数理解析研究所講究録
1513巻 2006年 131-143

Differentiation and Integration in Takebe Katahiro's Mathematics

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Abstract

Takebe Katahiro (建部賢弘, 1664 – 1739) studied carefully the *Suanxue Qimeng* (算学啓蒙) written by Zhu Shijie (朱世傑) in 1299 and learned from this Chinese classic how to deal with (one-variable) polynomials with numerical coefficients (天元術). Then he and his master Seki Takakazu (関孝和, ca.1640 – 1708) developed a method to handle (one-variable) polynomials with variable coefficients (傍書法) and applied this new method to solve many problems in the *Hatsubi Sanpō Endan Genkai* (発微算法演段諺解, 1685). Takebe published an annotated translation of the *Suanxue Qimeng* in 1690 for mathematical students. For the mathematics of Seki see his collected work (see [1]).

Takebe's exploit in the real variable calculus was his discovery of the Taylor expansion formula for the inverse trigonometric function $(\arcsin t)^2$. As was described in his book *Tetsujutsu Sankei* (綴術算経, 1722) he obtained this result by numerical calculation without knowing any theory of differentiation and integration as is presented in today's textbooks of real variable calculus.

In the *Tetsujutsu Sankei* we cannot find any notion of the Cartesian plane, which is basic in the modern advanced calculus, but we can find some primitive ideas of differentiation and integration.

1) Takebe knew that if a polynomial $P(x)$ takes a maximum value then the derivative $P'(x)$ vanishes.

2) Takebe could derive the formula of the surface area of a sphere from the formula of the volume with numerical differentiation.

3) Takebe could derive the formula of the volume of a sphere by the integration by partition.

*This paper was read at the Sixth International Symposium on the History of Mathematics & Mathematical Education Using Chinese Characters (ISHME) August 4–7, 2005, University of Tokyo, Komaba Campus.

1 Modern Mathematics and Japanese Mathematics

In 1722, Takebe Katahiro (建部賢弘) wrote the *Tetsujutsu Sankei* (綴術算経, Mathematical Treatise on the Technique of Linkage) to explain how mathematical research could be done in accordance of one's inclination, based on 12 examples of mathematical investigation. We shall consider this monograph and related works of Takebe as an example of Japanese mathematics in the 18th century.

To understand the situation in perspective we contrast the Japanese mathematics with modern mathematics.

(The most informative literature on the history of Japanese mathematics is the 5 volumes *History of Japanese Mathematics before the Meiji Restoration* [5] but it is written in Japanese. Horiuchi [2] and Ogawa [7] treat some accounts of the history of Japanese mathematics in western languages. We refer the reader to Martzloff [3] for the history of Chinese mathematics written in English.)

1.1 Numbers

Although Japanese mathematicians of the 18th century were able to manipulate fractions easily, they did not distinguish the rational and the irrational numbers. For them the number was something which could be represented and manipulated on an abacus with eventually infinite length. This means, numbers for Japanese mathematicians were real numbers represented by (infinite) decimals.

They distinguished exhaustible numbers (finite decimals) and inexhaustible numbers (infinite decimals). Although they did not hesitate to use inexhaustible numbers, they felt uneasy and tried to convert infinite decimals into approximate fractions using the "method of residual division" (零約法, i.e. the Euclidean algorithm).

1.2 Symbols and Suffix Notation

To name mathematical objects, Japanese mathematicians employed Chinese characters instead of alphabets, for example, the 10 "gan" (十干, a set of ordered 10 Chinese characters) and the 12 "zhi" (十二支, a set of 12 Chinese characters). If these sets were not sufficient, they employed the ordered set of 28 Chinese characters for constellations.

Like their contemporaries in Europe, Japanese mathematicians of the 18th century were not able to use the parameter suffix notation to represent general terms of a sequence.

1.3 Cartesian Plane

Japanese mathematicians of the 18th century were ignorant of Cartesian plane, consequently no idea of the graph of a function nor of the tangent. This means they had no basis for the notion of differentiation initiated by Newton and Leibniz.

1.4 Numerical Solution of an Algebraic Equation

In Chinese mathematics, an algebraic equation with numerical coefficients could be solved numerically digit by digit. The equation was represented on a counting board with counting rods.

