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Some explicit upper bounds for residues of zeta functions of number fields taking into account the behavior of the prime 2

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Abstract

We recall the known explicit upper bounds for the residue at $s=1$ of the Dedekind zeta function of a number field K. Then, we improve upon these previously known upper bounds by taking into account the behavior of the prime 2 in K. We finally give several examples showing how such improvements yield better bounds on the absolute values of the discriminants of CM-fields of a given relative class number. In particular, we will obtain a 4000-fold improvement on our previous bound for the absolute values of the discriminants of the non-normal sextic CM-fields with relative class number one.

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1 Introduction

The solutions to some class number one problems for CM-fields are sometimes difficult and rely heavily on good upper bounds for residues at $s = 1$ of Dedekind zeta functions of totally real number fields (e.g. see [Lou98, Section 5] for the construction of a very short list of CM-fields containing all the normal CM-fields of degree 24, of Galois group the special linear group over the finite field with three elements $SL_2(\mathbf{F}_3)$, and of class number one. Then, see [Lou01a, Theorem 12] for the solution to this class number one problem). At the moment, one difficult class number problem which is not yet completely solved is the determination of all the non-isomorphic non-normal sextic CM-fields with class number one which do not contain neither an imaginary quadratic subfield nor a real cyclic cubic subfield (however, see [BL, Corollary 17] for the conjectural complete list of these non-isomorphic CM-fields). The solution to this class number one problem will rely heavily on improvements on known upper bounds for residues at $s = 1$ of Dedekind zeta functions of non-normal totally real cubic number fields. The aim of this paper is to provide in Theorem 9 such improvements and to apply them to the solution to this difficult class number one problem (see Corollary 10). The main results arrived at in this paper are a new proof of Theorem 2 and Theorems 5, 6, 9, 16 and 24.

Let d_K and $\zeta_K(s)$ denote the absolute value of the discriminant and the Dedekind zeta function of a number field K of degree $m > 1$. The best general upper bound for the residues $\kappa_K := \mathrm{Res}_{s=1}(\zeta_K(s))$ at $s = 1$ of the Dedekind zeta function of number fields K of a given degree $m > 1$ is:

Theorem 1 *(See* [Lou00, Theorem 1] *and* [Lou01b, Theorem 1]*). Let K be a number field of degree $m > 1$. Then*

$$\kappa_K \leq \left(\frac{e \log d_K}{2(m-1)}\right)^{m-1}.$$

However, for some totally real number fields an improvement on this bound is known (see [BL] and [Oka] for applications):

Theorem 2 *(See* [Lou01b, Theorem 2]*). Let K range over a family of totally real number fields of a given degree $m > 1$ for which $\zeta_K(s)/\zeta(s)$ is entire (which holds true if $K/\mathbf{Q}$ is normal or if K is cubic). There exists C_m (computable) such that $d_K \geq C_m$ implies*

$$\kappa_K \leq \frac{\log^{m-1} d_K}{2^{m-1}(m-1)!} \leq \frac{1}{\sqrt{2\pi(m-1)}}\left(\frac{e \log d_K}{2(m-1)}\right)^{m-1}.$$

In fact, it is known that $\zeta_K(s)/\zeta(s)$ is entire (i) for any normal number field K (see [MM, Chapter 2, Theorem 3]), and (ii) for any number field K for which the Galois group of its normal closure is solvable (see [Uch], [vdW] and [MM, Chapter 2, Corollary 4.2]), e.g. for any cubic or quartic number field.

For totally real cubic number fields we have a slightly better bound than the one given in Theorem 2:

Theorem 3 *Let K be a totally real cubic number field. Set $\lambda := 2 + 2\gamma - 2\log(4\pi) = -1.90761\cdots$. Then,*

$$\kappa_K \leq \frac{1}{8}(\log d_K + \lambda)^2.$$

Let us finally point out that in the case that $K/\mathbf{Q}$ is abelian we have an even better bound (use [Ram1, Corollary 1] and notice that if K is imaginary, then $m/2$ of the m characters in the group of primitive Dirichlet characters associated with K are odd):

Theorem 4 *Let K be an abelian number field of degree $m > 1$. Set $\lambda_m = 0$ if K is real and $\lambda_m = \frac{m}{m-1}(\frac{5}{4} - \frac{1}{2}\log 6)$ if K is imaginary. Then,*

$$\kappa_K \leq \left(\frac{\log d_K}{2(m-1)} + \lambda_m\right)^{m-1}.$$

Now, we showed in [Lou03] how taking into account the behavior of the prime 2 in CM-fields can greatly improve upon the upper bounds on the root numbers of the normal CM-fields with abelian maximal totally real subfields of a given (relative) class number. The aim of this paper is, by taking into account the behavior of the prime 2, to improve upon these four previously known upper bounds

$$\kappa_K \leq c_m(d_m \log d_K + \lambda_m)^{m-1}$$

for the residues κ_K of Dedekind zeta functions of number fields K given in Theorems 1, 2, 3 and 4 by the factor $\Pi_K(2)/\Pi_{\mathbf{Q}}^m(2)$:

$$\kappa_K \leq c_m \frac{\Pi_K(2)}{\Pi_{\mathbf{Q}}^m(2)}(d_m \log d_K + \lambda'_m)^{m-1}.$$

Here, K is a number field of degree $m > 1$, $p \geq 2$ is a prime and for $s > 0$ we have set

$$\Pi_K(p,s) := \prod_{\mathcal{P}_K | p} (1 - (N_{K/\mathbf{Q}}(\mathcal{P}_K))^{-s})^{-1} \leq \Pi_{\mathbf{Q}}^m(p,s),$$

(where $\mathcal{P}_K$ runs over all the primes ideals of K above p) and $\Pi_K(p) := \Pi_K(p,1)$. In particular, $\Pi_K(p)/\Pi_{\mathbf{Q}}^m(p) \leq 1$. However, if 2 is inert in K, then the factor $\Pi_K(2)/\Pi_{\mathbf{Q}}^m(2) = 1/(2^m - 1)$ is small. We give in Corollaries 7 and 10 two examples showing how useful such improvements are. See also [Lou05] for various other applications.

We also refer the reader to [LK] for a recent paper dealing with upper bounds on the degrees and absolute values of the discriminants of the CM-fields of class number one, under the assumption of the generalized Riemann hypothesis. The proof relies on a generalization of Odlyzko ([Odl]), Stark ([Sta]) and Bessassi's ([Bes]) upper bounds for residues of Dedekind zeta functions of totally real number fields of large degrees, this generalization taking into account the behavior of small primes. All these bounds are better than ours, but only for numbers fields of large degrees and small root discriminants, whereas ours are developed to deal with CM-fields of small degrees (see Corollary 10).

2 The abelian case

Theorem 5 *(Compare with Theorem 4). Let K be an abelian number field of degree $m > 1$. Set $\lambda_m = 2\log 2$ if K is real, and $\lambda_m = \frac{m(5/2+\log(8/3))-4\log 2}{2(m-1)}$ if K is imaginary. Then,*

$$\kappa_K \leq \frac{\Pi_K(2)}{\Pi_{\mathbf{Q}}^m(2)}\left(\frac{\log d_K}{2(m-1)} + \lambda_m\right)^{m-1}.$$

Proof. According to [Ram2, Corollary 2], for any primitive Dirichlet character χ of conductor $f_\chi > 1$ it holds that

$$\left|(1-\frac{\chi(2)}{2})L(1,\chi)\right| \leq \frac{1}{4}(\log f_\chi + \kappa_\chi),$$

where

$$\kappa_\chi = \begin{cases} 4\log 2 & \text{if } \chi \text{ is even,} \\ 5-2\log(3/2) & \text{if } \chi \text{ is odd.} \end{cases}$$

(See also [Lou04b] for slightly worse bounds). Now, by letting χ range over the $m-1$ non-trivial characters $\chi \neq 1$ of the group X_K of Dirichlet characters associated with K, by noticing that

$$\Pi_K(2) = \prod_{\chi\in X_K}(1-\frac{\chi(2)}{2})^{-1} = 2\prod_{1\neq\chi\in X_K}(1-\frac{\chi(2)}{2})^{-1}$$

and $\kappa_K = \prod_{\chi\in X_K} L(1,\chi)$, by using the fact that the geometric mean is lass than or equal to the arithmetic mean, by using the conductor-discriminant formula $d_K = \prod_{1\neq\chi\in X_K} f_\chi$, and by noticing that if K is imaginary, then $m/2$ of the characters $\chi \in X_K$ are odd and $m/2-1$ of the characters $1 \neq \chi \in X_K$ are even, we obtain the desired result. •

3 The general case

Theorem 6 *(Compare with Theorem 1). Let K be a number field of degree $m \geq 2$ and root discriminant $\rho_K = d_K^{1/m}$. Set $E(x) := (e^x-1)/x = 1+O(x)$ for $x \longrightarrow 0^+$, $\lambda_K = (\log 4)E(\frac{\log 4}{\log\rho_K}) = \log 4 + O(\log^{-1}\rho_K)$ and $v_m = (m/(m-1))^{m-1} \in [2,e)$. Then,*

$$\kappa_K \leq (e/2)^{m-1} v_m \frac{\Pi_K(2)}{\Pi_{\mathbf{Q}}^m(2)} (\log\rho_K + \lambda_K)^{m-1} \tag{1}$$

Moreover, $0 < \beta < 1$ and $\zeta_K(\beta) = 0$ imply

$$\kappa_K \leq (1-\beta)(e/2)^m \frac{\Pi_K(2)}{\Pi_{\mathbf{Q}}^m(2)} (\log\rho_K + \lambda_K)^m. \tag{2}$$

Proof. We only prove (1), the proof of (2) being similar. According to [Lou01b, Section 6.1] but using the bound

$$\zeta_K(s) = \prod_{p\geq 2}\Pi_K(p,s) \leq \Pi_K(2,s)\prod_{p\geq 3}\Pi_{\mathbf{Q}}^m(p,s) = \frac{\Pi_K(2,s)}{\Pi_{\mathbf{Q}}^m(2,s)}\zeta^m(s)$$

(for $s > 1$), instead of the bound $\zeta_K(s) \leq \zeta^m(s)$, we have

$$\kappa_K \leq \frac{\Pi_K(2)}{\Pi_{\mathbf{Q}}^m(2)}\left(\frac{e\log d_K}{2(m-1)}\right)^{m-1} g(s_K)$$

where $s_K = 1 + 2(m-1)/\log d_K \in [1,6]$ and

$$g(s) := \frac{\Pi_K(2,s)/\Pi_K(2)}{\Pi_{\mathbf{Q}}^m(2,s)/\Pi_{\mathbf{Q}}^m(2)} \leq h(s) := \Pi_{\mathbf{Q}}^m(2)/\Pi_{\mathbf{Q}}^m(2,s)$$

(for $\Pi_K(2,s) \leq \Pi_K(2,1) = \Pi_K(2)$ for $s \geq 1$). Now, $\log h(1) = 0$ and

$$(h'/h)(s) = \frac{m\log 2}{2^s - 1} \leq m\log 2$$

for $s \geq 1$. Hence,

$$\log h(s_K) \leq (s_K - 1)m\log 2 = \frac{(m-1)\log 4}{\log \rho_K},$$

$$g(s_K) \leq h(s_K) \leq \left(\exp(\frac{\log 4}{\log \rho_K})\right)^{m-1}$$

and (1) follows. •

Corollary 7 *(Compare with* [Lou01b, Theorems 12 and 14] *and* [Lou03, Theorem 9 and 22]*). Set* $c = 2(\sqrt{3}-1)^2 = 1.07\cdots$. *Let* N *be a normal CM-field of degree* $2m > 2$, *relative class number* h_N^- *and root discriminant* $\rho_N = d_N^{1/2m} \geq 650$. *Assume that* N *contains no imaginary quadratic subfield (or that the Dedekind zeta functions of the imaginary quadratic subfields of* N *have no real zero in the range* $1 - (c/\log d_N) \leq s < 1$*). Then,*

$$h_N^- \geq \frac{c}{2mv_m e^{c/2-1}}\left(\frac{\frac{4}{3}\sqrt{\rho_N}}{\pi e(\log \rho_N + (\log 4)E(\frac{\log 4}{\log \rho_N}))}\right)^m.$$

Hence, $h_N^- > 1$ *for* $m \geq 5$ *and* $\rho_N \geq 14607$, *or for* $m \geq 10$ *and* $\rho_N \geq 9150$. *Moreover,* $h_N^- \to \infty$ *as* $[N : \mathbf{Q}] = 2m \to \infty$ *for such normal CM-fields* N *of root discriminants* $\rho_N \geq 3928$.

