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# THE PRIME NUMBER THEOREM FOR RANKIN-SELBERG $L$-FUNCTIONS

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ABSTRACT

In this article, we survey and announce a recent unconditional proof of the prime number theorem for Rankin-Selberg $L$-functions attached to automorphic cuspidal representations of $GL_n$ over $\mathbb{Q}$. Applications of this prime number theorem to Selberg's orthogonality conjecture and factorization of automorphic $L$-functions will be given.

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## 1. PROBLEMS CONCERNING AUTOMORPHIC $L$-FUNCTIONS

Let $\pi$ be an irreducible unitary cuspidal representation of $GL_m(\mathbb{Q}_\mathbb{A})$, and $s=\sigma+it\in\mathbb{C}$. The global $L$-function attached to $\pi$ is given by products of local factors for $\sigma>1$ (Godement and Jacquet [5]):

$$L(s,\pi)=\prod_p L_p(s,\pi_p),$$

$$\Phi(s,\pi)=L_\infty(s,\pi_\infty)L(s,\pi),$$

where

$$L_p(s,\pi_p)=\prod_{j=1}^m\Big(1-\frac{\alpha_\pi(p,j)}{p^s}\Big)^{-1},$$

and

$$L_\infty(s,\pi_\infty)=\prod_{j=1}^m\Gamma_\mathbb{R}(s+\mu_\pi(j)).$$

Here $\Gamma_\mathbb{R}(s)=\pi^{-s/2}\Gamma(s/2)$, and $\alpha_\pi(p,j)$ and $\mu_\pi(j)$, $j=1,\dots,m$, are complex numbers associated with $\pi_p$ and $\pi_\infty$, respectively, according to the Langlands correspondence. Denote by

$$a_\pi(p^k)=\sum_{1\le j\le m}\alpha_\pi(p,j)^k \tag{1.1}$$

the Fourier coefficients of $\pi$. Then for $\sigma>1$, we have

$$\frac{d}{ds}\log L(s,\pi)=-\sum_{n=1}^\infty\frac{\Lambda(n)a_\pi(n)}{n^s}, \tag{1.2}$$

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where $\Lambda(n)$ is the von Mangoldt function: $\Lambda(p^k) = \log p$, and $= 0$ elsewhere. If $\pi'$ is an automorphic irreducible cuspidal representation of $GL_{m'}(\mathbb{Q}_{\mathbb{A}})$, we define $L(s,\pi')$, $\alpha_{\pi'}(p,i)$, $\mu_{\pi'}(i)$, and $a_{\pi'}(p^k)$ likewise, for $i = 1,\dots,m'$. If $\pi$ and $\pi'$ are equivalent, then $m = m'$ and $\{\alpha_\pi(p,j)\} = \{\alpha_{\pi'}(p,i)\}$ for any $p$. Hence, by (1.1), $a_\pi(n) = a_{\pi'}(n)$ for any $n = p^k$, when $\pi \cong \pi'$.

The Rankin-Selberg $L$-function $L(s,\pi\times\tilde{\pi}')$ was developed by Jacquet [7], Jacquet, Piatetski-Shapiro, and Shalika [8], Shahidi [29], and Moeglin and Waldspurger [19], where $\pi$ and $\pi'$ are automorphic irreducible cuspidal representations of $GL_m$ and $GL_{m'}$, respectively, with unitary central characters. In our case, this (finite-part) $L$-function is defined by

$$L(s,\pi\times\tilde{\pi}') = \prod_p L_p(s,\pi_p\times\tilde{\pi}'_p),$$

where

$$L_p(s,\pi_p\times\tilde{\pi}'_p) = \prod_{j=1}^{m}\prod_{k=1}^{m'}\Big(1 - \frac{\alpha_\pi(p,j)\bar{\alpha}_{\pi'}(p,k)}{p^s}\Big)^{-1}.$$