This algorithm to find a solution of an algebraic equation had been known since longtime in the name of generalized division. Its first occurrence was the extraction of the square and the cubic roots in the *Jiuzhang Suanshu* (九章算術, Nine Chapters of Arithmetic Arts) of the Chinese Han Dynasty (ca.1 century). The generalized division was described in detail in the *Suanxue Qimeng* (算学啓蒙, Introduction to Mathematics) of Zhu Shijie (朱世傑) in the Chinese Yuen Dynasty (1299).

1.5 One Variable Polynomial with Numerical Coefficients

Besides, in the *Suanxue Qimeng*, Zhu Shijie explained how to represent a one variable polynomial with numerical coefficients on a counting board and to formulate an algebraic equation. This method was called the **method of celestial element** (天元術, "tianyuan shu" in Chinese or "tengen jutsu" in Japanese).

Seki Takakazu (関孝和, ca.1640 – 1708) learned this method from the *Suanxue Qimeng* and formulated the theory of one variable polynomials with numerical coefficients, which we call the **counting board algebra**, in the *Kai-indai no hō* (解隠題之法, Method for Solving Hidden Problems) in ca.1685. In Chapter 2 of the *Tetsujutsu Sankei*, Takebe explained a background of the method of celestial element.

1.6 Side Writing Method

Seki allowed algebraic combinations of symbols as coefficients of polynomials. This method was called the **side writing method** (傍書法, "bōsho hō" in Japanese). With this new tool he opened a new horizon in the mathematical research making it possible to manipulate polynomials with several variables. (See Chapter 6 of the *Tetsujutsu Sankei* and *Hatsubi Sanpō Endan Genkai*.)

1.7 Derived Function of a Polynomial

In Chapter 6 of the *Tetsujutsu Sankei*, Takebe treated a cubic function (polynomial of degree 3) and stated a proposition which meant the following: if a polynomial takes an extreme value at a certain point, then the derived polynomial vanishes at the point.

As remarked earlier, Takebe did not know the differentiation, how could he state this proposition in terms of the counting board algebra.

1.8 Numerical Differentiation

In Chapter 8 of the *Tetsujutsu Sankei*, Takebe used the numerical differentiation to obtain the formula for the surface area

$$S = 4\pi r^2$$

of a sphere with radius r from the formula for the volume

$$V = \frac{4\pi r^3}{3}$$

of the sphere.

1.9 Numerical Integration

In Chapter 9 of the *Tetsujutsu Sankei*, Takebe examined several ways to calculate the formulas for the circumference $L = 2\pi r$ of a circle with radius r and for the volume $V = \frac{4\pi r^3}{3}$ of a sphere with radius r.

The circumference was approximated by piecewise linear curves to find its length as the limit of the length of approximate piecewise linear curves.

To find the volume of the sphere he approximated the sphere by a pile of truncated cones.

1.10 Infinite Series

In Chapter 12 of the *Tetsujutsu Sankei*, Takebe found the Taylor series expansion of the inverse trigonometric function

$$f(t) = (\arcsin t)^2$$

and two approximation formulas of $f(t)$ by rational functions.

The meaning of these formulas was explained in detail in Morimoto-Ogawa [4].

2 Chapter 9 of the *Tetsujutsu Sankei*

In this section we shall examine Chapter 9 of the *Tetsujutsu Sankei*, which is entitled "Investigating Numbers Stemming from the Decomposition" and the first chapter in the 3rd Part "Four Examples on Numbers" of the monograph.

This chapter treats numerical integration and is composed of 4 sections. The 1st section is an introduction, where Takebe states his understanding about the numerical calculation. In the following sections, Takebe states two partitioning methods, one to find the circumference of a circle (the 2nd section) and the other, the volume of a sphere (the 3rd section). Then in the last 4th section, he examines the merits and demerits of these two methods in relationship to the natural attributes of the respective objects (circle or sphere.)

We shall present an English translation section by section and provide with some mathematical notes to understand succinct statements of Takebe.

