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Functional relations for various multiple zeta-functions

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1. Introduction

The Euler-Zagier multiple zeta-function of depth r is defined by

$$\zeta_{EZ,r}(s_1,\ldots,s_r)=\sum_{m_1,\ldots,m_r=1}^{\infty}\frac{1}{m_1^{s_1}(m_1+m_2)^{s_2}\cdots(m_1+\cdots+m_r)^{s_r}}. \tag{1.1}$$

Originally, Euler studied the values of double zeta-function at positive integers, and gave the relation formulas among them such as

$$\zeta_{EZ,2}(1,2)=\zeta(3), \tag{1.2}$$

$$\sum_{j=2}^{k-1}\zeta_{EZ,2}(k-j,j)=\zeta(k) \tag{1.3}$$

for $k\in\mathbb{N}$ with $k\geq 3$, which are called the *sum formulas for double zeta-values* (see [7]).

In early 1990's, Zagier ([29]) and Hoffman ([10]) studied the values of $\zeta_{EZ,r}$ at positive integers independently, which are called the "multiple zeta-values" (MZVs) or the "Euler-Zagier sums". Following their works, many relation formulas for MZVs have been discovered by a lot of authors. Furthermore a recent aim of the study about MZVs is to investigate the structure of $\mathbb{Q}$-algebra generated by MZVs (see details, [4]).

On the other hand, in late 1990's, $\zeta_{EZ,r}(s_1,\ldots,s_r)$ has been continued meromorphically to the whole complex space $\mathbb{C}^r$ by, for example Essouabri ([5, 6]), Akiyama-Egami-Tanigawa ([1]), Arakawa-Kaneko ([2]), Zhao ([30]) and the first-named author ([13, 14, 15]). The first-named author made use of the Mellin-Barnes integral formula. This method was inspired by Katsurada's work about the mean square of Dirichlet L-series and Hurwitz-Lerch zeta functions.

Based on these researches, we would like to think the following problem presented by the first-named author a few years ago:

Problem. *Are the known relation formulas for multiple zeta-values valid only at positive integers, or valid also at other values?*

As an answer to this problem, we can give the following "Harmonic product relation" by

$$\zeta(s_1)\zeta(s_2) = \zeta_{EZ,2}(s_1, s_2) + \zeta_{EZ,2}(s_2, s_1) + \zeta(s_1 + s_2), \tag{1.4}$$

which can be given by the well-known division of summation as

$$\sum_{m,n\geq 1} = \sum_{1\leq m<n} + \sum_{m>n\geq 1} + \sum_{1\leq m=n} .$$

We see that (1.4) holds for all $(s_1, s_2) \in \mathbb{C}^2$ except for the singularities of each function on both sides of (1.4). In particular when $(s_1, s_2) = (2, 2)$, we have the relation formula

$$\zeta_{EZ,2}(2,2) = \frac{1}{2}\left\{\zeta(2)^2 - \zeta(4)\right\}\left(= \frac{1}{120}\pi^4\right).$$

Hence we can say that (1.4) is an answer to the above problem, though it can be obtained trivially. So we would like to give non-trivial answers. More specifically we consider the following natural question:

Question. *Is there any functional relation which gives non-trivial Euler's formula $\zeta_{EZ,2}(1,2) = \zeta(3)$?*

Note that, for example, we can numerically check that

$$\zeta_{EZ,2}(s_1, s_2) \neq \zeta(s_1 + s_2) \quad (s_1, s_2 \in \mathbb{C})$$

as a relation for complex functions.

The main aim of this note is to give some non-trivial answers to the above Problem. Furthermore we introduce certain functional relations among Witten zeta-functions associated with semisimple Lie algebras (see [18]). Considering their special values, we can give new relation formulas among their values at positive integers, which can be regarded as analogues of Witten's results. Finally we give certain functional relations among the double L-series (see [26]).

2. Euler-Zagier and Mordell-Tornheim multiple zeta-functions

In order to answer the problem in Section 1, we need to consider the Mordell-Tornheim multiple zeta-functions defined by

$$\zeta_{MT,r}(s_1,\ldots,s_r,s_{r+1})=\sum_{m_1,\ldots,m_r=1}^{\infty}\frac{1}{m_1^{s_1}\cdots m_r^{s_r}(m_1+\cdots+m_r)^{s_{r+1}}} \tag{2.1}$$

(see [16]). Indeed, the first-named author proved that $\zeta_{MT,r}(s_1,\ldots,s_{r+1})$ can be continued meromorphically to $\mathbb{C}^{r+1}$ (see [16]).

