}$ defined over $\mathbb{Q}$ which splits over L is a quotient group scheme of G!

$$\text{`anisotropic'} \iff \mathrm{Hom}_{\mathbb{Q}}(\tilde{G}, \mathbb{G}_{\mathrm{m}}) = 0$$

$$\text{`split}/L\text{'} \iff \tilde{G} \times_{\mathbb{Q}} L \simeq \mathbb{G}_{\mathrm{m}} \times_{\mathbb{Q}} L$$

In the statement, the word 'anisotropic' implies that $\tilde{G}$ has no character over $\mathbb{Q}$ other than the trivial one. The phrase 'split over L' signifies $\tilde{G}$ becomes isomorphic over L to the standard (1-dimensional) multiplicative group $\mathbb{G}_{\mathrm{m}}$.

We next give the detail of the relation of $\tilde{G}$ to G.

<u>Explicit str. of</u> $\tilde{G}$ (omitted in the speech)

$\exists\ a, b \in \mathbb{Q}$ s.t.

$$\tilde{G} \simeq \mathrm{Spec}\left(\mathbb{Q}[T_0, T_1]/(T_1^2 + aT_0T_1 + bT_0^2 - 1)\right)$$

$$T_1^2 + aT_0T_1 + bT_0^2 = (T_1 - \alpha T_0)(T_1 - \beta T_0)$$

$\alpha, \beta \in L$

$$\tilde{G}(\mathbb{Q}) \simeq \{x \in \mathbb{Q}(\alpha)^\times \mid \mathrm{Norm}(x) = 1\}$$

An <u>explicit structure</u> of such a group $\tilde{G}$ as in the theorem is known. We recall it (omitted in the speech).

There exist two rational numbers a and b that give an isomorphism of $\tilde{G}$ to the spectrum of ...

On our assumption, we have in $L[T_0, T_1]$ a factorization of the quadratic form ... into a product of linear forms ... The numbers α and β are necessarily real and quadratic.

In this connection, the group of $\mathbb{Q}$-rational points of $\tilde{G}$ is identified with the multiplicative subgroup of the quadratic field $\mathbb{Q}(\alpha)$ whose elements are of norm 1.

<u>Character of</u> $\tilde{G} \times_{\mathbb{Q}} L$ (also omitted in the speech)

$\mathbb{Q}[T, T^{-1}]$: ring of fn on $\mathbb{G}_{\mathrm{m}}$

$$T \mapsto T_1 - \alpha T_0$$

$$\chi\colon \mathbb{G}_{\mathrm{m}} \times_{\mathbb{Q}} L \xleftarrow{\sim} \tilde{G} \times_{\mathbb{Q}} L$$

$$\mathrm{Hom}\left(\tilde{G} \times_{\mathbb{Q}} L, \mathbb{G}_{\mathrm{m}} \times_{\mathbb{Q}} L\right) = \mathbb{Z}\chi$$

Using this description, we fix a <u>character</u> of $\tilde{G} \times_{\mathbb{Q}} L$. It's needed for our purpose.

As usual we regard $\mathbb{Q}[T, T^{-1}]$ as the ring of functions over $\mathbb{Q}$ of $\mathbb{G}_{\mathrm{m}}$. Here T is another indeterminate.

Over the field L, we map T to $T_1 - \alpha T_0$ in the ring of functions on the scalar extension to L of $\tilde{G}$. This yields an isomorphism χ over L of $\tilde{G} \times_{\mathbb{Q}} L$ onto $\mathbb{G}_{\mathrm{m}} \times_{\mathbb{Q}} L$. The character group of $\tilde{G}$ over L is the infinite cyclic group generated by χ.

Filtr. on rep. sp

V: fin. dim. rep./$\mathbb{Q}$ of $\tilde{G}$

$V \otimes_{\mathbb{Q}} L \supset V_w$: eigensp. of $w\chi$ ($w \in \mathbb{Z}$)

$$V \otimes_{\mathbb{Q}} L = \oplus_{w \in \mathbb{Z}} V_w$$

$$V^i = \oplus_{w \geq i} V_w$$

Example (conti.; omitted in the speech)

$$\begin{aligned} V &= \mathbb{Q}T_0 \oplus \mathbb{Q}T_1 \\ &\hookrightarrow \mathbb{Q}[T_0, T_1]/(T_1^2 + aT_0T_1 + bT_0^2 - 1) \\ &= \text{ring of fn on } \tilde{G} \end{aligned}$$

Two filtr. coincide!

