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Smoothed GPY Sieve

Talk Report

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Hereby I shall give a report of the contents of my talk delivered on October 18, 2005 at the workshop on Analytic Number Theory held at RIMS, Kyoto University. I first gave an account of my recent joint work with M. Jutila [6][7] on uniform bounds for Hecke L-functions and the Rankin–Selberg L-functions, explaining the methodological connection between these subjects and the issue of small gaps between prime numbers, as I envisage that the investigation on the latter should eventually lead us to a refined treatment of the additive divisor sums either in the original ((4) below) or in a variety of extended senses via the reasoning represented by Linnik's dispersion method [8]. At the end of my talk I introduced, without proof, a smoothing device into the now famous sieve procedure [3] of D.A. Goldston, J. Pintz, and C.Y. Yıldırım combining the arguments developed in [2] and [11] (see also [9]). From my assertion (Lemma 4 below), it transpires that the bounded differences between prime numbers, i.e., the existence of an absolute constant B such that

$$\liminf\,(p_{n+1}-p_n)<B \tag{0}$$

should follow from any improvement of the Bombieri–Friedlander–Iwaniec prime number theorem [1] that admits *varying* residue classes, the number of which is relatively small.

1. Thus, let $\Gamma=\mathrm{PSL}_2(\mathbb{Z})$ and $G=\mathrm{PSL}_2(\mathbb{R})$. Let V be a Hecke invariant irreducible cuspidal Γ-automorphic representation occurring in $L^2(\Gamma\backslash G)$ such that $T(n)|_V=\tau_V(n)\cdot 1$, and $\Omega|_V=(\nu_V^2-\frac{1}{4})\cdot 1$, where $T(n)$ and Ω are Hecke and Casimir operators, respectively. The Hecke L-function attached to V is defined by

$$H_V(s)=\sum_{n=1}^{\infty}\tau_V(n)n^{-s},\quad \mathrm{Re}\,s>1. \tag{1}$$

With this, the main assertion of [6] is the uniform bound

$$H_V\left(\tfrac{1}{2}+it\right)\ll(t+|\nu_V|)^{1/3+\varepsilon},\quad t\ge 0, \tag{2}$$

where the implied constant depends on ε at most. This is a wide extension of the classical bound

$$\zeta\left(\tfrac{1}{2}+it\right)\ll t^{1/6}\log^2 t,\quad t\ge 2. \tag{3}$$

The implements that were employed in the proof of (2) are the Voronoï formula, the Spectral–Kloosterman sum formula of Bruggeman and Kuznetsov, and the spectral decomposition [10] of

$$\sum_{n=1}^{\infty}\sigma_\alpha(n)\sigma_\beta(n+f)W(n/f), \tag{4}$$

where $\sigma_\alpha(n)=\sum_{d|n}d^\alpha$, and W is a smooth function supported compactly on the positive reals. It should be noted that the proofs of (3) depends on an idea of Weyl and van der

Corput, and on the harmonic analysis over the unit circle or the Fourier analysis. The sum (4) can be considered as an extension of the basic idea (i.e., a lifting of a one-dimensional sum to a two-dimensional) of Weyl and van der Corput; and the Fourier analysis is replaced by the spectral theory of sums of Kloosterman sums. The last is, in fact, equivalent to the harmonic analysis on $\Gamma\backslash G$ or the theory of Γ-automorphic representations of G.

A partial extension of (2) to Rankin–Selberg L-functions is given in [7]. Thus, let

$$L(s, V\otimes V_0)=\zeta(2s)\sum_{n=1}^{\infty}\tau_V(n)\tau_{V_0}(n)n^{-s} \tag{5}$$

be the Rankin–Selberg L-function associated with irreducible representations V, V_0 as above. Then, a special case of the main assertion of [7] is

$$L\left(\tfrac{1}{2}+it, V\otimes V_0\right)\ll |\nu_V|^{2/3+\varepsilon},\quad 0\le t\le |\nu_V|^{2/3}, \tag{6}$$

where the implied constant depends on V_0 and ε at most. This time, the sum (4) is replaced by

$$\sum_{n=1}^{\infty}\tau_{V_0}(n)\tau_{V_0}(n+f)W(n/f), \tag{7}$$

on which investigation corresponding to [10] is developed in [5], for instance (see the latter for the history on this problem).