The Archimedean local factor $L_\infty(s,\pi_\infty\times\tilde{\pi}'_\infty)$ is defined by

$$L_\infty(s,\pi_\infty\times\tilde{\pi}'_\infty) = \prod_{j=1}^{m}\prod_{k=1}^{m'}\Gamma_{\mathbb{R}}(s+\mu_{\pi\times\tilde{\pi}'}(j,k))$$

where the complex numbers $\mu_{\pi\times\tilde{\pi}'}(j,k)$ satisfy the trivial bound

$$\mathrm{Re}\ \mu_{\pi\times\tilde{\pi}'}(j,k) > -1.$$

Denote

$$\Phi(s,\pi\times\tilde{\pi}') = L_\infty(s,\pi_\infty\times\tilde{\pi}'_\infty)L(s,\pi\times\tilde{\pi}').$$

Also, we have for $\sigma > 1$ that

$$\frac{d}{ds}\log L(s,\pi\times\tilde{\pi}') = -\sum_{n=1}^{\infty}\frac{\Lambda(n)a_\pi(n)\bar{a}_{\pi'}(n)}{n^s}. \tag{1.3}$$

The Prime Number Theorem (PNT) for Rankin-Selberg $L$-functions is the following

**Problem 1.1 (PNT for Rankin-Selberg $L$-functions).** *Let $\pi$ and $\pi'$ be irreducible unitary cuspidal representations of $GL_m(\mathbb{Q}_{\mathbb{A}})$ and $GL_{m'}(\mathbb{Q}_{\mathbb{A}})$, respectively. Determine the asymptotic behavior of*

$$\sum_{n\le x}\Lambda(n)a_\pi(n)\bar{a}_{\pi'}(n). \tag{1.4}$$

PNT with $\pi$ and $\pi'$ being homomorphic cusp forms has been studied by several authors. Rankin [25] proved a PNT for $\pi \cong \pi'$ being homomorphic cusp forms for the modular group, Perelli [24] generalized this to arithmetic progressions, while Laurinčikas and Matsumoto [13] proved a PNT in arithmetic progressions for $\pi \cong \pi'$ being homomorphic cusp forms for congruence groups. Ichihara [6] established a PNT in arithmetic progressions for homomorphic cusp forms $\pi$ and $\pi'$, not necessarily equivalent.

A statement easier than Problem 1.1 is the Selberg Orthogonality Conjecture (SOC); see Selberg [28] and Ram Murty [22] [23].

**Conjecture 1.2 (SOC).** *Let $\pi$ and $\pi'$ be given as in Problem 1.1. Then*

$$\sum_{n\le x}\frac{\Lambda(n)a_\pi(n)\bar a_{\pi'}(n)}{n}=\begin{cases}\log x+O(1) & \textit{if } \pi'\cong\pi;\\ O(1) & \textit{if } \pi'\not\cong\pi.\end{cases}$$

Problem 1.1 can be compared with the classical PNT

$$\sum_{n\le x}\Lambda(n)\sim x, \tag{1.5}$$

while SOC is similar to Mertens' theorem that

$$\sum_{n\le x}\frac{\Lambda(n)}{n}=\log x+O(1). \tag{1.6}$$

It is known that (1.6) is weaker than (1.5).

The following statement is still weaker than SOC.

**Conjecture 1.3 (Weighted SOC).** *Let $\pi$ and $\pi'$ be given as in Problem 1.1. Then*

$$\sum_{n\le x}\Big(1-\frac{n}{x}\Big)\frac{\Lambda(n)a_\pi(n)\bar a_{\pi'}(n)}{n}=\begin{cases}\log x+O(1) & \textit{if } \pi'\cong\pi;\\ O(1) & \textit{if } \pi'\not\cong\pi.\end{cases} \tag{1.7}$$

Rudnick and Sarnak [26] proved Conjecture 1.3 in the case $\pi'\cong\pi$, and then deduced Conjecture 1.2 in the case $\pi'\cong\pi$, using the fact that the left side of (1.7) is a sum of non-negative terms.

## 2. Weighted SOC and factorization of automorphic $L$-functions

In [15] and [16], we proved Conjecture 1.3 in the case $\pi'\not\cong\pi$.