Prop. V: $\forall$ rep. $\Rightarrow$ s.-s. of slope 0

$$\mathrm{Obj}\left(\underline{\mathrm{Rep}}_{\mathbb{Q}}(\tilde{G})\right) \to \mathrm{Obj}\,(\underline{\mathcal{C}}_0^{ss})$$

$$\mathrm{Hom}\left(\underline{\mathrm{Rep}}_{\mathbb{Q}}(\tilde{G})\right) \to \mathrm{Hom}\,(\underline{\mathcal{C}}_0^{ss})$$

Thm The functor

$$\underline{\mathrm{Rep}}_{\mathbb{Q}}(\tilde{G}) \to \underline{\mathcal{C}}_0^{ss} \simeq \underline{\mathrm{Rep}}_{\mathbb{Q}}(G)$$

is fully faithful.

Cor.

$$\tilde{G} \twoheadleftarrow G\colon \text{epi}$$

The choice of the character χ makes it possible to define a filtration on representation spaces.

Call V a finite dimensional representation over $\mathbb{Q}$ of $\tilde{G}$. As is well-known, the representation is diagonalized over L, Namely, denoting by V_w the eigensubspace of the character $w\chi$, we have a direct sum decomposition of $V \otimes_{\mathbb{Q}} L$ by V_w's.

Along with the decomposition, we get a filtration on $V \otimes_{\mathbb{Q}} L$, whose i-th filter is the partial sum of V_w's with the index w at least i.

The $\mathbb{Q}$-vector space V of our example is embedded in the ring of functions on $\tilde{G}$. This is a representation space of $\tilde{G}$ by the translation of $\tilde{G}$. Thus we have another filtration on $V \otimes_{\mathbb{Q}} L$ defined by $\tilde{G}$ as above. Then the old filtration defined by linear inequalities and the filtration defined by the representation coincide! (This paragraph is omitted in the speech.)

We are able to obtain a proposition which states that if a $\mathbb{Q}$-vector space V filtered over L is derived from a representation of $\tilde{G}$ as above, then it is semi-stable of slope 0.

In this way, the objects of the category $\underline{\mathrm{Rep}}_{\mathbb{Q}}(\tilde{G})$ of finite dimensional representations over $\mathbb{Q}$ of $\tilde{G}$ are mapped to objects of $\underline{\mathcal{C}}_0^{ss}$.

In addition, $\tilde{G}$-homomorphisms preserve eigenspaces, so filtrations as well are preserved. Thus the morphisms of $\underline{\mathrm{Rep}}_{\mathbb{Q}}(\tilde{G})$ are also mapped to morphisms of $\underline{\mathcal{C}}_0^{ss}$.

Obviously the correspondence is functorial and faithful. In fact, we can prove a theorem. It says that the functor is fully faithful. In particular, the category $\underline{\mathrm{Rep}}_{\mathbb{Q}}(\tilde{G})$ is regarded as a full subcategory of $\underline{\mathrm{Rep}}_{\mathbb{Q}}(G)$. The restriction on objects of $\underline{\mathrm{Rep}}_{\mathbb{Q}}(\tilde{G})$ of the action of the defining group G of $\underline{\mathrm{Rep}}_{\mathbb{Q}}(G)$ factors through the action of $\tilde{G}$. The group $\tilde{G}$ is a quotient group of G.

The proof of the theorem depends on a simple GALOIS argument.

<u>Rem.</u> (omitted in the speech) In more general settings,

$$\forall \text{ anisotropic torus}/\forall \text{ number field}$$

appears [4].

<u>Rem.</u> (also omitted in the speech) $\underline{\mathrm{Rep}}_{\mathbb{Q}}(\tilde{G})$'s with varying $\tilde{G}$ do not cover $\underline{\mathrm{Rep}}_{\mathbb{Q}}(G)$.

(The following paragraphs are entirely cut in the speech.)

I would like to close my talk by making a few remarks.

In more general settings, any anisotropic torus of any dimension appears. Base fields can also be arbitrary [4].

On the other hand, the subcategories $\underline{\mathrm{Rep}}_{\mathbb{Q}}(\tilde{G})$ with varying $\tilde{G}$ do not cover the total category $\underline{\mathrm{Rep}}_{\mathbb{Q}}(G)$.

The concluding remark: the whole story may result in a variant of a classical fact in Number Theory. But we are hoping that the story gives us a wider view and a deeper understanding into the theory!

参考文献

[1] M. Fujimori. On systems of linear inequalities. In *New Aspects of Analytic Number Theory*, 数理研講究録, 第 1274 巻, pp. 29–34, 2002.

[2] M. Fujimori. On systems of linear inequalities. *Bull. Soc. Math. France*, Vol. 131, No. 1, pp. 41–57, 2003.

[3] M. Fujimori. Corrigenda: 'On systems of linear inequalities'. *Bull. Soc. Math. France*, Vol. 132, No. 4, pp. 613–616, 2004.

[4] M. Fujimori. On the defining group of a category of systems of linear inequalities. Submitted, 7 pp., June 2004.