Proof. Follow the proof of [Lou01b, Theorems 12 and 14] and [Lou03, Theorems 9 and 12], but now make use of Theorem 6 instead of [Lou01b, Theorem 1] and use the following lower bound with $p = 2$:

$$\frac{\Pi_N(p)}{\Pi_K(p)/\Pi_{\mathbf{Q}}^m(p)} = \left(\frac{p}{p-1}\right)^m \frac{\Pi_N(2)}{\Pi_K(2)} = \left(\frac{p}{p-1}\right)^m \prod_{\mathcal{P}_K|(p)} \frac{1}{1 - \frac{\chi_{N/K}(\mathcal{P}_K)}{N_{K/\mathbf{Q}}(\mathcal{P}_K)}} \geq \left(\frac{p^2}{p^2-1}\right)^m$$

(here $\chi_{N/K}$ is the quadratic character associated with the quadratic extension N/K, and $\mathcal{P}_K$ ranges over the primes ideals of K lying above the prime 2). •

Remark 8 *It may be worth noticing that if instead of simply considering the prime 2 we fix a finite set* S *of primes, then (1) and (2) still hold true with the* $\log 4$ *term in* λ_K *being replaced (twice) by* $2(\sum_{p\in S}\frac{\log p}{p-1})$ *and the factor* $\Pi_K(2)/\Pi_{\mathbf{Q}}^m(2)$ *being replaced by the product* $\prod_{p\in S}(\Pi_K(p)/\Pi_{\mathbf{Q}}^m(p))$. *However, choose* $S = \{2,3\}$. *Then the terms* $\frac{4}{3}$ *and* $\log 4$ *(twice) in the lower bound in Corollary 7 are changed into* $\frac{3}{2} = \frac{2^2}{2^2-1}\frac{3^2}{3^2-1}$ *and* $\log(12) = 2(\frac{\log 2}{2-1} + \frac{\log 3}{3-1})$ *(twice), and we have a better asymptotic lower bound for* h_K^-. *However, this better asymptotic lower bound yields only* $h_N^- > 1$ *for* $m \geq 5$ *and* $\rho_N \geq 14496$, *or for* $m \geq 10$ *and* $\rho_N \geq 9208$, *and* $h_N^- \to \infty$ *as* $[N : \mathbf{Q}] = 2m \to \infty$ *for such normal CM-fields* N *of root discriminants* $\rho_N \geq 4072$.

4 The non-normal cubic case

It follows from [Lou04a, Corollary 2] that if F is a real quadratic number field, then we have an explicit upper bound of the type

$$\kappa_F \leq \frac{\Pi_F(2)}{2\Pi_{\mathbf{Q}}^2(2)}(\log d_F + \lambda_F).$$

More precisely, set

$$\begin{cases} \lambda_1 = 2 + \gamma - \log(4\pi) = 0.04619\cdots, \\ \lambda_2 = 2 + \gamma - \log \pi = 1.43248\cdots, \\ \lambda_3 = 2 + \gamma - \log(\pi/4) = 2.81878\cdots. \end{cases}$$

Then,

$$\kappa_F \leq \begin{cases} (\log d_F + \lambda_1)/2 & \text{if } (2) = \mathcal{P}_1\mathcal{P}_2 \text{ in } F, \\ (\log d_F + \lambda_2)/4 & \text{if } (2) = \mathcal{P}^2 \text{ in } F, \\ (\log d_F + \lambda_3)/6 & \text{if } (2) = \mathcal{P} \text{ in } F. \end{cases}$$

Moreover, O. Ramaré proved in [Ram1, Corollaries 1 and 3] and [Ram2] that this result still holds true with the better following values: $\lambda_1 = 0$, $\lambda_2 = 2\log 2 = 1.38629\cdots$ and $\lambda_3 = 4\log 2 = 2.77258\cdots$. In the same way, if F is an abelian cubic number field, then F is (totally) real, 2 is inert or splits completely in F, and according to Theorems 4 and 5 we have the desired types of explicit bounds:

$$\kappa_F \leq \begin{cases} (\log d_F)^2/16 & \text{if } (2) = \mathcal{P}_1\mathcal{P}_2\mathcal{P}_3 \text{ in } F, \\ (\log d_F + 8\log 2)^2/112 & \text{if } (2) = \mathcal{P} \text{ in } F. \end{cases}$$

One of the main result of this paper is the following one which gives an explicit upper bound of the type

$$\kappa_F \leq \frac{\Pi_F(2)}{8\Pi_{\mathbf{Q}}^3(2)}(\log d_F + \lambda_F)^2$$

for non-normal totally real cubic number fields F:

Theorem 9 *Set*

$$\begin{cases} \lambda_1 = 2 + 2\gamma - 2\log \pi - 4\log 2 = -1.90761\cdots, \\ \lambda_2 = 2 + 2\gamma - 2\log \pi - 2\log 2 = -0.52132\cdots, \\ \lambda_3 = 2 + 2\gamma - 2\log \pi + 4\log 2 = 3.63756\cdots, \\ \lambda_4 = 2 + 2\gamma - 2\log \pi = 0.86497\cdots, \\ \lambda_5 = 2 + 2\gamma - 2\log \pi + 2\log 6 = 4.44849\cdots. \end{cases}$$

Let F be a non-normal totally real cubic number field. Then,

$$\kappa_F \leq \begin{cases} (\log d_F + \lambda_1)^2/8 & \text{if } (2) = \mathcal{P}_1\mathcal{P}_2\mathcal{P}_3 \text{ in } F, \\ (\log d_F + \lambda_2)^2/16 & \text{if } (2) = \mathcal{P}_1\mathcal{P}_2^2 \text{ in } F, \\ (\log d_F + \lambda_3)^2/24 & \text{if } (2) = \mathcal{P}_1\mathcal{P}_2 \text{ in } F, \\ (\log d_F + \lambda_4)^2/32 & \text{if } (2) = \mathcal{P}^3 \text{ in } F, \\ (\log d_F + \lambda_5)^2/56 & \text{if } (2) = \mathcal{P} \text{ in } F. \end{cases}$$

This result will follow from Theorem 3 (in the case that $(2) = \mathcal{P}_1\mathcal{P}_2\mathcal{P}_3$ in F), Theorem 23 (in the cases that $(2) = \mathcal{P}_1\mathcal{P}_2^2$ or $(2) = \mathcal{P}^3$ in F) and Theorem 29 (in the cases that $(2) = \mathcal{P}_1\mathcal{P}_2$ or $(2) = \mathcal{P}$ in F) which are special cases of more general results (see Theorem 2, and Theorems 16 and 24 below). However, we first give an important consequence of these new bounds, that is a 4000-fold improvement on our previous bound on the absolute values of the discriminants of the non-normal sextic CM-fields of class number one:

Corollary 10 *Let K be a non-normal sextic CM-field such that K contains no imaginary quadratic subfield and the totally real cubic subfield F of K is not normal. Assume that $d_K \geq 4 \cdot 10^{20}$, which implies $\rho_K = d_K^{1/6} \geq 2683$. Then,*

$$h_K^- \geq \frac{d_K^{1/4}}{C_K(\log d_K + \lambda_K)^3}. \tag{3}$$

Hence, $h_K^- > 1$ for $d_K \geq B_K$ or for $d_F \geq B_F := \sqrt{B_K/3}$, with C_K, λ_K, B_K and B_F as follows:

splitting of 2 *in* F	*frequency*	C_K	λ_K	B_K	B_F
$(2) = \mathcal{P}_1\mathcal{P}_2\mathcal{P}_3$	2/21	14.9	−2.90	$50 \cdot 10^{24}$	$4.1 \cdot 10^{12}$
$(2) = \mathcal{P}_1\mathcal{P}_2^2$	6/21	10.0	−1.06	$11 \cdot 10^{24}$	$2.0 \cdot 10^{12}$
$(2) = \mathcal{P}_1\mathcal{P}_2$	6/21	8.27	4.49	$17 \cdot 10^{24}$	$2.4 \cdot 10^{12}$
$(2) = \mathcal{P}^3$	3/21	6.62	0.79	$22 \cdot 10^{23}$	$8.6 \cdot 10^{11}$
$(2) = \mathcal{P}$	4/21	4.96	5.57	$18 \cdot 10^{23}$	$7.8 \cdot 10^{11}$

Remark 11 *This greatly improves upon the lower bound (obtained in [BL, Theorem 12])*

$$h_K^- \geq \frac{d_K^{1/4}}{68 \log^3 d_K},$$

which implies $h_K^- > 1$ for $d_K \geq 2 \cdot 10^{29}$ or $d_F \geq 3 \cdot 10^{14}$. Notice that the number $N(X)$ of non-isomorphic non-normal totally real cubic number fields F of discriminants $d_F \leq X$ is asymptotic to $X/(12\zeta(3))$ (H. Davenport and H. Heilbronn). Hence, our 73-fold improvement on B_F would considerably alleviate the amount of numerical computation required to rigorously solve the class number one problem for the non-normal CM-sextic fields. In this respect, let us mention that all the non-isomorphic non-normal totally real cubic number fields F of discriminants $d_F \leq 10^{11}$ have been determined in [Bel].

Proof. Let N denote the normal closure of K. Then $[N : \mathbf{Q}] = 48$, d_K^8 divides d_N and d_N divides d_K^{24} (see [BL, Lemmas 10 and 11]). Set $c := 2(\sqrt{3} - 1)^2 = 1.07\cdots$. The Dedekind zeta function $\zeta_N(s)$ of a number field N has at most two real zeros in the range $1 - (c/\log d_N) \leq s < 1$ (see [LLO, Lemma 15]). Since any complex zero of $\zeta_K(s)/\zeta_F(s)$ is at least a triple zero of $\zeta_N(s)$ (see [BL, Lemma 11]), it follows that $\zeta_K(s)/\zeta_F(s)$ has no real zero in the range $1 - (2/\log d_K) \leq 1 - (c/8 \log d_K) \leq 1 - (c/\log d_N) \leq s < 1$. Finally, recall from [Lou03, Theorem 1(4)] that if $\rho_K \geq 2683$, $1 - (2/\log d_K) \leq \beta < 1$ and $\zeta_K(\beta) \leq 0$, then

$$\kappa_K \geq (1 - \beta) d_K^{(\beta-1)/2} \Pi_K(2).$$

Now, there are two cases to consider.

1. First, assume that $\zeta_F(s)$ has a real zero β in $[1-(c/\log d_N),1)$. Then, $\zeta_K(\beta)=0$ and

$$\kappa_K \geq (1-\beta)d_K^{(\beta-1)/2}\Pi_K(2) \geq (1-\beta)e^{-c/16}\Pi_K(2),$$

$$\kappa_F \leq \frac{1-\beta}{48}\log^3 d_F \leq \frac{1-\beta}{96}(\log d_F)^2\log d_K \leq \frac{1-\beta}{768}(\log d_F)^2\log d_N$$

(by [Lou01b, (2) and (7)]), and

$$h_K^- = \frac{Q_K w_K}{(2\pi)^3}\sqrt{\frac{d_K}{d_F}}\frac{\kappa_K}{\kappa_F} \geq \frac{192\Pi_K(2)\sqrt{d_K/d_F}}{\pi^3 e^{c/16}(\log d_F)^2\log d_N} \tag{4}$$

(where $w_K \geq 2$ is the number of complex roots of unity contained in K and $Q_K \in \{1,2\}$ is the Hasse unit index of K).
2. Second, assume that $\zeta_F(s)$ has no real zero β in $[1-(c/\log d_N),1)$. Then, $\zeta_K(s)=\zeta_F(s)(\zeta_K(s)/\zeta_F(s))$ has no real zero β in $[1-(c/\log d_N),1)$, hence $\zeta_K(1-(c/\log d_N)) \leq 0$, which yields

$$\kappa_K \geq \frac{c}{e^{c/16}\log d_N}\Pi_K(2).$$

Using the five bounds in Theorem 9 which we write $\kappa_F \leq \frac{1}{8c_F}(\log d_F+\lambda_F)^2$ with $c_F \in \{1,2,3,4,7\}$, we obtain

$$h_K^- = \frac{Q_K w_K}{(2\pi)^3}\sqrt{\frac{d_K}{d_F}}\frac{\kappa_K}{\kappa_F} \geq \frac{2cc_F\Pi_K(2)\sqrt{d_K/d_F}}{\pi^3 e^{c/16}(\log d_F+\lambda_F)^2\log d_N}. \tag{5}$$