2.1 Introductory Section

The first introductory section reads as follows:

> If we want to investigate according to principles, there is the rule of element placement, which unifies all the procedures.
>
> If we want to investigate according to numbers, there is no way other than the **procedure of decomposition**; furthermore, there is no definite rule and the paths to a solution are different according to thousands of rules. 31a
>
> This means, the **[procedure of] decomposition** is the basis of determining numbers and discerning principles, the way of investigation, and the method to find rules and procedures. Therefore, if we decompose according to the **form and attribute** and investigate deeply to determine numbers, we surely understand the rule and procedure. In this manner, we state its meaning and witness its importance.

Takebe proposes here the **procedure of decomposition** as a basis of numerical calculation.

2.2 Circumference of a Circle

The 2nd part reads as follows:

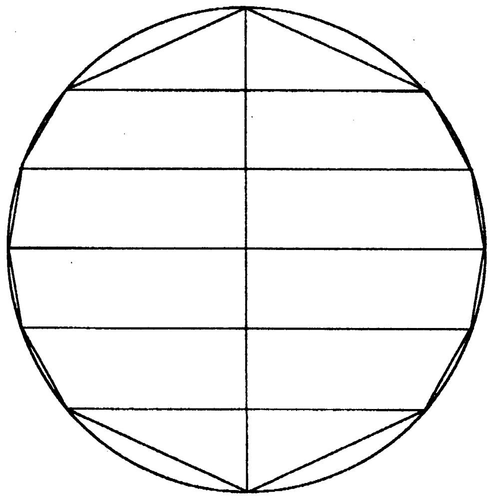
> If he who decomposes the circumference of a circle cuts the diameter equally and horizontally into thin slices, seeks the [length of the] right and left oblique chords cut by the horizontal lines and adds the oblique chords to seek the [approximate] circular circumference, then the parts of circumference are not equal even if he cuts the diameter equally.

Therefore, if he seeks the circumference doubling the sections of the diameter, these numbers being disobedient to the attribute, he stagnates in determining 31b
the extreme number and never obtain a basis to understand the attribute of circle.

On the other hand, when he cuts the circumference into the four angular forms [i.e., by an inscribed square] and further doubling angles [i.e., forming an inscribed octagon, etc.], the circumference is cut into equal length and the numbers are obedient to the attribute of circumference.

Therefore, doubling the number of angles and seeking the angular circumferences at each step, by the repeated application of the procedure of incremental divisor he can determine the extreme number rapidly and obtain a basis to understand the attribute of a circle.

Method of equal division of the diameter Mark $n-1$ points of a radius which divide it into n equal segments. Draw chords perpendicular to the radius through these $n-1$ points, and join consecutive points on the circle with chords.



The length of this piecewise linear curve Γ can be calculated by what Takebe calls the procedure of the right-angled triangle (i.e., Pythagoras' Theorem.) Because the length of the k-th half chord perpendicular to the radius is given by $rh_k = r\sqrt{1-(k/n)^2}$, the length of the k-th chord of Γ is equal to $r\sqrt{(1/n)^2+(h_k-h_{k-1})^2}$. The chords which approximate the semicircle come in pairs (left and right), so the n-th approximation of the full circumference is given by

$$S_n = 4r\sum_{k=1}^{n}\sqrt{(1/n)^2+(h_k-h_{k-1})^2}.$$

Doubling the partitioning number $n = 2, 4, 8, \cdots$, we obtain the following values with $r = 1/2$. (To apply recursive computation, Japanese mathematicians must have done the calculation in this way):

n	S_n	P.I.D.
2	3.03528	
4	3.1045	
8	3.12854	3.14134700
16	3.13699	3.14156089
32	3.13997	3.14158800
64	3.14102	3.14159191

(P.I.D. stands for the Procedure of Incremental Divisor.)

Procedure of Incremental Divisor In this chapter, Takebe uses extensively the so-called the procedure of incremental divisor, an acceleration method to find the limit of a sequence a_n.

If the given sequence satisfies $a_n = C_0 + C_1 s^n$, with $|s| < 1$ for $n = 1, 2, 3, \cdots$, the extreme value (i.e. the limit) is equal to C_0. In this special case, C_0 can be determined by a_1, a_2 and a_3:

$$C_0 = \frac{a_1 a_3 - a_2^2}{a_1 + a_3 - 2a_2}$$

If the first N terms of a sequence is given: a_n $(n = 1, 2, 3, ..., N)$, we form the first $N-2$ terms of a new sequence A_n $(n = 3, 4, 5, ..., N)$:

$$A_n = \frac{a_{n-2}a_n - a_{n-1}^2}{a_{n-2} + a_n - 2a_{n-1}}$$

It can be expected the limits of a_n and of A_n coincide and that the convergence of A_n is faster than that of a_n This procedure was called the procedure of incremental divisor.