In 1950's, Tornheim and Mordell independently studied the values of

$$\zeta_{MT,2}(s_1,s_2,s_3)=\sum_{m_1,m_2=1}^{\infty}\frac{1}{m_1^{s_1}m_2^{s_2}(m_1+m_2)^{s_3}}$$

at positive integers and gave some relation formulas (see [20, 22]). Concretely Tornheim showed that $\zeta_{MT,2}(p,q,r)$ can be expressed as a polynomial on $\{\zeta(j+1)\mid j\in\mathbb{N}\}$ with $\mathbb{Q}$-coefficients when p,q,r are nonnegative integers with $p+q+r\geq 3$ and $p+q+r$ is odd. For example,

$$\zeta_{MT,2}(2,2,3)=6\zeta(2)\zeta(5)-10\zeta(7). \tag{2.2}$$

Mordell showed that $\zeta_{MT,2}(2k,2k,2k)\in\mathbb{Q}\cdot\pi^{6k}$ for any $k\in\mathbb{N}$. For example,

$$\zeta_{MT,2}(2,2,2)=\frac{4}{3}\zeta(2)\zeta(4)-2\zeta(6). \tag{2.3}$$

Note that

$$\zeta_{MT,2}(s_1,0,s_3)=\zeta_{MT,2}(0,s_1,s_3)=\zeta_{EZ,2}(s_1,s_3)$$

Now we give a certain answer to the question in Section 1 as follows:

Proposition 2.1.

$$\zeta_{EZ,2}(1,s+1)-\zeta_{MT,2}(s,1,1)+\zeta(s+2)=0 \tag{2.4}$$

holds for all $s\in\mathbb{C}$ except for singularities of three functions on the left-hand side.

We can prove Proposition 2.1 by a kind of double analogue of Hardy's method of proving the functional equation for $\zeta(s)$ (see [9]), as mentioned later.

Let $s=1$ in (2.4). By using the well-known relation

$$\frac{1}{mn}=\frac{1}{m+n}\left(\frac{1}{m}+\frac{1}{n}\right), \tag{2.5}$$

we have $\zeta_{MT,2}(1,1,1)=2\zeta_{EZ,2}(1,2)$. Hence (2.4) in the case $s=1$ gives Euler's formula $\zeta_{EZ,2}(1,2)=\zeta(3)$.

Furthermore, Proposition 2.1 in the case $s = k-2$ ($k \geq 3$) gives the sum formula for double zeta values (1.3) proved by Euler:

$$\sum_{j=2}^{k-1} \zeta_{EZ,2}(k-j,j) = \zeta(k).$$

Indeed, considering partial fraction (2.5), we inductively see that

$$\begin{aligned}\zeta_{MT,2}(k-2,1,1) &= \zeta_{MT,2}(k-3,1,2) + \zeta_{EZ,2}(k-2,2) \\ &= \zeta_{MT,2}(0,1,k-1) + \sum_{j=2}^{k-1} \zeta_{EZ,2}(k-j,j).\end{aligned}$$

On the other hand, it follows from Proposition 2.1 that

$$\zeta_{MT,2}(k-2,1,1) = \zeta_{EZ,2}(1,k-1) + \zeta(k).$$

Hence we obtain (1.3).

More generally we can obtain the following results (see [25]).