**Theorem 2.1.** *For any automorphic irreducible cuspidal representations $\pi$ and $\pi'$ of $GL_m(\mathbb{Q}_\mathbb{A})$ and $GL_{m'}(\mathbb{Q}_\mathbb{A})$, respectively,*

$$\sum_{n\le x}\Big(1-\frac{n}{x}\Big)\frac{\Lambda(n)a_\pi(n)\bar a_{\pi'}(n)}{n}\ll 1, \tag{2.1}$$

*if $\pi$ is not equivalent to $\pi'$.*

In fact, when $\pi$ and $\pi'$ are not twisted equivalent, i.e., when $\pi\not\cong\pi'\otimes\alpha^{it}$ for any $t\in\mathbb{R}$, where $\alpha(g)=|\det(g)|$, (2.1) was proved in [15]. In the remaining case when $m=m'$ and $\pi\cong\pi'\otimes\alpha^{i\tau_0}$ for some non-zero $\tau_0\in\mathbb{R}$, (2.1) was established in [16].

It is a far-reaching conjecture of Langlands [12] that the most general $L$-function is indeed the $L$-function $L(s,\Pi)$ attached to an automorphic representation $\Pi$ of $GL_n$ over an algebraic number field. It was further conjectured that this $L(s,\Pi)$ is equal to a product of $L$-functions $L(s,\pi_j)$ attached to automorphic irreducible cuspidal representations $\pi_j$ of $GL_{m_j}$ over $\mathbb{Q}$ in a unique way:

$$L(s,\Pi)=L(s,\pi_1)\cdots L(s,\pi_k). \tag{2.2}$$

These $L(s,\pi_j)$ are called principal or primitive $L$-functions over $\mathbb{Q}$ in the sense that they are supposed to be $L$-functions that cannot be factorized further. They are believed to be the building blocks of all $L$-functions.

A known special case of the unique factorization (2.2) is for $\Pi$ being an automorphic irreducible cuspidal representation of $GL_n$ over a cyclic algebraic number field $F$, when $\Pi$ is invariant under the action of the $\mathrm{Gal}(F/\mathbb{Q})$. According to Arthur and Clozel [1], such a representation $\Pi$ is the base change of $\pi \otimes \eta$, where $\pi$ is an automorphic irreducible cuspidal representation of $GL_n$ over $\mathbb{Q}$, and $\eta$ is any idele class character of $\mathbb{Q}$ trivial on $\mathrm{N}_{F/\mathbb{Q}}(F_{\mathbb{A}}^{\times})$. In terms of $L$-functions, we have the factorization

$$L(s, \Pi) = \prod_{\eta} L(s, \pi \otimes \eta)$$

uniquely.

In [16], we proved the uniqueness of the factorization in (2.2). That is, if any general $L$-function can be written as a product of principal $L$-functions $L(s, \pi_j)$ for $GL_{m_j}$ over $\mathbb{Q}$, we showed that this factorization is unique.

**Theorem 2.2.** *Let $\pi_j$ and $\pi_i'$, $j = 1, \ldots, k$, $i = 1, \ldots, l$, be automorphic irreducible cuspidal representations of $GL_{m_j}(\mathbb{Q}_{\mathbb{A}})$ and $GL_{m_i'}(\mathbb{Q}_{\mathbb{A}})$ with unitary central characters, respectively. Then*

$$L(s, \pi_1) \cdots L(s, \pi_k) = L(s, \pi_1') \cdots L(s, \pi_l') \tag{2.3}$$

*cannot hold, if there is a $\pi_j$ that is not equivalent to any $\pi_i'$.*

By taking $k = 1$, Theorem 2.2 implies that $L(s, \pi_1)$ cannot be factorized further.

**Corollary 2.3.** *The $L$-function $L(s, \pi)$ attached to an automorphic irreducible cuspidal representation $\pi$ of $GL_m(\mathbb{Q}_{\mathbb{A}})$ cannot be factorized into $L(s, \pi_1') \cdots L(s, \pi_l')$ nontrivially, where $\pi_i'$ is an automorphic irreducible cuspidal representation of $GL_{m_i'}(\mathbb{Q}_{\mathbb{A}})$ with unitary central character.*

Unique factorization of $L$-functions in the Selberg class (Selberg [28]) was studied by Conrey and Ghosh [2] and Ram Murty [22], under SOC. For automorphic $L$-functions, Ram Murty [23] proved that $L(s, \pi)$ is primitive, i.e., cannot be factorized further, when $\pi$ is an automorphic irreducible cuspidal representation of $GL_2(\mathbb{Q}_{\mathbb{A}})$, under the Generalized Ramanujan Conjecture (GRC, Conjecture 3.1 below). Our Theorem 2.2 and Corollary 2.3 are unconditional.