Since (4) is better than (5), this latter lower bound (5) always holds true. Using $d_N \leq (d_K/d_F)^{24}$ (see [BL, proof of Lemma 11]), $d_K/d_F \geq \sqrt{3d_K}$ (see [BL, Proposition 1]) and consequently $d_F \leq \sqrt{d_K/3}$, we obtain $(\log d_F+\lambda_F)^2\log d_N \leq 24(\log d_F+\lambda_F)^2\log(d_K/d_F) \leq 24(\log(\sqrt{d_K/3})+\lambda_F)^2\log(\sqrt{3d_K})$ $= 3(\log(d_K/3)+2\lambda_F)^2\log(3d_K) \leq 3(\log d_K+(4\lambda_F-\log 3)/3)^3$, from which (3) follows with

$$C_K = \frac{3\pi^3 e^{c/16}}{2\cdot 3^{1/4}\cdot cc_F\Pi_K(2)} \text{ and } \lambda_K = \frac{4\lambda_F-\log 3}{3}.$$

Finally, we notice that

$$\Pi_K(2) = \prod_{\mathcal{P}_F|(2)} \frac{1}{1-\frac{1}{N_{F/\mathbb{Q}}(\mathcal{P}_K)}}\frac{1}{1-\frac{\chi_{K/F}(\mathcal{P}_F)}{N_{F/\mathbb{Q}}(\mathcal{P}_K)}} \geq \prod_{\mathcal{P}_F|(2)} \frac{1}{1-\frac{1}{N_{F/\mathbb{Q}}(\mathcal{P}_K)^2}} = \Pi_F(2,2)$$

(where $\chi_{K/F}$ is the quadratic character associated with the quadratic extension K/F, and $\mathcal{P}_F$ ranges over the primes ideals of F lying above the prime 2). Using the following Table:

splitting of 2 in F	c_F	$\Pi_K(2) \geq \Pi_F(2,2) =$	$\lambda_K =$	$C_K =$
$(2)=\mathcal{P}_1\mathcal{P}_2\mathcal{P}_3$	1	64/27	$-2.90969\cdots$	$14.87387\cdots$
$(2)=\mathcal{P}_1\mathcal{P}_2^2$	2	16/9	$-1.06130\cdots$	$9.91591\cdots$
$(2)=\mathcal{P}_1\mathcal{P}_2$	3	64/45	$4.48387\cdots$	$8.26326\cdots$
$(2)=\mathcal{P}^3$	4	4/3	$0.78709\cdots$	$6.61061\cdots$
$(2)=\mathcal{P}$	7	64/63	$5.56511\cdots$	$4.95795\cdots$

we obtain the desired bounds on B_K and B_F. •

5 Proofs of Theorems 2 and 3

In this section, we give a clearer proof of Theorem 2 than the one given in [Lou01b, Proof of Theorem 2]. We will then adapt this clearer proof to prove Theorems 16 and 24 below. To begin with, we set some notation: K is a totally real number field of degree $m > 1$, and we assume that $\zeta_{K/\mathbf{Q}}(s) := \zeta_K(s)/\zeta(s)$ is entire. We set $A_{K/\mathbf{Q}} := \sqrt{d_K/\pi^{m-1}}$ and $F_{K/\mathbf{Q}}(s) := A_{K/\mathbf{Q}}^s \Gamma^{m-1}(s/2)\zeta_{K/\mathbf{Q}}(s)$. Under our assumption, $F_{K/\mathbf{Q}}(s)$ is entire and satisfies the functional equation $F_{K/\mathbf{Q}}(1-s) = F_{K/\mathbf{Q}}(s)$. Let

$$S_{K/\mathbf{Q}}(x) := \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F_{K/\mathbf{Q}}(s)x^{-s}\mathrm{d}s \quad (c > 1 \text{ and } x > 0) \tag{6}$$

denote the inverse Mellin transform of $F_{K/\mathbf{Q}}(s)$. Since $F_{K/\mathbf{Q}}(s)$ is entire, it follows that $S_{K/\mathbf{Q}}(x)$ satisfies the functional equation

$$S_{K/\mathbf{Q}}(x) = \frac{1}{x}S_{K/\mathbf{Q}}(\frac{1}{x}) \tag{7}$$

(shift the vertical line of integration $\Re(s) = c > 1$ in (6) leftwards to the vertical line of integration $\Re(s) = 1 - c < 0$, then use the functional equation $F_{K/\mathbf{Q}}(1-s) = F_{K/\mathbf{Q}}(s)$ to come back to the vertical line of integration $\Re(s) = c > 1$). For $\Re(s) > 1$,

$$F_{K/\mathbf{Q}}(s) = \int_0^\infty S_{K/\mathbf{Q}}(x)x^s\frac{\mathrm{d}x}{x}$$

is the Mellin transform of $S_{K/\mathbf{Q}}(x)$. Using (7), we obtain

$$F_{K/\mathbf{Q}}(s) = \int_1^\infty S_{K/\mathbf{Q}}(x)(x^s + x^{1-s})\frac{\mathrm{d}x}{x} \tag{8}$$

on the whole complex plane.

Now, set

$$F_{m-1}(s) := \left(\pi^{-s/2}\Gamma(s/2)\zeta(s)\right)^{m-1}, \tag{9}$$

$A_{m-1} := \pi^{-(m-1)/2}$ and

$$d := \sqrt{d_K} = A_{K/\mathbf{Q}}/A_{m-1}. \tag{10}$$

Then,

$$\sqrt{d_K}\kappa_K = d\kappa_K = F_{K/\mathbf{Q}}(1) = \int_1^\infty S_{K/\mathbf{Q}}(x)(1+\frac{1}{x})\mathrm{d}x. \tag{11}$$

Let

$$S_{m-1}(x) := \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F_{m-1}(s)x^{-s}\mathrm{d}s \quad (c > 1 \text{ and } x > 0) \tag{12}$$

denote the inverse Mellin transform of $F_{m-1}(s)$. Here, $F_{m-1}(s)$ has two poles, at $s = 1$ and $s = 0$, the functional equation $F_{m-1}(1-s) = F_{m-1}(s)$ yields

$$\mathrm{Res}_{s=0}(F_{m-1}(s)x^{-s}) = -\mathrm{Res}_{s=1}(F_{m-1}(s)x^{s-1})$$

and

$$S_{m-1}(x) = \mathrm{Res}_{s=1}\{F_{m-1}(s)(x^{-s} - x^{s-1})\} + \frac{1}{x}S_{m-1}(\frac{1}{x}) \tag{13}$$

(shift the vertical line of integration $\Re(s) = c > 1$ in (12) leftwards to the vertical line of integration $\Re(s) = 1 - c < 0$, notice that you pick up residues at $s = 1$ and $s = 0$, then use the functional equation $F_{m-1}(1-s) = F_{m-1}(s)$ to come back to the vertical line of integration $\Re(s) = c > 1$). Finally, we set

$$H_{m-1}(x) := \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \Gamma^{m-1}(s/2)x^{-s}\mathrm{d}s \quad (c > 1 \text{ and } x > 0).$$

Notice that $H_{m-1}(x) > 0$ and $S_{m-1}(x) > 0$ for $x > 0$ (see [Lou00, Proof of Theorem 2] [1]). This observation is of paramount importance for obtaining our bounds, e.g. for obtaining (14), (18), (20) and (27) below.

5.1 An upper bound on κ_K

Now, write

$$\zeta_{K/\mathbf{Q}}(s) := \zeta_K(s)/\zeta(s) = \sum_{n\geq 1} a_{K/\mathbf{Q}}(n)n^{-s}$$

and

$$\zeta^{m-1}(s) = \sum_{n\geq 1} a_{m-1}(n)n^{-s}.$$

Then, $|a_{K/\mathbf{Q}}(n)| \leq a_{m-1}(n)$ for all $n \geq 1$ (see [Lou01b, Lemma 26] or use (17) below). Since

$$S_{K/\mathbf{Q}}(x) = \sum_{n\geq 1} a_{K/\mathbf{Q}}(n)H_{m-1}(nx/A_{K/\mathbf{Q}})$$

and

$$0 \leq S_{m-1}(x) = \sum_{n\geq 1} a_{m-1}(n)H_{m-1}(nx/A_{m-1}),$$

we obtain

$$|S_{K/\mathbf{Q}}(x)| \leq S_{m-1}(x/d), \tag{14}$$

by (10). Using (11) and (14), we obtain:

$$d\kappa_K \leq \int_1^\infty S_{m-1}(x/d)(1+\frac{1}{x})\mathrm{d}x. \tag{15}$$

Now, we compute the integral in (15) to obtain the following key Proposition:

Proposition 12 *For α and $D > 0$ real, it holds that*

$$\begin{aligned}\int_1^\infty S_{m-1}(x/D)x^{-\alpha}\mathrm{d}x &= \mathrm{Res}_{s=1}\left\{F_{m-1}(s)\Big(\frac{D^s}{s+\alpha-1}+\frac{D^{1-s}}{s-\alpha}\Big)\right\} \\ &\quad -D^{1-\alpha}\int_D^\infty S_{m-1}(x)x^{\alpha-1}\mathrm{d}x \\ &\leq \mathrm{Res}_{s=1}\left\{F_{m-1}(s)\Big(\frac{D^s}{s+\alpha-1}+\frac{D^{1-s}}{s-\alpha}\Big)\right\}.\end{aligned}$$

[1] Notice the misprints in [Lou00, page 273, line 1] and [Lou01b, Theorem 20] where one should read

$$(M_1 \star M_2)(x) = \int_0^\infty M_1(x/t)M_2(t)\frac{dt}{t}.$$

Proof.

$$\int_1^\infty S_{m-1}(x/D)x^{-\alpha}\mathrm{d}x = D^{1-\alpha}\int_{1/D}^\infty S_{m-1}(x)x^{-\alpha}\mathrm{d}x$$

$$= D^{1-\alpha}\int_1^\infty S_{m-1}(x)x^{-\alpha}\mathrm{d}x + D^{1-\alpha}\int_{1/D}^1 S_{m-1}(x)x^{-\alpha}\mathrm{d}x$$

$$= D^{1-\alpha}\int_1^\infty S_{m-1}(x)x^{-\alpha}\mathrm{d}x + D^{1-\alpha}\int_1^D \frac{1}{x}S_{m-1}(\frac{1}{x})x^{\alpha-1}\mathrm{d}x$$

$$= D^{1-\alpha}\int_1^\infty S_{m-1}(x)(x^{-\alpha}+x^{\alpha-1})\mathrm{d}x - D^{1-\alpha}\int_D^\infty S_{m-1}(x)x^{\alpha-1}\mathrm{d}x$$

$$-D^{1-\alpha}\int_1^D \mathrm{Res}_{s=1}\{F_{m-1}(s)(x^{-s}-x^{s-1})\}x^{\alpha-1}\mathrm{d}x \quad \text{(by (13))}$$

$$= D^{1-\alpha}\int_1^\infty S_{m-1}(x)(x^{-\alpha}+x^{\alpha-1})\mathrm{d}x - D^{1-\alpha}\int_D^\infty S_{m-1}(x)x^{\alpha-1}\mathrm{d}x$$

$$-D^{1-\alpha}\mathrm{Res}_{s=1}\Big\{F_{m-1}(s)\int_1^D (x^{-s}-x^{s-1})x^{\alpha-1}\mathrm{d}x\Big\}$$

(compute these residues as contour integrals along a circle of center 1 and of small radius, and use Fubini's theorem)

$$= D^{1-\alpha}\Big(\Big(\int_1^\infty S_{m-1}(x)(x^{-\alpha}+x^{\alpha-1})\mathrm{d}x - \mathrm{Res}_{s=1}\Big\{F_{m-1}(s)(\frac{1}{s-\alpha}+\frac{1}{s+\alpha-1})\Big\}\Big)$$

$$+\mathrm{Res}_{s=1}\Big\{F_{m-1}(s)(\frac{D^{1-s}}{s-\alpha}+\frac{D^s}{s+\alpha-1})\Big\} - D^{1-\alpha}\int_D^\infty S_{m-1}(x)x^{\alpha-1}\mathrm{d}x.$$

The desired result now follows from Lemma 13 below. •

Lemma 13 *For α real and $D > 0$, it holds that*

$$\int_1^\infty S_{m-1}(x)(x^{-\alpha}+x^{\alpha-1})\mathrm{d}x = \mathrm{Res}_{s=1}\Big\{F_{m-1}(s)(\frac{1}{s-\alpha}+\frac{1}{s+\alpha-1})\Big\}.$$