Proposition 2.2. *For $k, l \in \mathbb{N} \cup \{0\}$,*

$$\begin{aligned}&\zeta_{MT,2}(k,l,s) + (-1)^k \zeta_{MT,2}(s,k,l) + (-1)^l \zeta_{MT,2}(s,l,k) \qquad (2.6)\\ &= 2 \sum_{\substack{j=0 \\ j \equiv k\ (2)}}^{k} \left(2^{1-k+j}-1\right) \zeta(k-j) \\ &\quad \times \sum_{\mu=0}^{[j/2]} \frac{(i\pi)^{2\mu}}{(2\mu)!} \binom{l-1+j-2\mu}{j-2\mu} \zeta(l+j+s-2\mu) \\ &- 4 \sum_{\substack{j=0 \\ j \equiv k\ (2)}}^{k} \left(2^{1-k+j}-1\right) \zeta(k-j) \sum_{\mu=0}^{[(j-1)/2]} \frac{(i\pi)^{2\mu}}{(2\mu+1)!} \\ &\quad \times \sum_{\substack{\nu=0 \\ \nu \equiv l\ (2)}}^{l} \zeta(l-\nu) \binom{\nu-1+j-2\mu}{j-2\mu-1} \zeta(\nu+j+s-2\mu)\end{aligned}$$

holds for all $s \in \mathbb{C}$ except for singularities of functions on both sides of (2.6).

Remark. We can immediately see that (2.6) contains Mordell's result (mentioned above)

$$\zeta_{MT,2}(2k,2k,2k) \in \mathbb{Q} \cdot \pi^{6k}.$$

On the other hand, for example, (2.6) gives

$$\begin{aligned}&\zeta_{MT,2}(3,s,2) - \zeta_{MT,2}(3,2,s) - \zeta_{MT,2}(2,s,3) \\ &= 10\zeta(s+5) - 6\zeta(2)\zeta(s+3).\end{aligned}$$

In particular when $s=2$, we have Tornheim's (2.2). Furthermore, from (2.6), we can rediscover Tornheim's main result in [22] as mentioned above.

Now we give the sketch of the proof of Proposition 2.1. Before that, we recall Hardy's method of proving the functional equation for $\zeta(s)$ ([9], see also [21] § 2.2) as follows.

Let

$$f(x) := \sum_{n=0}^{\infty} \frac{\sin(2n+1)x}{2n+1} \quad (x>0). \tag{2.7}$$

From the well-known Fourier expansion, we have

$$f(x) = (-1)^m \frac{\pi}{4} \tag{2.8}$$

for $m\pi < x < (m+1)\pi$ $(m=0,1,2,\ldots)$. For $s \in \mathbb{R}$ with $0<s<1$, put

$$I := \int_0^{\infty} x^{s-1} f(x) dx. \tag{2.9}$$

Since the right-hand side of (2.7) is boundedly convergent, we see that the term-by-term integration on the right-hand side of (2.9) can be justified. Using the well-known functional relation for $\Gamma(x)$ and $\sin x$, we have

$$I = \Gamma(s) \sin\frac{s\pi}{2} \left(1-2^{-s-1}\right) \zeta(s+1)$$

On the other hand, it follows from (2.8) that

$$I = \frac{\pi^{s+1}}{2s} \left(1-2^{s+1}\right) \zeta(-s),$$

this means the functional equation for $\zeta(s)$.

Now we aim to consider the double analogue of this method. Let

$$f_2(x) := \sum_{m=1}^{\infty} \frac{\sin(mx)}{m^3} + \sum_{m,n=1}^{\infty} \frac{\sin((m+n)x)}{m(m+n)^2} - \sum_{m,n=1}^{\infty} \frac{\sin(mx)}{mn(m+n)}$$

for $x>0$. By the same consideration as $f(x)$, we can prove

$$f_2(x) = 0 \quad (0 < x < 2\pi),$$

namely $f_2(x)=0$ for all $x>0$. Hence, for $0<s<1$,

$$\begin{aligned} 0 &= \int_0^{\infty} x^{s-1} f_2(x) dx \\ &= \sin\frac{\pi s}{2} \Gamma(s) \Big\{ \zeta(s+3) + \zeta_{EZ,2}(1,s+2) - \zeta_{MT,2}(s+1,1,1) \Big\}. \end{aligned}$$

Since $\sin\frac{\pi s}{2}\Gamma(s)\neq 0$ when $0<s<1$, we can remove this term. Thus we obtain

$$\zeta(s+3)+\zeta_{EZ,2}(1,s+2)-\zeta_{MT,2}(s+1,1,1)=0.$$

Note that this holds for $s\in\mathbb{R}$ with $0<s<1$. Since the functions on the left-hand side are continued meromorphically, this result means Proposition 2.1.

