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Fourier coefficients of modular forms

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In this short survey article we will report on recent results regarding sign changes of Fourier coefficients of elliptic modular forms f. In particular this applies when f is a normalized Hecke eigenform and so the Fourier coefficients are the Hecke eigenvalues.

The following result seems to be "well-known".

Theorem. *Let f be a non-zero cusp form of even integral weight k on the Hecke congruence subgroup $\Gamma_0(N)$ and suppose that its Fourier coefficients $a(n)$ are real for all $n \geq 1$. Then there are infinitely many n such that $a(n) > 0$ and there are infinitely many n such that $a(n) < 0$.*

For the reader's convenience, we shall briefly recall the proof. It is sufficient to assume that $a(n) \geq 0$ for all but finitely many n and to derive a contradiction. Let

$$L(f,s) = \sum_{n\geq 1} a(n)n^{-s} \qquad (\Re(s) >> 0)$$

be the Hecke L-series of f. Our assumption and a classical result of Landau then imply that either $L(f,s)$ converges everywhere or $L(f,s)$ has a singularity at the real point of its line of convergence.

However, according to a classical result of Hecke, $L(f,s)$ extends to an entire a function. Hence we conclude in particular that $a(n) << n^c$ and hence also $a(n)^2 << n^c$ for all real c. As a consequence, the Rankin-Selberg zeta function

$$R_f(s) = \sum_{n\geq 1} a(n)^2 n^{-s} \qquad (\Re(s) >> 0)$$

must converge everywhere, hence is entire. However, by classical result of Rankin the latter has a pole at $s = k$ of residue proportional to the square of the Petersson norm of f which is non-zero, since f is non-zero, a contradiction. This completes the proof.

According to the above Theorem, it makes quite sense to ask if it is possible to obtain information when the first sign change takes place. Of course, in general this seems to be a difficult question.

Note that if $f \neq 0$, then by the valence formula for modular forms there must exist n in the range

$$1 \leq n \leq \frac{k}{12}[\Gamma(1) : \Gamma_0(N)]$$

such that $a(n) \neq 0$. Hence being optimistic, one might hope for a sign change in the range

$$1 \leq n \leq \frac{k}{12}[\Gamma(1) : \Gamma_0(N)] + 1.$$

In a very special case, this indeed follows from work of Siegel [3]. To formulate the result, suppose that $k \geq 4$ is even and denote by d_k the dimension of the space $M_k(\Gamma(1))$ of modular forms of weight k on $\Gamma(1)$. (Recall that d_k essentially is $[\frac{k}{12}]$.) Then Siegel showed that there are explicitly computable rational numbers c_n ($n = 0, 1, \ldots, d_k$), depending on k, such that

$$\sum_{n=0}^{d_k} c_n a_f(n) = 0 \qquad (\forall\, f \in M_k(\Gamma(1)).$$

Moreover, it follows from Siegel's explicit formulas that if $k \equiv 2 \pmod 4$, then all the c_n are strictly positive.

Since a cusp form of weight k on $\Gamma(1)$ is determined by its first $d_k - 1$ Fourier coefficients, Siegel's result immediately shows that if $k \equiv 2 \pmod 4$, then there must be a sign change of the $a(n)$ in the range $1 \leq n \leq d_k$.

Unfortunately, if $k \equiv 0 \pmod 4$ or if $N > 1$, then Siegel's arguments do not work any longer, and so one has to look for other devices.

In the following, we shall suppose that f is a normalized Hecke eigenform that is a newform. Recall that "normalized" means that $a(1) = 1$ and "newform" essentially means that the exact level of f is N. In this case, the Fourier coefficients are equal to the Hecke eigenvalues.

Theorem [2]. *Suppose that f is a normalized Hecke eigenform of even integral weight k and squarefree level N that is a newform. Then one has $a(n) < 0$ for some n with*

$$n << k\sqrt{N} \cdot \log^{8+\epsilon}(kN), \quad (n, N) = 1 \qquad (\epsilon > 0).$$

Note that it is reasonable to assume that $(n, N) = 1$, since the eigenvalues $a(p)$ with $p|N$ are explicitly known by Atkin-Lehner theory.

The proof of the above result uses techniques from analytic number theory (e.g. Perron's formula, a strong convexity principle, ...) and properties of the symmetric square L-function of f.

Very recently, the above result was improved as follows.

Theorem [1]. Suppose that f is a normalized Hecke eigenform of level N (not necessariy squarefree) and even integral weight k. Then $a(n) < 0$ for some n with

$$n << (k^2 N)^{\frac{29}{60}}, \quad (n, N) = 1.$$

The proof is "elementary" in the sense that it completely avoids the use of the symmetric square L-function. Instead, the Hecke relations for the eigenvalues are exploited.

References

[1] H. Iwaniec, W. Kohnen and J. Sengupta: The first negative Hecke eigenvalue. Preprint 2006

[2] W. Kohnen and J. Sengupta: On the first sign change of Hecke eigenvalues of newforms. To appear in Math. Z.

[3] C.L. Siegel: Berechnung von Zetafunktionen an ganzzahligen Stellen. Nachr. Akad. Wiss. Göttingen Math.-Phys. Kl., II (1969), 87-102

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