## 3. PNT and SOC under GRC

In this section, we will need GRC.

**Conjecture 3.1 (GRC).** *Let $\pi$ be an irreducible unitary cuspidal representation of $GL_m(\mathbb{Q}_{\mathbb{A}})$. For any unramified $p$, we have*

$$|\alpha_\pi(p, j)| = 1.$$

Note that in Conjecture 3.1, we do not include the Archimedean Ramanujan conjecture, $\mathrm{Re}\, \mu_\pi(j) = 0$.

As a consequence of GRC, we proved in [17] the following PNT. Denote $\alpha(g) = |\det(g)|$.

**Theorem 3.2.** *Assume GRC. Let $\pi$ and $\pi'$ be irreducible unitary cuspidal representations of $GL_m(\mathbb{Q}_{\mathbb{A}})$ and $GL_{m'}(\mathbb{Q}_{\mathbb{A}})$, respectively. Assume that $\pi$ and $\pi'$ are self contragredient: $\pi \cong \tilde{\pi}$*

*and* $\pi' \cong \tilde{\pi}'$. *Then*

$$\sum_{n\le x}\Lambda(n)a_\pi(n)\bar{a}_{\pi'}(n)$$
$$=\begin{cases}\dfrac{x^{1+i\tau_0}}{(1+i\tau_0)}+O\{x\exp(-c\sqrt{\log x})\} \\ \qquad \text{if } \pi'\cong\pi\otimes|\det|^{i\tau_0} \text{ for some } \tau_0\in\mathbb{R}; \\ O\{x\exp(-c\sqrt{\log x})\} \\ \qquad \text{if } \pi'\not\cong\pi\otimes|\det|^{it} \text{ for any } t\in\mathbb{R}.\end{cases}$$

*Here and throughout, $c$ is a positive constant, not necessarily the same at each occurrence.*

**Corollary 3.3.** *Assume GRC. Let $\pi$ and $\pi'$ be given as in Theorem 3.2. We have*

$$\sum_{n\le x}\frac{\Lambda(n)a_\pi(n)\bar{a}_{\pi'}(n)}{n}$$
$$=\begin{cases}\log x+c_1+O\{\exp(-c\sqrt{\log x})\} \\ \qquad \text{if } \pi'\cong\pi; \\ \dfrac{x^{i\tau_0}}{i\tau_0(1+i\tau_0)}+c_2+O\{\exp(-c\sqrt{\log x})\} \\ \qquad \text{if } \pi'\cong\pi\otimes|\det|^{i\tau_0} \text{ for some } \tau_0\in\mathbb{R}^\times; \\ c_2+O\{\exp(-c\sqrt{\log x})\} \\ \qquad \text{if } \pi'\not\cong\pi\otimes|\det|^{it} \text{ for any } t\in\mathbb{R}.\end{cases}$$

*Here $c_1$ and $c_2$ are constants depending on $\pi$ and $\pi'$:*

$$c_1=\lim_{s\to0}\Big(-\frac{L'}{L}(s+1,\pi\times\tilde{\pi}')-\frac{1}{s}\Big)-1,\qquad c_2=-\frac{L'}{L}(1,\pi\times\tilde{\pi}').$$

A remarkable feature of this corollary is that it describes the orthogonality of $a_\pi(n)$ and $a_{\pi'}(n)$ in three cases with different main terms. It is thus in a more precise form than Selberg's Conjecture 1.2. Moreover, one can see from the last case of Corollary 3.3 that the Dirichlet series on the right side of (1.3) converges to $L'/L(s,\pi\times\tilde{\pi}')$ on Re $s=1$, when $\pi$ and $\pi'$ are not twisted equivalent.