We can generalize this method to the multiple case. Let

$$Z(s)=\sum_{m=1}^{\infty}\frac{a_m}{m^s}, \tag{2.10}$$

where $\{a_m\}\subset\mathbb{C}$. Let $\Re(s)=\rho$ ($\rho\in\mathbb{R}$) be the abscissa of convergence of $Z(s)$, and assume $0\leq\rho<1$.

Proposition 2.3. *Assume that*

$$\sum_{m=1}^{\infty}a_m\sin(mt)=0 \tag{2.11}$$

or

$$\sum_{m=1}^{\infty}a_m\cos(mt)=0 \tag{2.12}$$

is boundedly convergent for $t>0$ *and that, for* $\rho<s<1$,

$$\lim_{\lambda\to\infty}\sum_{m=1}^{\infty}a_m\int_{\lambda}^{\infty}t^{s-1}\sin(mt)dt=0 \tag{2.13}$$

(if we assume (2.11)) or

$$\lim_{\lambda\to\infty}\sum_{m=1}^{\infty}a_m\int_{\lambda}^{\infty}t^{s-1}\cos(mt)dt=0 \tag{2.14}$$

(if we assume (2.12)). Then $Z(s)$ *can be continued meromorphically to* $\mathbb{C}$, *and actually* $Z(s)=0$ *for all* $s\in\mathbb{C}$.

From this result, we can construct certain functional relations for "multiple" zeta-functions.

For $k\in\mathbb{N}$, let $V_k:=\{\sigma=(\sigma_1,\ldots,\sigma_k)\in\{\pm1\}^k\,|\,\sigma_1=1\}$ and

$$\sigma(X_1,\ldots,X_k):=\sigma_1X_1+\cdots+\sigma_kX_k.$$

For $p\in\mathbb{N}\cup\{0\}$ and $\sigma=(\sigma_j)\in V_{2p+1}$, let

$$\Delta_\sigma=(-1)^p\prod_{j=1}^{2p+1}\sigma_j\in\{\pm1\}.$$

Then we can define

$$f(t) := 2 \sum_{\sigma \in V_{2p+1}} \Delta_\sigma \sum_{m_1,\dots,m_{2p+1}=1}^{\infty} \frac{\sin(\sigma(m_1,\dots,m_{2p+1})t)}{m_1 \cdots m_{2p+1}} + \sum_{j=0}^{p} \beta_{pj} \sum_{m=1}^{\infty} \frac{\sin(mt)}{m^{2j+1}},$$

where $\{\beta_{pj} \in \mathbb{Q}[\pi^2] \mid 0 \le j \le p\}$ can be calculated explicitly, such that $f(t) = 0$ for $t > 0$.

Corresponding to $f(t)$, we define

$$\mathcal{Z}_{2p+1}(s) = 2 \sum_{\sigma \in V_{2p+1}} \Delta_\sigma \Bigg\{ \sum_{\substack{m_1,\dots,m_{2p+1} \ge 1 \\ \sigma(m_1,\dots,m_{2p+1})>0}} \frac{1}{m_1 \cdots m_{2p+1} \sigma(m_1,\dots,m_{2p+1})^s} - \sum_{\substack{m_1,\dots,m_{2p+1} \ge 1 \\ \sigma(m_1,\dots,m_{2p+1})<0}} \frac{1}{m_1 \cdots m_{2p+1} (-\sigma(m_1,\dots,m_{2p+1}))^s} \Bigg\} + \sum_{j=0}^{p} \beta_{pj} \zeta(s+2j+1)$$

for $s \in \mathbb{C}$ with $\Re s > 1$. $\mathcal{Z}_{2p+1}(s)$ can be continued meromorphically to $\mathbb{C}$.

From Proposition 2.3, we obtain,

Proposition 2.4. *For* $p \in \mathbb{N} \cup \{0\}$, $\mathcal{Z}_{2p+1}(s) = 0$ *for all* $s \in \mathbb{C}$.

Let $p = 1$. Then $\mathcal{Z}_3(s) = 0$ implies

$$\begin{aligned} &2\zeta_{EZ,3}(1,1,s+1) - \zeta_{MT,3}(s,1,1,1) + 2\zeta_{MT,2}(1,2,s) \\ &\quad + 2\zeta_{MT,2}(s,2,1) - 2\zeta(2)\zeta(s+1) + 4\zeta(s+3) = 0 \end{aligned}$$

holds for all $s \in \mathbb{C}$ except for the singularities of all functions on the left-hand side.