Takebe recommended to apply repeatedly this procedure; that is, put further

$$B_n = \frac{A_{n-2}A_n - A_{n-1}^2}{A_{n-2} + A_n - 2A_{n-1}}, \quad n = 5, 6, \cdots, N-4.$$

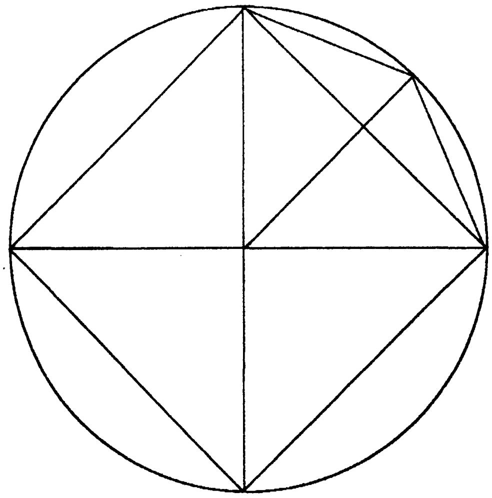
Then B_n converges to the limit C_0 of the sequence a_n faster, and so on.

This statement can be verified if the original sequence has the following form:

$$a_n = C_0 + C_1 s^n + C_2 s^{2n} + \cdots, \quad n = 1, 2, 3, \cdots$$

with some $|s| < 1$.

Method of equal division of the circle Divide the circle equally into 4 parts and connect the dividing points to obtain the inscribed square. Then the length of a side is equal to $a_4 = \sqrt{2}r$ and the length of the circumference of the inscribed regular square is equal to $4a_4 = 4\sqrt{2}r$. By the procedure of the right angled triangle, the length of a side of the inscribed regular octagon a_8 is given as follows: $a_8 = \sqrt{(r - \sqrt{r^2 - (a_4/2)^2})^2 + (a_4/2)^2}$. This relation holds in general.



Let a_n be the length of a side of the inscribed regular n-gone. Then we have

$$a_{2n} = \sqrt{(r - \sqrt{r^2 - (a_n/2)^2})^2 + (a_n/2)^2}.$$

If we put $a_2 = 2r$, this holds even for $n = 2$. Therefore, if we know the length of the circumference of the inscribed square, we can calculate, recursively, the length of the circumference of the inscribed regular octagon, 16-gon, 32-gon, 64-gon $\cdots$. The numerical calculation with $r = 1/2$ gives us

n	na_n	P.I.D.
2	2.000000	
4	2.828443	
8	3.061467	3.15268277
16	3.121445	3.14223140
32	3.136548	3.14163181
64	3.134033	3.14159509

Comparison of two methods Takebe considers the latter method to be more natural for a circle. But the numerical calculation by computer shows that there are no significant difference between them. We are not sure if Takebe really executed the former calculation or not. In fact, a Japanese mathematician of the late *Edo* period calculated the length of circular length using the former method.

2.3 Volume of a Sphere

The 3rd part reads as follows:

> He who decomposes the volume of a ball, slices the diameter of the ball equally and makes each slice into the shape of a circular platform.

Because the sum of widths of these slices is the sagitta of a arc, we can calculate the chord of the arc, which we take as the diameters of the upper and the lower ends of the platform; the width of the slice is the height of the platform.

By the procedure to seek the volume of a circular platform, one finds the volume of each slice and summing these slices forms the cut out volume. 32a

(If he omits the circular coefficient in seeking the volume of a platform, he can obtain the volume of a square platform.)

Further, doubling the number of slices and seeking the cut out volume at each step, investigating the obtained numbers to determine the incremental divisors, according to the procedure, we find the extreme number of the true volume.

Because this does not disobey the principle of volume seeking, he does not stagnate in determining the extreme number.

But further investigating deeply, we find that the procedure to find the volume of a platform seems good as a principle but the numbers do not converge well.