Proof. Let $I_{m-1}(\alpha)$ denote this left hand side integral. By (12) with c large enough (namely with $c > 1$, $c > 1-\alpha$ and $c > -\alpha$) and by Fubini's theorem, we have

$$I_{m-1}(\alpha) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F_{m-1}(s)\Big(\int_1^\infty (x^{-s-\alpha}+x^{-s+\alpha-1})\mathrm{d}x\Big)\mathrm{d}s$$

$$= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} G_{m-1,\alpha}(s)\mathrm{d}s,$$

where

$$G_{m-1,\alpha}(s) := F_{m-1}(s)(\frac{1}{s-\alpha}+\frac{1}{s+\alpha-1}).$$

The functional equation $G_{m-1,\alpha}(1-s) = -G_{m-1,\alpha}(s)$ yields

$$I_{m-1}(\alpha) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} G_{m-1,\alpha}(s)\mathrm{d}s$$

$$= \mathrm{Res}_{s=1}(G_{m-1,\alpha}(s)) + \mathrm{Res}_{s=0}(G_{m-1,\alpha}(s)) + \frac{1}{2\pi i}\int_{1-c-i\infty}^{1-c+i\infty} G_{m-1,\alpha}(s)\mathrm{d}s$$

$$= 2\mathrm{Res}_{s=1}(G_{m-1,\alpha}(s)) - I_{m-1}(\alpha),$$

from which the desired result follows. •

Hence, using (15) and Proposition 12 with $D = d$ and $\alpha = 0$, and with $D = d$ and $\alpha = 1$, we have proved:

Proposition 14 *Let K be a totally real number field of degree $m > 1$, and assume that $\zeta_K(s)/\zeta(s)$ is entire. Set $d = \sqrt{d_K}$. Then, $\kappa_K \leq \rho_{m-1}(d)$, where*

$$\rho_{m-1}(d) := \mathrm{Res}_{s=1}\Big\{\big(\pi^{-s/2}\Gamma(s/2)\zeta(s)\big)^{m-1}\big(\frac{1}{s}+\frac{1}{s-1}\big)(d^{s-1}+d^{-s})\Big\}.$$

5.2 Proof of Theorem 2

Lemma 15 *It holds that*

$$\pi^{-s/2}\Gamma(s/2)\zeta(s) = \frac{1}{s-1} - a_0 + a_1(s-1) + O((s-1)^2)$$

with $a_0 = (\log(4\pi)-\gamma)/2 = 0.97690\cdots$ *and*

$$a_1 = -\gamma(1) + \pi^2/16 - \gamma^2/2 + (\log(4\pi)-\gamma)^2/8 = 1.00024\cdots,$$

where

$$\gamma(1) = \lim_{m\to\infty}\big(\sum_{k=1}^{m}\frac{\log k}{k}\big) - \frac{1}{2}\log^2 m = -0.072815\cdots.$$

Let us now complete the proof of Theorem 2. It is clear that

$$\mathrm{Res}_{s=1}\Big\{\big(\pi^{-s/2}\Gamma(s/2)\zeta(s)\big)^{m-1}\big(\frac{1}{s}+\frac{1}{s-1}\big)d^{s-1}\Big\} = P_{m-1}(\log d)$$

for some polynomial $P_{m-1}(x)$ of degree $m-1$. Then,

$$\begin{aligned}\mathrm{Res}_{s=1}\Big\{\big(\pi^{-s/2}\Gamma(s/2)\zeta(s)\big)^{m-1}\big(\frac{1}{s}+\frac{1}{s-1}\big)d^{-s}\Big\} &= \frac{1}{d}P_{m-1}(-\log d)\\ &= O_m\big(\frac{\log^{m-1} d}{d}\big).\end{aligned}$$

Since by [Ram1, Corollary 1] Theorem 2 holds true for $m = 2$, we may assume that $m \geq 3$. Using Lemma 15, we obtain

$$\rho_{m-1}(d) = \frac{1}{(m-1)!}\log^{m-1} d - \frac{c_m}{(m-2)!}\log^{m-2} d + O_m(\log^{m-3} d) \qquad (16)$$

with $c_m := (m-1)a_0 - 1 > 0$ for $m \geq 3$. The desired result follows.

5.3 Proof of Theorem 3

We have just proved that

$$\kappa_K \leq \rho_{m-1}(d) \leq \frac{1}{(m-1)!}\big(\log d - c_m\big)^{m-1} + O_m(\log^{m-3} d).$$

In the special case that $m = 3$, we want to prove that this error term $O_m(\log^{m-3} d)$ is less than or equal to zero. We have $c_2 = 2a_0 - 1 = \log(4\pi) - \gamma - 1 = -\lambda_1/2 =$

0.95380⋯ and, setting $c_2' = 3+2a_0^2-4a_1 = 3+2\gamma^2+4\gamma(1)-\pi^2/4 = 0.90769\cdots$, we obtain (and this result can be checked using Maple)

$$\rho_2(d) = \frac{1}{2}\big((\log d - c_2)^2 - c_2'\big) + \frac{1}{2d}\big((\log d + c_2)^2 - c_2'\big),$$

from which it follows that $\rho_2(d) \le (\log d - c_2)^2/2$ for $(d+1)c_2' \ge (\log d + c_2)^2$, hence for $d = \sqrt{d_K} \ge \sqrt{148}$ (notice that 148 is the least discriminant of a non-normal totally real cubic number field).

6 First bound for κ_K taking into account the behavior of the prime 2, when $\zeta_K(s)/\zeta(s)$ is entire

The proof of Theorem 2 stems from the bound $|a_{K/\mathbf{Q}}(n)| \le a_{m-1}(n)$, which yields (14). To improve upon Theorem 2 we will give better bounds on the $a_{K/\mathbf{Q}}(n)$'s (see Lemma 18 below) in order to obtain in (20) a better bound than (14). This will enable us to prove the following bound:

Theorem 16 *(Compare with Theorem 2). Let K range over a family of totally real number fields of a given degree $m \ge 3$ for which $\zeta_K(s)/\zeta(s)$ is entire (which holds true if $K/\mathbf{Q}$ is normal or if K is cubic). Then,*

$$\kappa_K \le \frac{1}{2^{m-g}} \frac{(\log d_K + \lambda_{m,g})^{m-1}}{2^{m-1}(m-1)!} + O_m(\log^{m-3} d_K)$$

with

$$g = \begin{cases} l & \text{if } \exists i \in \{1,\cdots,l\} \text{ such that } N_{K/\mathbf{Q}}(\mathcal{P}_i) = 2, \\ l+1 & \text{otherwise,} \end{cases}$$

where $(2) = \mathcal{P}_1^{e_1}\cdots\mathcal{P}_l^{e_l}$ in K, and $\lambda_{m,g} = 2+(m-1)(\gamma-\log\pi)-2(g-1)\log 2$.

Remark 17 *1. Notice that $1 \le g \le m$. Moreover, if $m \ge 3$ and if 2 does not split completely in K, then $1 \le g < m$ and Theorem 16 yields a better upper bound than Theorem 2. Indeed, if none of the $N_{K/\mathbf{Q}}(\mathcal{P}_i)$ is equal to 2, then $2^m = N_{K/\mathbf{Q}}(2) \ge 4^l$ implies $g \le l+1 \le m/2+1 \le m$ for $m \ge 2$, and $g < m$ for $m \ge 3$.*

2. We have $\Pi_K(2)/\Pi_{\mathbf{Q}}^m(2) \le 1/2^{m-g}$, and $1/2^{m-g} = \Pi_K(2)/\Pi_{\mathbf{Q}}^m(2)$ if and only if $(2) = \mathcal{P}_1^{e_1}\cdots\mathcal{P}_l^{e_l}$ in K with $N(\mathcal{P}_1) = \cdots = N(\mathcal{P}_l) = 2$. For example, let K be a non-normal totally real number field of prime degree $p \ge 3$ whose normal closure is a real dihedral number field of degree $2p$. Then, $\zeta_K(s)/\zeta(s)$ is entire. Assume that 2 is ramified in K. Then, $(2) = \mathcal{P}^p$ or $(2) = \mathcal{P}_1\mathcal{P}_2^2\cdots\mathcal{P}_{(p+1)/2}^2$, with $N(\mathcal{P}) = N(\mathcal{P}_1) = \cdots = N(\mathcal{P}_{(p+1)/2}) = 2$ (see [Mar, Théorème III.2]).

6.1 Bound on $S_{K/\mathbf{Q}}(x)$ taking into account the behavior of the prime 2

Lemma 18 *(Compare with* [Lou01b, Lemma 26]*). Let K be a number field of degree $m > 1$, let $p \ge 2$ be a prime, let $(p) = \mathcal{P}_1^{e_1}\cdots\mathcal{P}_l^{e_l}$ be the prime ideal factorization in K of the principal ideal (p). Set*

$$g = \begin{cases} l & \text{if } \exists i \in \{1,\cdots,l\} \text{ such that } N_{K/\mathbf{Q}}(\mathcal{P}_i) = p, \\ l+1 & \text{otherwise.} \end{cases}$$

Hence $1 \leq g \leq m$. Define the $a_{K/\mathbf{Q}}(p^k)$ and $a_n(p^k)$ by means of

$$\frac{\Pi_K(p,s)}{\Pi_{\mathbf{Q}}(p,s)} = (1-p^{-s})\prod_{i=1}^{l}(1-N_{K/\mathbf{Q}}(\mathcal{P}_i)^{-s})^{-1} = \sum_{k\geq 0} a_{K/\mathbf{Q}}(p^k)p^{-ks}$$

and

$$\Pi_{\mathbf{Q}}^{n}(p,s) = (1-p^{-s})^{-n} = \sum_{k\geq 0} a_n(p^k)p^{-ks}.$$

Then, $|a_{K/\mathbf{Q}}(p^k)| \leq a_{g-1}(p^k)$, which implies $|a_{K/\mathbf{Q}}(p^k)| \leq a_{m-1}(p^k)$ and

$$|a_{K/\mathbf{Q}}(n)| \leq a_{m-1}(n) \qquad (n \geq 1). \tag{17}$$

Finally,

$$\Pi_{\mathbf{Q}}^{-(m-1)}(p,s)\sum_{k\geq 0}\frac{a_{g-1}(p^k)}{p^{ks}} = (1-p^{-s})^{m-g} = \sum_{k=0}^{m-g}(-1)^k\binom{m-g}{k}p^{-ks}$$

is a finite Dirichlet series.

Proof. Set

$$E_k = \{(x_1,\cdots,x_l);\ \sum_{i=1}^{l} f_i x_i = k\},$$

where $N_{K/\mathbf{Q}}(\mathcal{P}_i) = p^{f_i}$,

$$F_k = \{(x_1,\cdots,x_l);\ \sum_{i=1}^{l} x_i = k\}$$

and define the $a_K(p^k)$ by means of $\Pi_K(p,s) = \sum_{k\geq 0} a_K(p^k)p^{-ks}$. Then, $\#F_k = \binom{l-1+k}{k} = a_l(p^k)$. Since $(x_1,\cdots,x_l) \in E_k \mapsto (f_1x_1,\cdots,f_lx_l) \in F_k$ is injective, we have $a_K(p^k) = \#E_k \leq \#F_k$. Moreover, $a_{K/\mathbf{Q}}(p^k) = a_K(p^k) - a_K(p^{k-1})$.

1. First, assume that there exists $i \in \{1,\cdots,l\}$ such that $N_{K/\mathbf{Q}}(\mathcal{P}_i) = p$. We may assume that $N_{K/\mathbf{Q}}(\mathcal{P}_1) = p$. Then, $g = l$, $f_1 = 1$ and

$$\begin{aligned}
a_{K/\mathbf{Q}}(p^k) &= a_K(p^k) - a_K(p^{k-1}) \\
&= \#\{(x_1,\cdots,x_l);\ x_1 + \sum_{i=2}^{l} f_i x_i = k\} \\
&\quad -\#\{(x_1,\cdots,x_l);\ x_1 + \sum_{i=2}^{l} f_i x_i = k-1\} \\
&= \sum_{j=0}^{k}\#\{(x_2,\cdots,x_l);\ \sum_{i=2}^{l} f_i x_i = k-j\} \\
&\quad -\sum_{j=0}^{k-1}\#\{(x_2,\cdots,x_l);\ \sum_{i=2}^{l} f_i x_i = k-1-j\} \\
&= \#\{(x_2,\cdots,x_l);\ \sum_{i=2}^{l} f_i x_i = k\} \\
&\leq \#\{(x_2,\cdots,x_l);\ \sum_{i=2}^{l} x_i = k\} = \binom{l-2+k}{k} = a_{l-1}(p^k) = a_{g-1}(p^k).
\end{aligned}$$

2. Otherwise, $g-1=l$, $l\leq m-1$ and using

$$0\leq a_K(p^k)=\#E_k\leq\#F_k=\binom{l-1+k}{k}$$

and

$$0\leq a_K(p^{k-1})=\#E_{k-1}\leq\#F_{k-1}=\binom{l-1+k-1}{k-1}\leq\binom{l-1+k}{k},$$

we obtain

$$|a_{K/\mathbf{Q}}(p^k)|=|a_K(p^k)-a_K(p^{k-1})|\leq\binom{l-1+k}{k}=a_l(p^k)=a_{g-1}(p^k).$$