In particular when $s = 1$, from $\zeta_{MT,3}(1,1,1,1) = 6\zeta_{EZ,3}(1,1,2)$, we obtain the well-known relation (see [10]):

$$\zeta_{EZ,3}(1,1,2) = \zeta(4).$$

3. Witten zeta-functions

For the details of the results in this section, see [18].

For any semisimple Lie algebra $\mathfrak{g}$, Zagier defined the *Witten zeta-function* (1994) by

$$\zeta_{\mathfrak{g}}(s) = \sum_{\rho} (\dim \rho)^{-s} \quad (s \in \mathbb{C}),$$

where ρ runs over all finite dimensional irreducible representations of $\mathfrak{g}$ (see [29]). For example,

$$\zeta_{\mathfrak{sl}(2)}(s) = \zeta(s), \quad \zeta_{\mathfrak{sl}(3)}(s) = 2^s \zeta_{MT,2}(s,s,s),$$
$$\zeta_{\mathfrak{so}(5)}(s) = 6^s \sum_{m,n=1}^{\infty} \frac{1}{m^s n^s (m+n)^s (m+2n)^s}.$$

It follows from Witten's work ([27]) about calculation of the volumes of certain moduli spaces that

$$\zeta_{\mathfrak{g}}(2k) \in \mathbb{Q}\pi^{2kl} \quad (k \in \mathbb{N}),$$

where l is the number of positive roots of $\mathfrak{g}$. Note that the case $\mathfrak{g} = \mathfrak{sl}(2)$ means well-known Euler's formula about $\zeta(2k)$ and the case $\mathfrak{g} = \mathfrak{sl}(3)$ means Mordell's result mentioned above.

As generalizations of Zagier's Witten zeta-function, we define the Witten zeta-function associated with $\mathfrak{sl}(r+1)$ of several variables by

$$\zeta_{\mathfrak{sl}(r+1)}(\overrightarrow{\mathbf{s}}) = \sum_{m_1,\ldots,m_r=1}^{\infty} \prod_{j=1}^{r} \prod_{k=1}^{r-j+1} \left(\sum_{\nu=k}^{j+k-1} m_\nu \right)^{-s_{jk}}. \tag{3.1}$$

We can prove the meromorphic continuation of $\zeta_{\mathfrak{sl}(r+1)}(\overrightarrow{\mathbf{s}})$ for

$$\overrightarrow{\mathbf{s}} = (s_{jk})_{1\le j\le r;\, 1\le k\le r-j+1} \in \mathbb{C}^{\frac{r(r+1)}{2}},$$

using the Mellin-Barnes method which was established by the first-named author in his previous works [16, 17] (for details, see [18] Theorem 2.2).

For example,

$$\zeta_{\mathfrak{sl}(2)}(s) = \zeta(s), \quad \zeta_{\mathfrak{sl}(3)}(s_1,s_2,s_3) = \zeta_{MT,2}(s_1,s_2,s_3)$$
$$\zeta_{\mathfrak{sl}(4)}(s_1,\ldots,s_6) = \sum_{l,m,n=1}^{\infty} \frac{1}{l^{s_1} m^{s_2} n^{s_3} (l+m)^{s_4} (m+n)^{s_5} (l+m+n)^{s_6}}.$$

Remark. $\zeta_{\mathfrak{sl}(4)}(s_1,\ldots,s_6)$ can be continued meromorphically to the whole complex space $\mathbb{C}^6$, and all of its possible singularities are located on the subsets of $\mathbb{C}^6$ defined by one of the equations (see [18] Theorem 3.5):

$$\begin{cases} s_1+s_4+s_6=1-l \quad (l\in\mathbb{N}\cup\{0\}); \\ s_3+s_5+s_6=1-l \quad (l\in\mathbb{N}\cup\{0\}); \\ s_2+s_4+s_5+s_6=1-l \quad (l\in\mathbb{N}\cup\{0\}); \\ s_1+s_2+s_4+s_5+s_6=2-l \quad (l\in\mathbb{N}\cup\{0\}); \\ s_1+s_3+s_4+s_5+s_6=2-l \quad (l\in\mathbb{N}\cup\{0\}); \\ s_2+s_3+s_4+s_5+s_6=2-l \quad (l\in\mathbb{N}\cup\{0\}); \\ s_1+s_2+s_3+s_4+s_5+s_6=3. \end{cases}$$