Note that in Theorem 3.2 and Corollary 3.3, we have to assume that $\pi\cong\tilde{\pi}$ and $\pi'\cong\tilde{\pi}'$. This is because a standard zero-free region of the type of de la Vallée Poussin is only available for self contragredient representations (Moreno [20] [21], Sarnak [27],and Gelbart, Lapid, and Sarnak [3]). On the other hand, our Theorems 2.1 and 2.2, together with Corollary 2.3, hold for all representations, not necessarily self contragredient, as we did not use zero-free regions in their proofs.

## 4. SOC without GRC

In [14], we proved SOC without GRC. To this end, we firstly proved the following weighted PNT.

**Theorem 4.1.** *Let $\pi$ and $\pi'$ be irreducible unitary cuspidal representations of $GL_m(\mathbb{Q}_{\mathbb{A}})$ and $GL_{m'}(\mathbb{Q}_{\mathbb{A}})$, respectively. Assume that $\pi$ and $\pi'$ are self contragredient. Then*

$$\sum_{n\le x}\Big(1-\frac{n}{x}\Big)\Lambda(n)a_\pi(n)\bar{a}_{\pi'}(n)$$
$$=\begin{cases}\dfrac{x^{1+i\tau_0}}{(1+i\tau_0)(2+i\tau_0)}+O\{x\exp(-c\sqrt{\log x})\} \\ \qquad \text{if } \pi'\cong\pi\otimes\alpha^{i\tau_0} \text{ for some } \tau_0\in\mathbb{R}; \\ O\{x\exp(-c\sqrt{\log x})\} \\ \qquad \text{if } \pi'\not\cong\pi\otimes\alpha^{it} \text{ for any } t\in\mathbb{R}.\end{cases}$$

If $\tau_0=0$, i.e. if $\pi\cong\pi'$, then $a_\pi(n)=a_{\pi'}(n)$, and hence Theorem 4.1 states that

$$\sum_{n\le x}\Big(1-\frac{n}{x}\Big)\Lambda(n)|a_\pi(n)|^2=\frac{x}{2}+O\{x\exp(-c\sqrt{\log x})\}. \tag{4.1}$$

Now $\Lambda(n)|a_\pi(n)|^2$ is non-negative. By a classical argument of de la Vallée Poussin, we can remove the weight $1-n/x$ from (4.1), to get the following PNT for automorphic representations.

**Corollary 4.2.** *Let $\pi$ be as in Theorem 4.1. Then*

$$\sum_{n\le x}\Lambda(n)|a_\pi(n)|^2=x+O\{x\exp(-c\sqrt{\log x})\}.$$

In general, we could not remove the weight $1-n/x$ from Theorem 4.1. But similar to Theorem 4.1, in [14] we established

$$\sum_{n\le x}\Big(1-\frac{n}{x}\Big)\frac{\Lambda(n)a_\pi(n)\bar{a}_{\pi'}(n)}{n}$$
$$=\begin{cases}\log x+c_1+O\{\exp(-c\sqrt{\log x})\} \\ \qquad \text{if } \pi'\cong\pi; \\ \dfrac{x^{i\tau_0}}{i\tau_0(1+i\tau_0)}+c_2+O\{\exp(-c\sqrt{\log x})\} \\ \qquad \text{if } \pi'\cong\pi\otimes\alpha^{i\tau_0} \text{ for some } \tau_0\in\mathbb{R}^\times; \\ c_2+O\{\exp(-c\sqrt{\log x})\} \\ \qquad \text{if } \pi'\not\cong\pi\otimes\alpha^{it} \text{ for any } t\in\mathbb{R}.\end{cases} \tag{4.2}$$

Here $c_1$ and $c_2$ are as in Corollary 3.3. This is more precise than Conjecture 1.3.

Using Corollary 4.2 and an idea of Landau [11], we were able to remove the weight $1-n/x$ from (4.2), to get SOC.