This proves the first point of this Lemma. Finally, (17) follows from the fact that $n\mapsto a_{K/\mathbf{Q}}(n)$ and $n\mapsto a_{m-1}(n)$ are multiplicative. •

Lemma 19 *Let the notation be as in Lemma 18 and, in accordance with Lemma 18, assume that* $|a_{K/\mathbf{Q}}(2^k)|\leq b(2^k)$ *where*

$$\Pi_{\mathbf{Q}}^{-(m-1)}(2,s)\sum_{k\geq 0}\frac{b(2^k)}{2^{ks}}=(1-\frac{1}{2^s})^{m-1}\sum_{k\geq 0}\frac{b(2^k)}{2^{ks}}=\sum_{k=0}^{r}\frac{c(2^k)}{2^{ks}}$$

is a finite Dirichlet series. Then,

$$|S_{K/\mathbf{Q}}(x)|\leq\sum_{k=0}^{r}c(2^k)S_{m-1}(2^kx/d). \tag{18}$$

Proof. We have

$$\zeta_K(s)/\zeta(s)=\sum_{n\geq 1}a_{K/\mathbf{Q}}(n)n^{-s}=\frac{\Pi_K(2,s)}{\Pi_{\mathbf{Q}}(2,s)}\sum_{\substack{n'\geq 1\\ n'\ \mathrm{odd}}}a_{K/\mathbf{Q}}(n')n'^{-s}$$

and

$$\zeta^{m-1}(s)=\sum_{n\geq 1}a_{m-1}(n)n^{-s}=\Pi_{\mathbf{Q}}^{m-1}(2,s)\sum_{\substack{n'\geq 1\\ n'\ \mathrm{odd}}}a_{m-1}(n')n'^{-s}.$$

Now, define the $\tilde{a}_{m-1}(n)$ by means of

$$\begin{aligned}\Big(\sum_{k\geq 0}\frac{b(2^k)}{2^{ks}}\Big)\Pi_{\mathbf{Q}}^{-(m-1)}(2,s)\zeta^{m-1}(s) &= \Big(\sum_{k\geq 0}\frac{b(2^k)}{2^{ks}}\Big)\Big(\sum_{\substack{n'\geq 1\\ n'\ \mathrm{odd}}}a_{m-1}(n')n'^{-s}\Big)\\ &= \sum_{n\geq 1}\tilde{a}_{m-1}(n)n^{-s}.\end{aligned}$$

If $n=2^kn'$ with n' odd, then

$$|a_{K/\mathbf{Q}}(n)|=|a_{K/\mathbf{Q}}(2^k)||a_{K/\mathbf{Q}}(n')|\leq|a_{K/\mathbf{Q}}(2^k)|a_{m-1}(n')\leq b(2^k)a_{m-1}(n')$$

and

$$|a_{K/\mathbf{Q}}(n)|\leq\tilde{a}_{m-1}(n). \tag{19}$$

Hence, we obtain

$$\begin{aligned}|S_{K/\mathbf{Q}}(x)| &= \Big|\sum_{n\geq 1} a_{K/\mathbf{Q}}(n)H_{m-1}(nx/A_{K/\mathbf{Q}})\Big| \\ &\leq \sum_{n\geq 1}\tilde{a}_{m-1}(n)H_{m-1}(nx/dA_{m-1}) \text{ (use (19) and (10))} \\ &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\Gamma^{m-1}(s/2)\Big(\sum_{n\geq 1}\tilde{a}_{m-1}(n)(nx/dA_{m-1})^{-s}\Big)\mathrm{d}s \\ &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\Big(\sum_{n\geq 1}\frac{\tilde{a}_{m-1}(n)}{n^s}\Big)A_{m-1}^{s}\Gamma^{m-1}(s/2)(x/d)^{-s}\mathrm{d}s \\ &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\Pi_{\mathbf{Q}}^{-(m-1)}(2,s)\Big(\sum_{k\geq 0}\frac{b(2^k)}{2^{ks}}\Big)F_{m-1}(s)(x/d)^{-s}\mathrm{d}s \\ &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\Big(\sum_{k=0}^{r}\frac{c(2^k)}{2^{ks}}\Big)F_{m-1}(s)(x/d)^{-s}\mathrm{d}s \\ &= \sum_{k=0}^{r}c(2^k)S_{m-1}(2^kx/d) \text{ (by (12))},\end{aligned}$$

as desired. •

6.2 An upper bound on κ_K

By Lemmas 18 and 19, we obtain (compare with (14)):

$$|S_{K/\mathbf{Q}}(x)| \leq \sum_{k=0}^{m-g}(-1)^k\binom{m-g}{k}S_{m-1}(2^kx/d). \tag{20}$$

Using (11) and (20), we obtain: (compare with (15)):

$$d\kappa_K \leq \sum_{k=0}^{m-g}(-1)^k\binom{m-g}{k}\int_1^\infty S_{m-1}(2^kx/d)(1+\frac{1}{x})\mathrm{d}x. \tag{21}$$

Proposition 20 *(Compare with Proposition 14). Let K be a totally real number field of degree $m > 1$. Assume that $\zeta_K(s)/\zeta(s)$ is entire. Set $d = \sqrt{d_K}$ and let $F_{m-1}(s)$ be as in (9). Let g be as in Theorem 16. Then, $\kappa_K \leq \rho_{m-1,g}(d) - R_{m-1,g}(d)$, where*

$$\begin{aligned}&\rho_{m-1,g}(d) \\ &:= \operatorname{Res}_{s=1}\Big\{F_{m-1}(s)(\frac{1}{s}+\frac{1}{s-1})\Big((1-2^{-s})^{m-g}d^{s-1}+(1-2^{s-1})^{m-g}d^{-s}\Big)\Big\}\end{aligned}$$

and

$$R_{m-1,g}(d) := \frac{1}{d}\sum_{k=0}^{m-g}(-1)^k\binom{m-g}{k}\int_{d/2^k}^\infty S_{m-1}(x)(\frac{d}{2^kx}+1)\mathrm{d}x.$$

Proof. Use (21) and Proposition 12 with $D = d/2^k$ and $\alpha = 0$, and with $D = d/2^k$ and $\alpha = 1$. •

6.3 Proof of Theorem 16

Using Lemma 15 and Proposition 20, we obtain (compare with (16)):

$$\rho_{m-1,g}(d) = \frac{1}{2^{m-g}} \frac{\log^{m-1} d}{(m-1)!} - \frac{c_m(g)}{2^{m-g}} \frac{\log^{m-2} d}{(m-2)!} + O_m(\log^{m-3} d) \qquad (22)$$

with $c_m(g) = -1 + (m-1)a_0 - (m-g)\log 2$. Hence, Theorem 16 follows from the following bound (notice that $R_{m-1,m}(d) \geq 0$):

Proposition 21 *For $1 \leq g \leq m-1$, it holds that*

$$|R_{m-1,g}(d)| \leq (m-1)\frac{4^{m-g}\pi^{m-1}}{6^{m-1}d^2} \exp\Big(-\pi\big(\frac{d}{2^{m-g}}\big)^{2/(m-1)}\Big).$$

Proof. Noticing that

$$\int_{d/2^k}^{\infty} S_{m-1}(x)\big(\frac{d}{2^k x} + 1\big)\mathrm{d}x \leq 2\int_{d/2^{m-g}}^{\infty} S_{m-1}(x)\mathrm{d}x \qquad (0 \leq k \leq m-g)$$

and

$$\sum_{\substack{k=0 \\ k \text{ even}}}^{m-g} \binom{m-g}{k} = \sum_{\substack{k=0 \\ k \text{ odd}}}^{m-g} \binom{m-g}{k} = 2^{m-g-1},$$

we obtain

$$|R_{m-1,g}(d)| \leq \frac{2^{m-g}}{d} \int_{d/2^{m-g}}^{\infty} S_{m-1}(x)\mathrm{d}x.$$

The desired bound follows from Lemma 22 below. •

Lemma 22 *Set*

$$d_k(n) = \#\{(n_1, \cdots, n_k);\ n_i \geq 1 \text{ and } n = \prod_{i=1}^{k} n_i\}.$$

For $A > 0$ it holds that

$$\int_A^{\infty} S_k(x)\mathrm{d}x \leq \frac{k}{\pi^k A}\sum_{n\geq 1}\frac{d_k(n)}{n^2}e^{-\pi(nA)^{2/k}} \leq \frac{ke^{-\pi A^{2/k}}}{\pi^k A}\zeta^k(2) = \frac{k\pi^k}{6^k A}e^{-\pi A^{2/k}}.$$

Proof. Since $\zeta^k(s) = \sum_{n\geq 1} d_k(n)n^{-s}$, we have (by (9) and (12))

$$S_k(x) = \sum_{n\geq 1} d_k(n)\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} (\pi^{k/2}nx)^{-s}\Gamma^k(s/2)\mathrm{d}s.$$

Using

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{t^s}{s-1}\mathrm{d}s = \begin{cases} t & \text{if } t > 1 \\ 0 & \text{if } 0 < t < 1 \end{cases}$$

and Fubini's theorem we obtain

$$\begin{aligned}
&\int_A^\infty \left(\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} (\pi^{k/2}nx)^{-s}\Gamma^k(s/2)ds\right)dx \\
&= \frac{A}{2\pi i}\int_{c-i\infty}^{c+i\infty} (\pi^{k/2}nA)^{-s}\Gamma^k(s/2)\frac{ds}{s-1} \\
&= A\int\cdots\int \left(\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left(\frac{\sqrt{t_1\cdots t_k}}{\pi^{k/2}nA}\right)^s \frac{ds}{s-1}\right) e^{-(t_1+\cdots+t_k)}\frac{dt_1\cdots dt_k}{t_1\cdots t_k} \\
&= \frac{1}{\pi^{k/2}n}\int\cdots\int_{t_1\cdots t_k\geq\pi^k n^2A^2} e^{-(t_1+\cdots+t_k)}\frac{dt_1\cdots dt_k}{\sqrt{t_1\cdots t_k}} \\
&\leq \frac{1}{\pi^k n^2 A}\int\cdots\int_{t_1\cdots t_k\geq\pi^k n^2A^2} e^{-(t_1+\cdots+t_k)}dt_1\cdots dt_k \\
&\leq \frac{k}{\pi^k n^2 A}\int_{\pi(nA)^{2/k}}^\infty e^{-t}dt = \frac{k}{\pi^k n^2 A}e^{-\pi(nA)^{2/k}}
\end{aligned}$$

(if the product $t_1\cdots t_k$ is greater than or equal to $\pi^k n^2 A^2$, then at least one of the t_i's is greater than or equal to $\pi(nA)^{2/k}$). •

6.4 Proof of the first part of Theorem 9

Theorem 23 *Let K be a totally real cubic number field. Set $\lambda_2 = 2+2\gamma-2\log\pi-2\log 2 = -0.52132\cdots$ and $\lambda_4 = 2+2\gamma-2\log\pi = 0.86497\cdots$. Then,*

$$\kappa_K \leq \begin{cases} (\log d_K+\lambda_2)^2/16 & \text{if } (2)=\mathcal{P}_1\mathcal{P}_2^2,\ \mathcal{P}_1\mathcal{P}_2 \text{ or } \mathcal{P} \text{ in } K, \\ (\log d_K+\lambda_4)^2/32 & \text{if } (2)=\mathcal{P}^3 \text{ in } K. \end{cases}$$

Proof. If $(2) = \mathcal{P}_1\mathcal{P}_2^2$, $\mathcal{P}_1\mathcal{P}_2$, $\mathcal{P}$ or $\mathcal{P}^3$, then $g = 2, 2, 2$ or 1, respectively. Using Lemma 15, we obtain (and these results can be checked using Maple)

$$\rho_{2,1}(d) = \frac{1}{8}(\log d + 1 + \gamma - \log\pi)^2 - \kappa + \frac{\log^2 2}{d}$$

(with $\kappa := (3+4\log^2 2+2\gamma^2+4\gamma(1)-\pi^2/4)/8 = 0.35368\cdots$) and

$$\begin{aligned}
\rho_{2,2}(d) &= \frac{1}{4}(\log d+1+\gamma-\log(2\pi))^2 \\
&\quad -\kappa' + \frac{\log 2}{2d}(2\log d+3\log 2+2\log\pi-2-2\gamma)
\end{aligned}$$

(with $\kappa' := (3+2\log^2 2+2\gamma^2+4\gamma(1)-\pi^2/4)/4 = 0.46714\cdots$). The desired results follow from Proposition 21. •