Note that $\zeta_{\mathfrak{sl}(4)}(s)=12^s\zeta_{\mathfrak{sl}(4)}(s,s,s,s,s,s)$. Hence, from Witten's result,

$$\zeta_{\mathfrak{sl}(4)}(2,2,2,2,2,2)=\frac{23}{2554051500}\pi^{12}=\frac{23}{2764}\zeta(12). \tag{3.2}$$

Furthermore Gunnells and Sczech ([8]) recently gave the explicit formulas for $\zeta_{\mathfrak{sl}(4)}(2k,2k,2k,2k,2k,2k)$ $(k\in\mathbb{N})$. From these results, we have the following natural questions:

- *Is there any functional relation for* $\zeta_{\mathfrak{sl}(4)}(s_1,s_2,\ldots,s_6)$ *?*
- *What is the value* $\zeta_{\mathfrak{sl}(4)}(k_1,k_2,k_3,k_4,k_5,k_6)$ *at any positive integer point ?*

As certain answers to these questions, we obtain the following.

Proposition 3.1.

$$\begin{aligned} &2\zeta_{\mathfrak{sl}(4)}(s_1,s_2,2,s_3,0,2)+\zeta_{\mathfrak{sl}(4)}(2,0,s_2,s_1,2,s_3)+\zeta_{\mathfrak{sl}(4)}(s_1,0,2,2,s_2,s_3) \\ &=-6\zeta_{\mathfrak{sl}(3)}(s_1,s_2,s_3+4)-\zeta_{\mathfrak{sl}(3)}(s_1+2,s_2+2,s_3) \\ &\qquad+4\zeta_{\mathfrak{sl}(2)}(2)\zeta_{\mathfrak{sl}(3)}(s_1,s_2,s_3+2) \end{aligned}$$

holds for all $(s_1,s_2,s_3)\in\mathbb{C}^3$ *except for singularities of functions on both sides, where* $\zeta_{\mathfrak{sl}(2)}(s)=\zeta(s)$ *and* $\zeta_{\mathfrak{sl}(3)}(s_1,s_2,s_3)=\zeta_{MT,2}(s_1,s_2,s_3)$.

Remark. More generally, we can prove that, for $k,l\in\mathbb{N}$ and $q\in\{0,1\}$,

$$\begin{aligned} &\zeta_{\mathfrak{sl}(4)}(s_1,s_2,2k,s_3,0,2l+q)+(-1)^q\zeta_{\mathfrak{sl}(4)}(s_1,s_2,2l+q,s_3,0,2k) \\ &\quad+\zeta_{\mathfrak{sl}(4)}(2k,0,s_2,s_1,2l+q,s_3)+\zeta_{\mathfrak{sl}(4)}(s_1,0,2l+q,2k,s_2,s_3) \end{aligned}$$

is expressed as a polynomial on $\zeta_{\mathfrak{sl}(3)}(\mathbf{s})$ and $\zeta_{\mathfrak{sl}(2)}(s)$ with $\mathbb{Q}$-coefficients (for details, see [18] Theorems 4.9 and 4.10).

From Proposition 3.1 and Tornheim's results in [22], and using (2.5), we obtain

$$\begin{aligned}
\zeta_{\mathfrak{sl}(4)}(1,1,1,2,1,2) &= -\frac{29}{175}\zeta(2)^4 + \zeta(3)\zeta(5) - \frac{1}{2}\zeta_{EZ,2}(2,6); \\
\zeta_{\mathfrak{sl}(4)}(1,1,2,1,2,1) &= \frac{2683}{1050}\zeta(2)^4 + \frac{1}{2}\zeta(2)\zeta(3)^2 \\
&\quad - 16\zeta(3)\zeta(5) + \frac{29}{4}\zeta_{EZ,2}(2,6); \\
\zeta_{\mathfrak{sl}(4)}(1,1,1,2,1,3) &= \frac{2}{5}\zeta(2)^2\zeta(5) + 10\zeta(2)\zeta(7) - \frac{53}{3}\zeta(9),
\end{aligned}$$

which can be regarded as analogues of Witten's formula (3.2). However we can only obtain special cases of these evaluation formulas, because we can only obtain the special cases of functional relations like that in Proposition 3.1.