**Corollary 4.3.** *Conjecture 1.2 is true, provided that $\pi$ and $\pi'$ are self contragredient.*

The reason that we can remove $1-n/x$ from (4.2) is that now the main term is of order $\log x$ when $\pi'\cong\pi$, which is substantially bigger than the $O(1)$ of the case of $\pi'\not\cong\pi$.

## 5. PNT WITHOUT GRC

The classical Perron's formula gives us a formula for a sum of complex numbers $a_n$, $1 \le n \le x$, in terms of their Dirichlet series

$$f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

and bounds for individual terms $a_n$. Let $A(x) > 0$ be non-decreasing such that $a_n \ll A(n)$, and let

$$B(\sigma) = \sum_{n=1}^{\infty} \frac{|a_n|}{n^\sigma} \tag{5.1}$$

for $\sigma > \sigma_a$, the abscissa of absolute convergence of $\sum_{n=1}^{\infty} a_n/n^s$. Then the classical Perron's formula (see e.g. Titchmarsh [30]) states that, for $x = [x] + 1/2, b > \sigma_a$ and $T \ge 4$,

$$\sum_{n \le x} a_n = \frac{1}{2\pi i} \int_{b-iT}^{b+iT} f(s) \frac{x^s}{s} ds + O\Big(\frac{A(2x) x \log x}{T}\Big) + O\Big(\frac{x^b B(b)}{T}\Big). \tag{5.2}$$

When applying (5.2) to the Riemann zeta-function or Dirichlet $L$-functions, bounds for $a_n$ pose no problem. When applying this formula to other automorphic $L$-functions, however, bounds for $a_n$ often require an assumption of GRC. Examples include our Theorem 3.2. In proving Theorem 3.2 by (5.2), we start from (1.3), and let

$$a_n = \Lambda(n) a_\pi(n) \bar{a}_{\pi'}(n), \quad f(s) = -\frac{L'}{L}(s, \pi \times \tilde{\pi}'). \tag{5.3}$$

Therefore, by Rudnick and Sarnak [26], the upper bound function $A(n)$ for $|a_n|$ can be taken as

$$A(n) = n^{1-1/(m^2+1)-1/(m'^2+1)}. \tag{5.4}$$

Obviously, (5.4) will make the first $O$-term in (5.2) too big. If we assume GRC, then instead of (5.4), we can take

$$A(n) = mm' \log n,$$

from which we deduce Theorem 3.2.

In [18], we proved a revised version of Perron's formula (Theorem 5.1 below). Different from the classical (5.2), the new Perron formula produces a formula for $\sum_{n \le x} a_n$ in terms of a sum of $|a_n|$ over a short interval. While bounding individual Fourier coefficients $|a_\pi(n)|$ of an automorphic cuspidal representation is hard and may require GRC, estimation of a sum of $|a_\pi(n)|$ can usually be done by the Rankin-Selberg method. The new Perron's formula thus allows us to prove certain results for automorphic $L$-functions without assuming GRC.

**Theorem 5.1.** *Let $\{a_n\}_{n=1}^{\infty}$ be complex numbers and let the series $f(s) = \sum_{n=1}^{\infty} a_n/n^s$ be absolutely convergent for $\sigma > \sigma_a$. Let $B(\sigma)$ be as in (5.1). Then, for $x = [x] + 1/2, b > \sigma_a$ and $T \ge 4$,*

$$\sum_{n \le x} a_n = \frac{1}{2\pi i} \int_{b-iT}^{b+iT} f(s) \frac{x^s}{s} ds + O\Big\{ \sum_{x - x/\sqrt{T} < n \le x + x/\sqrt{T}} |a_n| \Big\} + O\Big\{ \frac{x^b B(b)}{\sqrt{T}} \Big\}.$$

We remark that Theorem 5.1 can be used to derive the classical PNT. In fact, taking $a_n = \Lambda(n)$, we have

$$\sum_{x-x/\sqrt{T}<n\le x+x/\sqrt{T}} |a_n| \ll \log x \sum_{x-x/\sqrt{T}<n\le x+x/\sqrt{T}} 1 \ll \frac{x\log x}{\sqrt{T}},$$

and, for $\sigma > \sigma_a = 1$,

$$B(\sigma) = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^\sigma} \ll \frac{1}{\sigma - 1}.$$