7 Second bound for κ_K taking into account the behavior of the prime 2, when $\zeta_K(s)/\zeta(s)$ is entire

Let $(2) = \mathcal{P}_1^{e_1}\cdots\mathcal{P}_l^{e_l}$ be the prime ideal factorization in K of the principal ideal (2). We noticed in Remark 17 that the bound on κ_K obtained in Theorem 16 is of the desired type

$$\kappa_K \leq \frac{\Pi_K(2)}{\Pi_{\mathbf{Q}}^m(2)} \frac{\log^{m-1} d_K}{2^{m-1}(m-1)!} + O_m(\log^{m-2} d_K)$$

only if $N_{K/\mathbf{Q}}(\mathcal{P}_1) = \cdots = N_{K/\mathbf{Q}}(\mathcal{P}_l) = 2$. The aim of this section is to obtain in Theorem 24 below such a desired bound. However, Remark 30 below will show that from a practical point of view this better desired asymptotic upper bound for κ_K is sometimes poorer than the one obtained in Theorem 16. We set

$$\frac{\Pi_{\mathbf{Q}}(2,s)}{\Pi_K(2,s)} = \frac{1}{1-\frac{1}{2^s}} \prod_{k=1}^{l} (1 - \frac{1}{2^{f_k s}}) = \sum_{k=0}^{r} c_k 2^{-ks}, \tag{23}$$

where $N_{K/\mathbf{Q}}(\mathcal{P}_k) = 2^{f_k}$ and $r = -1 + \sum_{k=1}^{l} f_k$. We also set

$$f_K(s) = \sum_{k=0}^{r} |c_k| 2^{-ks}. \tag{24}$$

Theorem 24 *(Compare with Theorems 2 and 16). Let K range over a family of totally real number fields of a given degree $m \geq 3$ for which $\zeta_K(s)/\zeta(s)$ is entire (which holds true if $K/\mathbf{Q}$ is normal or if K is cubic). Then,*

$$\kappa_K \leq \frac{\Pi_K(2)}{\Pi_{\mathbf{Q}}^m(2)} \frac{(\log d_K + \lambda_K)^{m-1}}{2^{m-1}(m-1)!} + O_m(\log^{m-3} d_K),$$

where $\lambda_K := 2 + (m-1)(\gamma - \log \pi) + 2(g-1)\log 2 + 2\log f_K(0)$.

Remark 25 *In the case that $(2) = \mathcal{P}_1^{e_1} \cdots \mathcal{P}_l^{e_l}$ with $N_{K/\mathbf{Q}}(\mathcal{P}_1) = \cdots = N_{K/\mathbf{Q}}(\mathcal{P}_l) = 2$, we have $g = l$, $1/2^{m-g} = \Pi_K(2)/\Pi_{\mathbf{Q}}^m(2)$ (see Remark 17) and $f_K(0) = 2^{l-1} = 2^{g-1}$. Hence, $\lambda_{m,g} \leq \lambda_K$ and the bound in Theorem 16 is better than the one given in Theorem 24.*

Another way to take into account the behavior of the prime 2 is to obtain bounds for the value at $s = 1$ of the Dirichlet series

$$\tilde{\zeta}_{K/\mathbf{Q}}(s) := \frac{\Pi_{\mathbf{Q}}(2,s)}{\Pi_K(2,s)}(\zeta_K(s)/\zeta(s)) = \prod_{p \geq 3} \frac{\Pi_K(p,s)}{\Pi_{\mathbf{Q}}(p,s)} = \sum_{\substack{n \geq 1 \\ n \text{ odd}}} a_{K/\mathbf{Q}}(n) n^{-s}.$$

We set

$$\tilde{F}_{K/\mathbf{Q}}(s) := A_{K/\mathbf{Q}}^s \Gamma^{m-1}(s/2) \tilde{\zeta}_{K/\mathbf{Q}}(s)$$

and

$$\tilde{S}_{K/\mathbf{Q}}(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \tilde{F}_{K/\mathbf{Q}}(s) x^{-s} \mathrm{d}s \qquad (c > 1 \text{ and } x > 0).$$

Then,

$$\tilde{F}_{K/\mathbf{Q}}(s) = \int_0^\infty \tilde{S}_{K/\mathbf{Q}}(x) x^s \frac{\mathrm{d}x}{x},$$

for $\Re(s) > 1$. Since $\tilde{F}_{K/\mathbf{Q}}(s)$ does not satisfy any simple functional equation, neither does $\tilde{S}_{K/\mathbf{Q}}(x)$, and we cannot readily obtain a simple integral representation for $\tilde{F}_{K/\mathbf{Q}}(s)$ valid on the whole complex plane, as in (8), or even valid at the point of interest $s = 1$.

7.1 The functional equation satisfied by $\tilde{S}_{K/\mathbf{Q}}(x)$

Lemma 26 *It holds that*

$$\tilde{S}_{K/\mathbf{Q}}(x) = \sum_{k=0}^{r} c_k S_{K/\mathbf{Q}}(2^k x) \qquad (x > 0).$$

Hence, by (7), it holds that

$$\frac{1}{x}\tilde{S}_{K/\mathbf{Q}}(\frac{1}{x}) = \sum_{k=0}^{r} c_k S_{K/\mathbf{Q}}(x/2^k) \qquad (x > 0).$$

Proof. We have

$$\begin{aligned}\tilde{S}_{K/\mathbf{Q}}(x) &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \tilde{F}_{K/\mathbf{Q}}(s)x^{-s}\mathrm{d}s \qquad (c > 1 \text{ and } x > 0)\\ &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} (\sum_{k=0}^{r} c_k 2^{-ks})F_{K/\mathbf{Q}}(s)x^{-s}x^{-s}\mathrm{d}s\\ &= \sum_{k=0}^{r} c_k \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F_{K/\mathbf{Q}}(s)(2^k x)^{-s}\mathrm{d}s,\end{aligned}$$

and the desired result follows, by (6). •

7.2 Integral representation of $\tilde{F}_{K/\mathbf{Q}}(s)$

Now, by Lemma 26, for any $a > 0$ (to be suitably chosen later on (see (30) below), we have

$$\begin{aligned}\tilde{F}_{K/\mathbf{Q}}(s) &= \int_0^{\infty} \tilde{S}_{K/\mathbf{Q}}(x)x^s\frac{\mathrm{d}x}{x}\\ &= \int_a^{\infty} \tilde{S}_{K/\mathbf{Q}}(x)x^s\frac{\mathrm{d}x}{x} + \int_{1/a}^{\infty} \frac{1}{x}\tilde{S}_{K/\mathbf{Q}}(\frac{1}{x})x^{1-s}\frac{\mathrm{d}x}{x}\\ &= \int_a^{\infty} \tilde{S}_{K/\mathbf{Q}}(x)x^s\frac{\mathrm{d}x}{x} + \sum_{k=0}^{r} c_k 2^{-k}\int_{1/a}^{\infty} S_{K/\mathbf{Q}}(x/2^k)x^{1-s}\frac{\mathrm{d}x}{x},\end{aligned}$$

and this representation is now valid on the whole complex plane, by Lemma 26 and since $|S_{K/\mathbf{Q}}(x)| \leq S_{m-1}(x/d)$, by (14). In particular,

$$\tilde{F}_{K/\mathbf{Q}}(1) = \int_a^{\infty} \tilde{S}_{K/\mathbf{Q}}(x)\mathrm{d}x + \sum_{k=0}^{r} c_k 2^{-k}\int_{1/a}^{\infty} S_{K/\mathbf{Q}}(x/2^k)\frac{\mathrm{d}x}{x}. \qquad (25)$$

7.3 An upper bound on κ_K

Set

$$\tilde{\zeta}(s) = (1 - \frac{1}{2^s})\zeta(s) = \sum_{\substack{n\geq 1\\ n \text{ odd}}} n^{-s}$$

and

$$\begin{aligned}\tilde{F}_{m-1}(s) &= (\pi^{-s/2}\Gamma(s/2)\tilde{\zeta}(s))^{m-1} \\ &= (1-\frac{1}{2^s})^{m-1}F_{m-1}(s) = \sum_{k=0}^{m-1}(-1)^k\binom{m-1}{k}2^{-ks}F_{m-1}(s).\end{aligned}$$

Then,

$$\tilde{S}_{m-1}(x) := \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\tilde{F}_{m-1}(s)x^{-s}\mathrm{d}s = \sum_{k=0}^{m-1}(-1)^k\binom{m-1}{k}S_{m-1}(2^k x). \quad (26)$$

Since

$$\tilde{S}_{K/\mathbf{Q}}(x) = \sum_{\substack{n\geq 1\\ n \text{ odd}}} a_{K/\mathbf{Q}}(n)H_{m-1}(nx/A_{K/\mathbf{Q}}),$$

using (17) and (10), we obtain

$$|\tilde{S}_{K/\mathbf{Q}}(x)| \leq \sum_{\substack{n\geq 1\\ n \text{ odd}}} a_{m-1}(n)H_{m-1}(nx/dA_{m-1}) = \tilde{S}_{m-1}(x/d). \quad (27)$$

Now, (25) yields

$$\tilde{F}_{K/\mathbf{Q}}(1) \leq \int_a^\infty \tilde{S}_{m-1}(x/d)\mathrm{d}x + \sum_{k=0}^{r}\frac{|c_k|}{2^k}\int_{1/a}^\infty |S_{K/\mathbf{Q}}(x/2^k)|\frac{\mathrm{d}x}{x}, \quad (28)$$

Using (26) and (20) and noticing that $\frac{\Pi_{\mathbf{Q}}(2)}{\Pi_K(2)}d\kappa_K = \tilde{F}_{K/\mathbf{Q}}(1)$, we finally obtain (Compare with 15 and 21):

$$\begin{aligned}\frac{\Pi_{\mathbf{Q}}(2)}{\Pi_K(2)}d\kappa_K &\leq \sum_{k=0}^{m-1}(-1)^k\binom{m-1}{k}a\int_1^\infty S_{m-1}(2^k ax/d)\mathrm{d}x \\ &+\sum_{k=0}^{r}\frac{|c_k|}{2^k}\sum_{l=0}^{m-g}(-1)^l\binom{m-g}{l}\int_1^\infty S_{m-1}(x/2^{k-l}ad)\frac{\mathrm{d}x}{x}\end{aligned} \quad (29)$$

Proposition 27 *(Compare with Propositions 14 and 20). Let K be a totally real number field of degree $m > 1$. Assume that $\zeta_K(s)/\zeta(s)$ is entire. Set $d = \sqrt{d_K}$, let $F_{m-1}(s)$, $f_K(s)$ and the c_k's be as in (9), (24) and (23). Then, for any $a > 0$ it holds that*

$$\kappa_K \leq \frac{\Pi_K(2)}{\Pi_{\mathbf{Q}}(2)}(\tilde{\rho}_K(d) - \tilde{R}_K(d)) = 2^{m-1}\frac{\Pi_K(2)}{\Pi_{\mathbf{Q}}^m(2)}(\tilde{\rho}_K(d) - \tilde{R}_K(d)),$$

where $\tilde{\rho}_K(d) = \tilde{\rho}_{1,K}(d) + \tilde{\rho}_{2,K}(d)$ with

$$\tilde{\rho}_{1,K}(d) = \mathrm{Res}_{s=1}\left\{F_{m-1}(s)\Big(\frac{a^{1-s}(1-2^{-s})^{m-1}}{s-1} + \frac{a^s f_K(1-s)(1-2^{-s})^{m-g}}{s}\Big)d^{s-1}\right\},$$

$$\tilde{\rho}_{2,K}(d) = \mathrm{Res}_{s=1}\left\{F_{m-1}(s)\left(\frac{a^s(1-2^{s-1})^{m-1}}{s} + \frac{a^{1-s}f_K(s)(1-2^{s-1})^{m-g}}{s-1}\right)d^{-s}\right\}$$

and

$$\begin{aligned}\tilde{R}_K(d) &= \sum_{k=0}^{m-1}\frac{(-1)^k}{2^k}\binom{m-1}{k}\int_{d/2^k a}^{\infty}S_{m-1}(x)\frac{\mathrm{d}x}{x}\\ &+\frac{1}{d}\sum_{k=0}^{r}\frac{|c_k|}{2^k}\sum_{l=0}^{m-g}(-1)^l\binom{m-g}{l}\int_{2^{k-l}ad}^{\infty}S_{m-1}(x)\mathrm{d}x.\end{aligned}$$

Proof. Use (29) and Proposition 12 with $D = d/2^k a$ and $\alpha = 0$, and with $D = 2^{k-l}ad$ and $\alpha = 1$. •

Proposition 28 *(Compare with Proposition 21). Set $a^* = \min(1/a, 2a)$. Assume that $2 \leq g \leq m$ (notice that if $g = 1$ then $(2) = \mathcal{P}^m$ in K and Theorem 16 already provides us with the desired bound for κ_K). Then,*

$$\tilde{R}_K(d) \geq -\left(2^{m-2}a + \frac{(3^{m-2}-1)f_K(1)}{2^{m-1}}\right)\frac{(m-1)\pi^{m-1}}{3^{m-1}a^*d^2}\exp\left(-\pi\left(\frac{a^*d}{2^{m-1}}\right)^{2/(m-1)}\right).$$