Remark. We are now studying the Witten zeta-function associated with any type of semisimple Lie algebras in a more general situation. We will report on these results in forthcoming papers (see [11, 12]).

4. Functional relations for Double L-functions

For a Dirichlet character χ, we define

$$L_{MT,2}^{\mathrm{III}}(s_1,s_2,s_3;\chi,\chi) = \sum_{m_1,m_2=1}^{\infty} \frac{\chi(m_1)\chi(m_2)}{m_1^{s_1}m_2^{s_2}(m_1+m_2)^{s_3}}; \tag{4.1}$$

$$L_{MT,2}^{*}(s_1,s_2,s_3;\chi,\chi) = \sum_{m_1,m_2=1}^{\infty} \frac{\chi(m_1)\chi(m_1+m_2)}{m_1^{s_1}m_2^{s_2}(m_1+m_2)^{s_3}}. \tag{4.2}$$

$L_{MT,2}^{\mathrm{III}}$ and its multiple analogues were considered by Wu in [28] (see also [17]). Note that

$$\begin{aligned}
L_{EZ,2}^{\mathrm{III}}(s_1,s_2;\chi,\chi) &= L_{MT,2}^{\mathrm{III}}(s_1,0,s_2;\chi,\chi) \\
&= \sum_{m_1,m_2=1}^{\infty} \frac{\chi(m_1)\chi(m_2)}{m_1^{s_1}(m_1+m_2)^{s_2}}
\end{aligned}$$

is called the *double L-function* of the Euler-Zagier type (see, for example, [3]).

We can obtain, for example,

$$L_{EZ,2}^{\mathrm{III}}(1,-1;\chi,\chi) = B_{1,\chi}^2 - \frac{1}{2}L(1;\chi)B_{2,\chi},$$

where χ is nontrivial and $\{B_{n,\chi}\}$ are the generalized Bernoulli numbers. This implies that the double L-function has some information about abelian number fields related to χ.

In particular, for $j = 3, 4$, we denote by χ_j the primitive Dirichlet character of conductor j, and χ_j^2 be defined by $\chi_j^2(m) = \{\chi_j(m)\}^2$. As χ-analogues of Proposition 2.1, we can prove

$$L_{MT,2}^{\mathrm{III}}(1, s, 1; \chi_4, \chi_4) + L_{MT,2}^{*}(1, 1, s; \chi_4, \chi_4) - L_{MT,2}^{*}(s, 1, 1; \chi_4, \chi_4) \tag{4.3}$$
$$= 2L(1; \chi_4)L(s+1; \chi_4) - L(s+2; \chi_4^2);$$
$$L_{MT,2}^{\mathrm{III}}(1, s, 2; \chi_3, \chi_3) + L_{MT,2}^{*}(1, 2, s; \chi_3, \chi_3) + L_{MT,2}^{*}(s, 2, 1; \chi_3, \chi_3) \tag{4.4}$$
$$= -L(s+3; \chi_3^2) + 3L(1; \chi_3)L(s+2; \chi_3) - \frac{3}{4}L(2; \chi_3^2)L(s+1; \chi_3^2)$$

for $s \in \mathbb{C}$ except for the singular points of each side. Letting $s = 1$ in (4.3) and $s = 2$ in (4.4), we obtain, for example,

$$L_{EZ,2}^{\mathrm{III}}(1, 2; \chi_4, \chi_4) = L(1; \chi_4)L(2; \chi_4) - L(3; \chi_4^2);$$
$$L_{MT,2}^{\mathrm{III}}(1, 2, 2; \chi_3, \chi_3) + L_{MT,2}^{*}(1, 2, 2; \chi_3, \chi_3) + L_{MT,2}^{*}(2, 2, 1; \chi_3, \chi_3)$$
$$= -L(5; \chi_3^2) + 3L(1; \chi_3)L(4; \chi_3) - \frac{3}{4}L(2; \chi_3^2)L(3; \chi_3^2).$$

Note that we can further give more general functional relations for the double L-functions (for details, see [24, 26]).

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