Turning to the subject indicated by the title of our talk, it should be stressed again that the above should have a basic relevance to such an extension of the main result of [1] as needed in the expected proof of (0), as far as one tries to apply the dispersion method. A typical instance was treated in [8] based on Estermann's result on the sum (4) with $\alpha=\beta=0$. Indeed, it was later extended by A.I. Vinogradov, and resulted in the Mean Prime Number Theorem, independently established by Bombieri. Hence, it appears natural to surmise that (4), (7), and their further extensions should be relevant to the investigation of the distribution of primes in arithmetic progressions of large moduli, which is most probably needed in the proof of (0).

2. Let N be a parameter increasing monotonically to infinity. There are four other basic parameters H, R, k, ℓ in our discussion. We impose the following conditions to them:

$$H\ll \log N\ll \log R\le \log N, \tag{8}$$

and

$$\text{integers } k,\ell>0 \text{ are sufficiently large but bounded.} \tag{9}$$

Let

$$\mathcal{H}=\{h_1,h_2,\ldots,h_k\}\subseteq[-H,H]\cap\mathbb{Z}, \tag{10}$$

with $h_i\neq h_j$ for $i\neq j$; and put, for a prime p,

$$\Omega(p)=\{\text{different residue classes among } -h(\operatorname{mod} p),\ h\in\mathcal{H}\} \tag{11}$$

and write $n\in\Omega(p)$ instead of $n\,(\operatorname{mod} p)\in\Omega(p)$. We call $\mathcal{H}$ admissible if $|\Omega(p)|<p$ for all p, and assume this unless otherwise stated. We extend Ω multiplicatively, so that $n\in\Omega(d)$ with square-free d if and only if $n\in\Omega(p)$ for all $p|d$. We put, with μ the Möbius function,

$$\lambda_R(d;\ell)=\begin{cases} 0 & \text{if } d>R,\\ \dfrac{\mu(d)}{(k+\ell)!}\left(\log\dfrac{R}{d}\right)^{k+\ell} & \text{if } d\le R,\end{cases} \tag{12}$$

and

$$\Lambda_R(n;\mathcal{H},\ell) = \sum_{n\in\Omega(d)} \lambda_R(d;\ell). \tag{13}$$

Also, let

$$E^*(y;r,q) = \vartheta^*(y;r,q) - \frac{y}{\varphi(q)}, \quad \vartheta^*(y;r,q) = \sum_{\substack{y<n\le 2y\\ n\equiv r \bmod q}} \varpi(n), \tag{13}$$

where φ is the Euler totient function; and $\varpi(n) = \log n$ if n is a prime, and $= 0$ otherwise. In all accounts [2]–[4] of the GPY sieve, it is assumed that

$$\sum_{q\le x^\theta} \max_{(r,q)=1} \max_{y\le x} |E^*(y;r,q)| \ll \frac{x}{(\log x)^C}, \tag{14}$$

with a certain $\theta \in (0,1)$ and an arbitrary fixed $C > 0$.

The following asymptotic formulas are the fundamental implements in the GPY sieve:

Lemma 1. *Provided* (8), (9), *and* $R \le N^{1/2}/(\log N)^C$ *hold with a sufficiently large* $C = C(k,l)$, *we have*

$$\sum_{N<n\le 2N} \Lambda_R(n;\mathcal{H},\ell)^2$$
$$= \frac{\mathfrak{S}(\mathcal{H})}{(k+2\ell)!}\binom{2\ell}{\ell} N(\log R)^{k+2\ell} + O(N(\log N)^{k+2\ell-1}(\log\log N)^c), \tag{15}$$

where $c = c(k,l)$ *is a certain constant, and*

$$\mathfrak{S}(\mathcal{H}) = \prod_p \left(1 - \frac{|\Omega(p)|}{p}\right)\left(1-\frac{1}{p}\right)^{-k}. \tag{16}$$