Proof. We have

$$\begin{aligned}\tilde{R}_K(d) &\geq \sum_{\substack{k=0\\ k\ odd}}\frac{(-1)^k}{2^k}\binom{m-1}{k}\frac{2^k a}{d}\int_{d/2^{m-1}a}^{\infty}S_{m-1}(x)\mathrm{d}x\\ &+\frac{1}{d}\sum_{k=0}^{r}\frac{|c_k|}{2^k}\sum_{\substack{l=0\\ l\ odd}}^{m-2}(-1)^l\binom{m-2}{l}\int_{ad/2^{m-2}}^{\infty}S_{m-1}(x)\mathrm{d}x \text{ (since } g\geq 2)\\ &= -\frac{2^{m-2}a}{d}\int_{d/2^{m-1}a}^{\infty}S_{m-1}(x)\mathrm{d}x - \frac{(3^{m-2}-1)f_K(1)}{2^{m-1}d}\int_{ad/2^{m-2}}^{\infty}S_{m-1}(x)\mathrm{d}x\\ &\geq -\left(\frac{2^{m-2}a}{d}+\frac{(3^{m-2}-1)f_K(1)}{2^{m-1}d}\right)\int_{a^*d/2^{m-1}}^{\infty}S_{m-1}(x)\mathrm{d}x.\end{aligned}$$

using Lemma 22, we obtain the desired bound. •

7.4 Proof of Theorem 24

To begin with, we notice that for a given degree $m > 1$, the $f_K(s)$ run over a finite family of functions. Hence, using Lemma 15, we obtain (compare with (16) and (22)):

$$\begin{aligned}\tilde{\rho}_K(d) &= \tilde{\rho}_{1,K}(d) + O\left(\frac{\log^{m-1}d}{d}\right)\\ &= \frac{\log^{m-1}d}{2^{m-1}(m-1)!} + c_m(a)\frac{\log^{m-2}d}{2^{m-1}(m-2)!} + O_m(\log^{m-3}d)\\ &= \frac{(\log d + c_m(a))^{m-1}}{2^{m-1}(m-1)!} + O_m(\log^{m-3}d),\end{aligned}$$

where $c_m(a) = (m-1)(\gamma - \log\pi)/2 - \log a + 2^{g-1}af_K(0)$. To have $c_m(a)$ as small as possible, we choose

$$a = 1/(2^{g-1}f_K(0)). \tag{30}$$

We obtain $c_m(a) = 1 + (m-1)(\gamma - \log\pi)/2 + (g-1)\log 2 + \log(f_K(0))$. Using Propositions 27 and 28, the proof of Theorem 24 is complete.

7.5 Proof of the second part of Theorem 9

We now prove:

Theorem 29 *Let K be a totally real cubic number field. Set $\lambda_3 = 2+2\gamma-2\log\pi+4\log 2 = 3.63756\cdots$ and $\lambda_5 = 2+2\gamma-2\log\pi+2\log 6 = 4.44849\cdots$. Then,*

$$\kappa_K \leq \begin{cases} (\log d_K+\lambda_3)^2/24 & \text{if } (2)=\mathcal{P}_1\mathcal{P}_2 \text{ in } K,\\ (\log d_K+\lambda_5)^2/56 & \text{if } (2)=\mathcal{P} \text{ in } K.\end{cases}$$

Proof. In the present situation, we have $m=3$, $g=2$, $r=2$ and $c_3(a) = 1+\gamma-\log\pi+\log 2+\log(f_K(0))$.
1. If $(2)=\mathcal{P}_1\mathcal{P}_2$ in K, then $\Pi_{\mathbf{Q}}(2,s)/\Pi_K(2,s) = 1-2^{-2s}$ and $f_K(s)=1+2^{-2s}$. Hence, $f_K(0)=2$, $a=1/4$ and

$$\begin{aligned}\tilde{\rho}_K(d) &= \frac{1}{8}(\log d+1+\gamma-\log\pi+2\log 2)^2\\ &\quad -\kappa+\frac{\log 2}{8d}(5\log d+10\log\pi-\log 2-10\gamma),\end{aligned}$$

with $\kappa := (2\gamma^2+4\gamma(1)+3+4\log^2 2-\pi^2/4+8\log 2)/8 = 0.80660\cdots$ (this result can easily be checked using Maple). Since $\tilde{R}_K(d)\geq -\frac{\pi^2}{2d^2}e^{-\pi d/8}$, by Proposition 28, we obtain $\tilde{\rho}_K(d)-\tilde{R}_K(d)\leq (\log d+1+\gamma-\log\pi+2\log 2)^2/8$ for $d>2$.
2. If $(2)=\mathcal{P}$ in K, then $\Pi_{\mathbf{Q}}(2,s)/\Pi_K(2,s) = 1+2^{-s}+2^{-2s} = f_K(s)$. Hence, $f_K(0)=3$, $a=1/6$ and

$$\begin{aligned}\tilde{\rho}_K(d) &= \frac{1}{8}(\log d+1+\gamma-\log\pi+\log 6)^2\\ &\quad -\kappa'+\frac{7\log 2}{4d}(\log d-\log\pi+\gamma-\frac{15}{14}\log 2+\log 3),\end{aligned}$$

with $\kappa' := (2\gamma^2+4\gamma(1)+3+4\log^2 2-\pi^2/4+4\log 6)/8 = 1.00934\cdots$ (this result can easily be checked using Maple). Since $\tilde{R}_K(d)\geq -\frac{29\pi^2}{32d^2}e^{-\pi d/12}$, by Proposition 28, $\tilde{\rho}_K(d)-\tilde{R}_K(d)\leq(\log d+1+\gamma-\log\pi+\log 6)^2/8$ for $d>2.5$. The proof is complete. •

Remark 30 *These bounds are better than the first one given in Theorem 23 for $d_K\geq\exp((\lambda_3-\sqrt{3/2}\lambda_2)/(\sqrt{3/2}-1))$, hence for $d_K\geq 2\cdot 10^8$, if $(2)=\mathcal{P}_1\mathcal{P}_2$, and for $d_K\geq\exp((\lambda_5-\sqrt{7/2}\lambda_2)/(\sqrt{7/2}-1))$, hence for $d_K\geq 507$, if $(2)=\mathcal{P}$.*

8 The case of Dirichlet L-functions

8.1 A bound on $|L(1,\chi)|$

Let $L(s,\chi)=\sum_{n\geq 1}\chi(n)n^{-s}$ be the Dirichlet series associated with a primitive even Dirichlet character χ of conductor $f>1$. It is known that

$$\Lambda(s,\chi) := A_\chi^s\Gamma(s/2)L(s,\chi),$$

with $A_\chi := \sqrt{f/\pi}$, is entire and satisfies the functional equation $\Lambda(s,\chi) = W_\chi \Lambda(1-s,\bar{\chi})$ for some complex number W_χ of absolute value equal to one (see [Dav, Chapter 9]). It follows that

$$S(x,\chi) := \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \Lambda(s,\chi)x^{-s}\mathrm{d}s = \sum_{n\geq 1}\chi(n)H_1(nx/A_\chi)$$

satisfies the functional equation $S(x,\chi) = \frac{W_\chi}{x}S(\frac{1}{x},\bar{\chi})$ and that

$$\Lambda(s,\chi) = \int_1^\infty S(x,\chi)x^{-s}\mathrm{d}x + W_\chi\int_1^\infty S(x,\bar{\chi})x^{s-1}\mathrm{d}x.$$

Since $|\chi(n)| \leq 1$, we obtain $|S(x,\chi)| = |S(x,\bar{\chi})| \leq S_1(x/d)$ and (to be compared with (15))

$$d|L(1,\chi)| \leq \int_1^\infty S_1(x/d)(1+\frac{1}{x})\mathrm{d}x,$$

where

$$d := A_\chi/A_1 = \sqrt{f}.$$

By Proposition 12 with $D=d$ and $\alpha=0$, and $D=d$ and $\alpha=1$, we obtain

$$\begin{aligned}|L(1,\chi)| &\leq \operatorname{Res}_{s=1}\left\{F_1(s)(\frac{1}{s}+\frac{1}{s-1})(d^{s-1}+d^{-s})\right\}\\ &= \frac{1}{2}(2\log d-\lambda) - \frac{1}{2d}(2\log d-\lambda) \leq \frac{1}{2}(2\log d-\lambda) = \frac{1}{2}(\log f-\lambda),\end{aligned}$$

where $\lambda := 2+\gamma-\log(4\pi) = 0.04619\cdots$ (for $d^2 = f > 3 > e^\lambda = 1.04727\cdots$). This is precisely the bound obtained in [Lou04a, Theorem 1] and [Lou04b, Theorem 1] for $S=\emptyset$.

8.2 First bound on $|L(1,\chi)|$ taking into account the behavior of the prime 2

Now, let us try to obtain an upper bound for $|(1-\frac{\chi(2)}{2})L(1,\chi)|$. Setting

$$\tilde{L}(s,\chi) := (1-\frac{\chi(2)}{2^s})L(s,\chi) = \sum_{\substack{n\geq 1\\ n \text{ odd}}}\chi(n)n^{-s},$$

$\tilde{\Lambda}(s,\chi) := A_\chi^s\Gamma(s/2)\tilde{L}(s,\chi)$ and

$$\tilde{S}(x,\chi) := \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\tilde{\Lambda}(s,\chi)x^{-s}\mathrm{d}s = \sum_{\substack{n\geq 1\\ n \text{ odd}}}\chi(n)H_1(nx/A_\chi),$$

for $\Re(s) > 1$ and for any $a>0$ to be suitably chosen below, we have

$$\tilde{\Lambda}(s,\chi) = \int_0^\infty \tilde{S}(x,\chi)x^s\frac{\mathrm{d}x}{x} = \int_a^\infty \tilde{S}(x,\chi)x^s\frac{\mathrm{d}x}{x} + \int_{1/a}^\infty \frac{1}{x}\tilde{S}(\frac{1}{x},\chi)x^{1-s}\frac{\mathrm{d}x}{x}.$$

Now, $\tilde{\Lambda}(s,\chi) = \Lambda(s,\chi) - \frac{\chi(2)}{2^s}\Lambda(s,\chi)$ yields $\tilde{S}(x,\chi) = S(x,\chi) - \chi(2)S(2x,\chi)$ and

$$\frac{1}{x}\tilde{S}(\frac{1}{x},\chi) = \frac{1}{x}S(\frac{1}{x},\chi) - \frac{\chi(2)}{2}\frac{2}{x}S(\frac{2}{x},\chi) = W_\chi(S(x,\bar{\chi}) - \frac{\chi(2)}{2}S(x/2,\bar{\chi})).$$

Hence, for any complex s we have

$$\tilde{\Lambda}(s,\chi)=\int_a^\infty \tilde{S}(x,\chi)x^s\frac{dx}{x}+W_\chi\int_{1/a}^\infty (S(x,\bar{\chi})-\frac{\chi(2)}{2}S(x/2,\bar{\chi}))x^{1-s}\frac{dx}{x}.$$

In particular, we obtain (to be compared with (25))

$$\tilde{\Lambda}(1,\chi)=\int_a^\infty \tilde{S}(x,\chi)dx+W_\chi\int_{1/a}^\infty (S(x,\bar{\chi})-\frac{\chi(2)}{2}S(x/2,\bar{\chi}))\frac{dx}{x}. \qquad (31)$$

Since $|S(x,\bar{\chi})|\leq S_1(x/d)$ and

$$|\tilde{S}(x,\chi)|=|\tilde{S}(x,\bar{\chi})|\leq \sum_{\substack{n\geq 1\\ n\ odd}} H_1(nx/A_\chi)=\tilde{S}_1(x/d)=S_1(x/d)-S_1(2x/d),$$

by (26), using Proposition 12, we obtain (to be compared with Proposition 27):

$$\begin{aligned}
|(1-\frac{\chi(2)}{2})L(1,\chi)| &= \frac{1}{d}|\tilde{\Lambda}(1,\chi)|\\
&\leq \frac{a}{d}\int_1^\infty S_1(ax/d)dx-\frac{a}{d}\int_1^\infty S_1(2ax/d)dx\\
&\quad +\frac{1}{d}\int_1^\infty S_1(x/ad)\frac{dx}{x}+\frac{1}{2d}\int_1^\infty S_1(x/2ad)\frac{dx}{x}\\
&= \mathrm{Res}_{s=1}\Big\{F_1(s)\Big(\frac{a^{1-s}(1-2^{-s})}{s-1}+\frac{a^s(1+2^{s-1})}{s}\Big)d^{s-1}\Big\}\\
&\quad +\mathrm{Res}_{s=1}\Big\{F_1(s)\Big(\frac{a^s(1-2^{s-1})}{s}+\frac{a^{1-s}(1+2^{-s})}{s-1}\Big)d^{-s}\Big\}+R_a(d)\\
&= \frac{1}{4}(2\log d+\gamma-\log\pi+8a-2\log a)\\
&\quad -\frac{1}{4d}(6\log d+3\log\pi-3\gamma-4\log 2)+R_a(d)
\end{aligned}$$

where

$$R_a(d)=-\int_{d/a}^\infty S_1(x)\frac{dx}{x}+\frac{1}{2}\int_{d/2a}^\infty S_1(x)\frac{dx}{x}-\frac{1}{d}\int_{ad}^\infty S_1(x)dx-\frac{1}{2d}\int_{2ad}^\infty S_1(x)dx.$$

Choosing $a=1/4$, we have

$$R_{1/4}(a)\leq \frac{1}{4d}\int_{2d}^\infty S_1(x)dx-\frac{1}{d}\int_{d/4}^\infty S_1(x)dx\leq 0$$

and

$$|(1-\frac{\chi(2)}{2})L(1,\chi)|\leq\frac{1}{4}(\log f+2+\gamma-\log\pi+4\log 2),$$

for $f\geq 2$. However, this bound is not as good as the one (33) below obtained in [Lou04a, Theorem 1] and [Lou04b, Theorem 1] for $S=\{2\}$.