Therefore, Theorem 5.1 with $b = 1 + 1/\log x$ gives

$$\sum_{n\le x} \Lambda(n) = \frac{1}{2\pi i}\int_{b-iT}^{b+iT} \Big\{-\frac{\zeta'(s)}{\zeta(s)}\Big\}\frac{x^s}{s}ds + O\Big\{\frac{x\log x}{\sqrt{T}}\Big\}.$$

We can take $T = \exp(\sqrt{\log x})$. The classical PNT now follows from the zero-free region of the Riemann zeta-function and a standard contour-integration argument.

As another application, we proved in [18] a prime number theorem (Theorem 5.2) unconditionally for Rankin-Selberg $L$-functions $L(s, \pi \times \tilde{\pi}')$, by removing the assumption of GRC in Theorem 3.2.

**Theorem 5.2.** *In Theorem 3.2, the assumption of GRC can be removed.*

To prove Theorem 5.2, we apply Corollary 4.2 to obtain a bound for the short sum

$$\sum_{x<n\le x+y} \Lambda(n)|a_\pi(n)|^2 \ll y$$

for $y \gg x\exp(-c\sqrt{\log x})$. Let $a_n$ be as in (5.3); then for the above $y$,

$$\begin{aligned}\sum_{x<n\le x+y} |a_n| &\ll \Big\{\sum_{x<n\le x+y} \Lambda(n)|a_\pi(n)|^2\Big\}^{1/2}\Big\{\sum_{x<n\le x+y} \Lambda(n)|a_{\pi'}(n)|^2\Big\}^{1/2} \\ &\ll y.\end{aligned}$$

Now let $T = \exp(\sqrt{\log x})$. Then

$$\sum_{x-x/\sqrt{T}<n\le x+x/\sqrt{T}} |a_n| \ll \frac{x}{\sqrt{T}}. \tag{5.5}$$

On the other hand,

$$B(\sigma) = \sum_{n=1}^{\infty}\frac{|a_n|}{n^\sigma} \ll \Big\{\sum_{n=1}^{\infty}\frac{\Lambda(n)|a_\pi(n)|^2}{n^\sigma}\Big\}^{1/2}\Big\{\sum_{n=1}^{\infty}\frac{\Lambda(n)|a_{\pi'}(n)|^2}{n^\sigma}\Big\}^{1/2}. \tag{5.6}$$

By Corollary 4.2,

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{\Lambda(n)|a_\pi(n)|^2}{n^\sigma} &= \int_1^\infty \frac{1}{u^\sigma}d\Big\{\sum_{n\le u}\Lambda(n)|a_\pi(n)|^2\Big\} \\ &= \int_1^\infty \frac{du}{u^\sigma} + \int_1^\infty \frac{1}{u^\sigma}dr(u), \end{aligned} \tag{5.7}$$

where $r(u) \ll u\exp(-c\sqrt{\log u})$. The last integral in (5.7) is $O(1)$, while the first one is $O\{1/(\sigma-1)\}$. Consequently (5.6) gives us

$$B(\sigma) \ll \frac{1}{\sigma-1}. \tag{5.8}$$

Without loss of generality, we may assume $x=[x]+1/2$. Now we may apply Theorem 5.1 with $b=1+1/\log x$ and $T=\exp(\sqrt{\log x})$ to $\sum_{n\le x} a_n$. By (5.5) and (5.8), we get

$$\sum_{n\le x} a_n = \frac{1}{2\pi i}\int_{b-iT}^{b+iT}\left\{-\frac{L'}{L}(s,\pi\times\tilde{\pi}')\right\}\frac{x^s}{s}ds + O\{x\exp(-c\sqrt{\log x})\}. \tag{5.9}$$

Now we can shift the contour in (5.9) to the left, apply the zero-free region of Moreno [20] [21], Sarnak [27], and Gelbart, Lapid, and Sarnak [3], and estimate the resulting sums over zeros and poles. Theorem 5.2 then follows.

**Corollary 5.3.** *In Corollary 3.3, the assumption of GRC can be removed.*

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