Lemma 2. *Provided* (8), (9), *and* (14), *we have for* $R \le N^{\theta/2}/(\log N)^C$ *with a sufficiently large* $C = C(k,l)$,

$$\sum_{N<n\le 2N} \varpi(n+h)\Lambda_R(n;\mathcal{H},\ell)^2$$
$$= \frac{\mathfrak{S}(\mathcal{H})}{(k+2\ell+1)!}\binom{2(\ell+1)}{\ell+1} N(\log R)^{k+2\ell+1} + O(N(\log N)^{k+2\ell}(\log\log N)^c), \tag{17}$$

whenever $h \in \mathcal{H}$.

3. It is known that the combination of (15), (17), and (14) with a $\theta > \frac{1}{2}$ gives rise to the assertion (0). The principal aim of my talk was to show a smoothed version of (15) and (17) to look into the possibility of replacing (14) with a $\theta > \frac{1}{2}$ by any less stringent hypothesis.

With this in mind, we shall hereafter assume that

$$H = H(k,\ell) \text{ is bounded.} \tag{18}$$

We begin with a smoothing of (15); and to this end, we follow [9][11]. Thus, let us put

$$R_0 = \exp\left(\frac{\log R}{(\log\log R)^{1/5}}\right), \quad R_1 = \exp\left(\frac{\log R}{(\log\log R)^{9/10}}\right), \quad \tau = (\log\log R)^{1/10}. \tag{19}$$

We divide the interval $[R_0, R_0R^J)$ into intervals $[R_0R_1^{j-1}, R_0R_1^j)$ $(j = 1, 2, \dots, J)$, denoting them by P, with or without suffix; and further put

$$J = \xi \cdot (\log\log R)^{9/10}, \tag{20}$$

with a sufficiently small $\xi > 0$; it appears that the choice $\xi = 1/\sqrt{k}$ is appropriate. Let D be a generic element of the commutative semi-group generated by all P's. When $D = P_1P_2\cdots P_r$, the notation $d \in D$ indicates that d has the prime decomposition $d = p_1p_2\cdots p_r$ with $p_j \in P_j$ $(1 \le j \le r)$. Note that we use the convention that $1 \in D$ if and only if D is the empty product. Further, we put $|P| = R_0R_1^j$ if $P = [R_0R_1^{j-1}, R_0R_1^j)$; and $|D| = |P_1|\cdots|P_r|$ if $D = P_1P_2\cdots P_r$. Naturally, $|D| = 1$ if D is empty. We put

$$\Delta(D) = \prod_{P|D}\left(\sum_{p\in P}\frac{|\Omega(p)|}{p}\right), \tag{21}$$

and

$$\Phi(D) = \prod_{P|D}\left(\sum_{p\in P}\frac{|\Omega(p)|}{p}\left(1 - \frac{|\Omega(p)|}{p}\right)\right)\left(\sum_{p\in P}\frac{|\Omega(p)|}{p}\right)^{-2}. \tag{22}$$

Also, modifying [2, (1.21)] and [11, (6)], we put

$$\tilde{\lambda}_R(D;\ell) = \frac{\mathfrak{S}(\mathcal{H})}{\ell! W(R_0)}\frac{\mu(D)}{\Delta(D)}\sum_{\substack{|K|<R\\ D|K}}\frac{\mu(K)^2}{\Phi(K)}\left(\log\frac{R}{|K|}\right)^{\ell}, \tag{23}$$

where μ is a natural extension of the Möbius function, $W(z) = \prod_{d<z}(1 - |\Omega(p)|/p)$; and the empty sum is to vanish, that is, $\tilde{\lambda}_R(D;\ell) = 0$ for $|D| \ge R$. Note that in (21) and (22) we have $p \ge R_0$, and thus $|\Omega(p)| = k$ always. We shall, however, keep the notation $|\Omega(p)|$, because of a future purpose.