8.3 Second bound on $|L(1,\chi)|$ taking into account the behavior of the prime 2

However, we can obtain a better result. Assume that the conductor f of χ is odd. Then $\chi(2) \neq 0$ and

$$\begin{aligned} S(x,\bar\chi) - \frac{\chi(2)}{2}S(x/2,\bar\chi) &= \sum_{n\geq 1}\bar\chi(n)H_1(nx/A_\chi) - \frac{\chi(2)}{2}\sum_{n\geq 1}\bar\chi(n)H_1(nx/2A_\chi) \\ &= \frac{1}{2}\sum_{n\geq 1}\bar\chi(n)H_1(nx/A_\chi) - \frac{\chi(2)}{2}\sum_{\substack{n\geq 1\\ n \text{ odd}}}\bar\chi(n)H_1(nx/2A_\chi) \end{aligned}$$

yields (by (26))

$$|S(x,\bar\chi) - \frac{\chi(2)}{2}S(x/2,\bar\chi)| \leq (S_1(x/d) + \tilde S_1(x/2d))/2 = \frac{1}{2}S_1(x/2d), \qquad (32)$$

instead of the trivial bound $|S(x,\bar\chi) - \frac{\chi(2)}{2}S(x/2,\bar\chi)| \leq S_1(x/d) + \frac{1}{2}S_1(x/2d)$ we have previously used. Plugging this bound in (31) and using Proposition 12, we end up with the bound

$$\begin{aligned} &|(1-\frac{\chi(2)}{2})L(1,\chi)| = \frac{1}{d}|\tilde\Lambda(1,\chi)| \\ \leq\ & \frac{a}{d}\int_1^\infty S_1(ax/d)\mathrm{d}x - \frac{a}{d}\int_1^\infty S_1(2ax/d)\mathrm{d}x + \frac{1}{2d}\int_1^\infty S_1(x/2ad)\frac{\mathrm{d}x}{x} \\ =\ & \mathrm{Res}_{s=1}\Big\{F_1(s)\Big(\frac{a^{1-s}(1-2^{-s})}{s-1} + \frac{a^s 2^{s-1}}{s}\Big)d^{s-1}\Big\} \\ & +\mathrm{Res}_{s=1}\Big\{F_1(s)\Big(\frac{a^s(1-2^{s-1})}{s} + \frac{a^{1-s}2^{-s}}{s-1}\Big)d^{-s}\Big\} + R_a(d) \\ =\ & \frac{1}{4}(2\log d + \gamma - \log\pi + 4a - 2\log a) \\ & -\frac{1}{4d}(2\log d - \gamma + \log\pi + 4\log 2 + 2\log a) + R_a(d) \end{aligned}$$

where

$$R_a(d) = -\int_{d/a}^\infty S_1(x)\frac{\mathrm{d}x}{x} + \frac{1}{2}\int_{d/2a}^\infty S_1(x)\frac{\mathrm{d}x}{x} - \frac{1}{2d}\int_{2ad}^\infty S_1(x)\mathrm{d}x.$$

Choosing $a = 1/2$, we have

$$R_{1/2}(d) \leq \frac{1}{2}\int_d^\infty S_1(x)\frac{\mathrm{d}x}{x} - \frac{1}{2d}\int_d^\infty S_1(x)\mathrm{d}x \leq 0$$

and

$$|(1-\frac{\chi(2)}{2})L(1,\chi)| \leq \frac{1}{4}(\log f + 2 + \gamma - \log\pi + 2\log 2), \qquad (33)$$

which is the bound obtained in [Lou04a, Theorem 1] and [Lou04b, Theorem 1] for $S = \{2\}$. Notice that we recover this previously known upper bound, but without making use of the technical Lemmas [Lou04a, Lemma 3] or [Lou04b, Lemma 2]. We simply apply the machinery developed in sections 5 and 7.

8.4 An improvement on Theorem 29

Assume that K is a totally real cubic number field in which $(2)=\mathcal{P}_1\mathcal{P}_2$. Then, $\Pi_{\mathbf{Q}}(2,s)/\Pi_K(2,s)=1-2^{-2s}$, $c_0=1$, $c_1=0$ and $c_2=-1$. Noticing that $a_{K/\mathbf{Q}}(2^k)=0$ or 1 according as k is odd or even, we obtain $a_{K/\mathbf{Q}}(4n)=a_{K/\mathbf{Q}}(n)$, $a_{K/\mathbf{Q}}(n)=0$ if $n\equiv 2 \pmod 4$ and

$$\begin{aligned}
\sum_{k=0}^{r} c_k 2^{-k} S_{K/\mathbf{Q}}(x/2^k) &= S_{K/\mathbf{Q}}(x)-\frac{1}{4}S_{K/\mathbf{Q}}(x/4)\\
&= \sum_{n\geq 1} a_{K/\mathbf{Q}}(n)H_{m-1}(nx/A_{K/\mathbf{Q}})-\frac{1}{4}\sum_{n\geq 1} a_{K/\mathbf{Q}}(n)H_{m-1}(nx/4A_{K/\mathbf{Q}})\\
&= \frac{3}{4}\sum_{n\geq 1} a_{K/\mathbf{Q}}(n)H_{m-1}(nx/A_{K/\mathbf{Q}})-\frac{1}{4}\sum_{\substack{n\geq 1\\ n\not\equiv 0 \pmod 4}} a_{K/\mathbf{Q}}(n)H_{m-1}(nx/4A_{K/\mathbf{Q}})\\
&= \frac{3}{4}\sum_{n\geq 1} a_{K/\mathbf{Q}}(n)H_{m-1}(nx/A_{K/\mathbf{Q}})-\frac{1}{4}\sum_{\substack{n\geq 1\\ n \text{ odd}}} a_{K/\mathbf{Q}}(n)H_{m-1}(nx/4A_{K/\mathbf{Q}})\\
&= \frac{3}{4}S_{K/\mathbf{Q}}(x)-\frac{1}{4}\tilde{S}_{K/\mathbf{Q}}(x/4).
\end{aligned}$$

Hence, instead of simply using (25) and the trivial bound

$$\Big|\sum_{k=0}^{r} c_k 2^{-k} S_{K/\mathbf{Q}}(x/2^k)\Big| \leq \sum_{k=0}^{r}\frac{|c_k|}{2^k}|S_{K/\mathbf{Q}}(x/2^k)| \leq |S_{K/\mathbf{Q}}(x)|+\frac{1}{4}|S_{K/\mathbf{Q}}(x/4)|$$

to obtain (28), we now use the better bound

$$\begin{aligned}
\Big|\sum_{k=0}^{r} c_k 2^{-k} S_{K/\mathbf{Q}}(x/2^k)\Big| &\leq \frac{3}{4}|S_{K/\mathbf{Q}}(x)|+\frac{1}{4}|\tilde{S}_{K/\mathbf{Q}}(x/4)|\\
&\leq \frac{3}{4}S_2(x/d)-\frac{3}{4}S_2(2x/d)+\frac{1}{4}\tilde{S}_2(x/4d)\\
&= S_2(x/d)-\frac{3}{4}S_2(2x/d)+\frac{1}{4}S_2(x/4d)-\frac{1}{2}S_2(x/2d)
\end{aligned}$$

(by (20), and since in our situation we have $m=3$ and $g=l=2$, by (27), and by (26)). Hence, instead of (29) we obtain

$$\begin{aligned}
\frac{\Pi_{\mathbf{Q}}(2)}{\Pi_K(2)}d\kappa_K &\leq a\int_1^{\infty}(S_2(ax/d)-2S_2(2ax/d)+S_2(4ax/d))\mathrm{d}x\\
&\quad +\int_1^{\infty}(S_2(x/ad)-\frac{3}{4}S_2(2x/ad)+\frac{1}{4}S_2(x/4ad)-\frac{1}{2}S_2(x/2ad))\frac{\mathrm{d}x}{x},
\end{aligned}$$

which in using $\Pi_{\mathbf{Q}}(2)/\Pi_K(2)=3/4$ and Proposition 12 yields (compare with Proposition 27)

$$\frac{3}{4}\kappa_K \leq \rho_1(d)+\rho_2(d)-R(d)$$

with

$$\begin{aligned}
\rho_1(d) &= \mathrm{Res}_{s=1}\left\{F_2(s)\Big(\frac{a^{1-s}(1-2^{-s})^2}{s-1}+\frac{a^s(1-\frac{3}{4}2^{-s}-\frac{1}{2}2^s+\frac{1}{4}2^{2s}}{s}\Big)d^{s-1}\right\}\\
&= \frac{1}{8}\log^2 d+\frac{1}{4}(\gamma-\log\pi+5a/2-\log a)\log d+\kappa,
\end{aligned}$$

where $\kappa = -0.54977\cdots$,

$$\begin{aligned}\rho_2(d) &= \mathrm{Res}_{s=1}\left\{F_2(s)\Big(\frac{a^s(1-2^{s-1})^2}{s} + \frac{a^{1-s}(1-\frac{3}{4}2^{s-1}-\frac{1}{2}2^{1-s}+\frac{1}{4}2^{2(1-s)})}{s-1}\Big)d^{-s}\right\}\\ &= \frac{3\log 2}{4d}\big(\log d - \gamma + \log\pi + \frac{11}{6}\log 2 + \log a\big),\end{aligned}$$

and

$$\begin{aligned}R(d) &= \int_{d/a}^{\infty} S_2(x)\frac{\mathrm{d}x}{x} - \int_{d/2a}^{\infty} S_2(x)\frac{\mathrm{d}x}{x} + \frac{1}{4}\int_{d/4a}^{\infty} S_2(x)\frac{\mathrm{d}x}{x}\\ &\quad +\frac{1}{d}\int_{ad}^{\infty} S_2(x)\mathrm{d}x - \frac{3}{4d}\int_{ad/2}^{\infty} S_2(x)\mathrm{d}x + \frac{1}{4d}\int_{4ad}^{\infty} S_2(x)\mathrm{d}x - \frac{1}{2d}\int_{2ad}^{\infty} S_2(x)\mathrm{d}x\\ &\geq -\frac{2a+3/4+1/2}{d}\int_{da^*}^{\infty} S_2(x)dx \geq -(2a+3/4+1/2)\frac{\pi^2}{18a^*d^2}e^{-\pi a^* d},\end{aligned}$$

by Lemma 22, where $a^* := \min(1/2a, a/2, 2a)$. Now, we choose $a = 2/5$. We obtain

$$\kappa_K \leq \frac{1}{6}(\log d + 1 + \gamma - \log\pi + \log(5/2))^2 + O(1),$$

and

Theorem 31 *Let K be a totally real cubic number field. Assume that $(2) = \mathcal{P}_1\mathcal{P}_2$ in K. Set $\lambda_3 = 2 + 2\gamma - 2\log\pi + 2\log(5/2) = 2.69755\cdots$. Then,*

$$\kappa_K \leq (\log d_K + \lambda_3)^2/24.$$

This bound is better than the first one given in Theorem 23 for $d_K \geq \exp((\lambda_3 - \sqrt{3/2}\lambda_2)/(\sqrt{3/2}-1))$, hence for $d_K \geq 3\cdot 10^7$.

In conclusion, using integral representations of Dedekind zeta functions of number fields is a powerful tool for obtaining good upper bounds on the residues at $s = 1$ of these Dedekind zeta functions.

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