As to the interval $[1, R_0)$, which is excluded in the above, we appeal to the Fundamental Lemma in the sieve method (see e.g., [12, Sections 3.2–3.5]). Thus, there exits a set of sieve weights $\varrho(d)$ such that $|\varrho(d)| \le 1$ for any $d \ge 1$, and $\varrho(d) = 0$ either if $d \ge R_0^\tau$ with τ as above or if d has a prime factor greater than or equal to R_0, and that

$$\gamma(n;\mathcal{H}) = \sum_{n\in\Omega(d)}\varrho(d) \ge 0 \quad \text{for any } n \ge 1 \tag{24}$$

as well as

$$\sum_d \frac{\varrho(d)}{d}|\Omega(d)| = W(R_0)\left\{1 + O\left(e^{-\tau}\right)\right\}, \tag{25}$$

with the implied constant being absolute.

Lemma 3. *A smoothed counterpart of* (12) *is defined to be*

$$\tilde{\Lambda}_R(n;\mathcal{H},\ell) = \sum_{D} \tilde{\lambda}_R(D;\ell) \sum_{\substack{d\in D\\ n\in\Omega(d)}} 1. \tag{26}$$

Under (9), (18), *we have*

$$\begin{aligned}&\sum_{N<n\le 2N} \gamma(n;\mathcal{H})\tilde{\Lambda}_R(n;\mathcal{H},\ell)^2\\ &= \frac{\mathfrak{S}(\mathcal{H})}{(k+2\ell)!}\binom{2\ell}{\ell} N(\log R)^{k+2\ell}\left(1+O(e^{-k\xi})\right) + O(R_0^{\tau}R^2(\log R)^c).\end{aligned} \tag{27}$$

Lemma 4. *Under* (9), (18), *we have, for any* $h\in\mathcal{H}$,

$$\begin{aligned}&\sum_{N<n\le 2N} \varpi(n+h)\gamma(n;\mathcal{H})\tilde{\Lambda}_R(n;\mathcal{H},\ell)^2\\ &\ge \frac{\mathfrak{S}(\mathcal{H})}{(k+2\ell+1)!}\binom{2(\ell+1)}{\ell+1} N(\log R)^{k+2\ell+1}\left(1+O(e^{-k\xi})\right) + E,\end{aligned} \tag{28}$$

where

$$E \ll (\log N)^c \sup_{\alpha,\beta}{}^{*} \left| \sum_{a<A,b<B} \alpha_a\beta_b \sum_{r\in\Omega^*(ab)} E^*(N;r,ab)\right|, \tag{29}$$

with $\Omega^*(p) = \Omega(p)\setminus\{-h \bmod p\}$. *Here* $AB = R_0^{\tau}R^2$; *and* α, β *are complex vectors such that* $|\alpha_a|\le 1$, $|\beta_b|\le 1$, *with the construction restricted by* (19), (20), *and* (24) *as indicated by* $\sup^*$.

Behind the appearance of the factor $1+O(e^{-k\xi})$ in (27)–(28) is a use of the Buchstab identity; specifically a prototype of (27) with $l=0$ has been known to me since early 1980's, together with the error term corresponding to (29) (i.e., without the twist by ϖ). Although I have not checked it fully yet, the inequality (28) may be replaced by an asymptotic identity as k tends to infinity. The assertion (29) is proved as in [11, pp. 1063–1064] but with a little more care, on noting the choice (20), which means that the set of sifting primes are $\le R_0R^{\xi}$ in place of R in [2]–[4]. We should note that (29) is not completely comparable with Iwaniec's error term for the linear sieve, but since k can be assumed to be sufficiently large, this blemish may turn out to be immaterial as far as the possible application to prove (0). Details are to be published elsewhere under the title '*On the high dimensional Selberg sieve*'.

Addendum (May 23, 2006): While attending the workshop '*Gaps between Primes*' (November 28–December 3, 2005) at the American Institute of Mathematics, I learnt that J. Pintz had obtained essentially the same assertion as Lemma 4 above. Accordingly, a joint work entitled '*A smoothed GPY sieve*' has been uploaded to arXiv (math.NT/0602599). Note also that the references below are updated at this opportunity.

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