Therefore, multiplying the sum of the square of the upper radius and the square of the lower radius by the height, and halving this to form the volume of the tubular slices and adding them up, we form the cut out volume of accumulated tubes.

If, doubling the number of slices and seeking the polyhedral volumes, we apply the procedure of incremental divisor to determine the extreme number, we can find the extreme number rapidly even with a very small number of slices. 32b

Truly it is not indeed the procedure to find the volume of a platform to find the volume of a tabular slice.

This is a **miraculous procedure** in the decomposition of the volume of a ball and follows the attribute of the decomposition of the volume of ball.

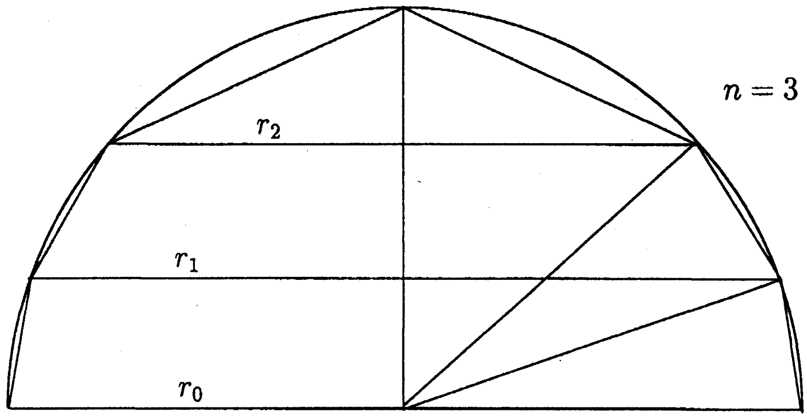
Volume of a circular platform The formula for the volume of a circular platform is quoted here. A circular platform is the cone truncated by a plane perpendicular to the axis. Let r_1 be the radius of the bottom, r_2 that of the top, h the height. Then the volume is given by

$$V = \frac{\pi h}{3}(r_1^2 + r_1 r_2 + r_2^2).$$

If $r_2 = 0$, it coincides with the volume of the circular cone $V = \frac{\pi h r_1^2}{3}$; if $r_1 = r_2$, it coincides with the volume of a cylinder.

Approximation by the superposed circular platforms Divide the radius of the sphere into n segments. Because the radius of the small circle perpendicular to the axis and passing through the k-th division point is given $r_k = r\sqrt{1-(k/n)^2}$, the volume of the k-th circular platform inscribed in the sphere is given by

$$V_k = \frac{\pi r}{3n}(r_{k-1}^2 + r_{k-1}r_k + r_k^2).$$



Therefore, the volume of the hemisphere is approximated by

$$V(n) = \frac{\pi r}{3n}\sum_{k=1}^{n}(r_{k-1}^2 + r_{k-1}r_k + r_k^2).$$

Because the (approximate) value of π is known, we calculate numerically $V(n)/\pi$ with $r = 1$.

Along with $V(n)$, we also calculate numerically, with $r = 1$, the following

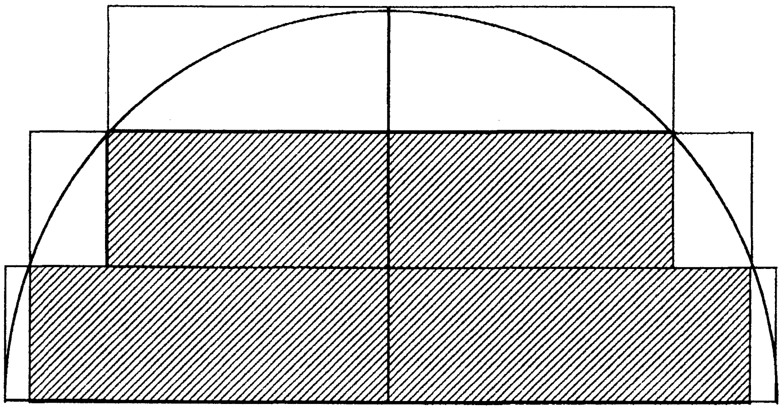
$$\bar{V}(n) = \frac{\pi r}{2n}\sum_{k=1}^{n}(r_{k-1}^2 + r_k^2).$$

The results are as follows:

n	$V(n)/\pi$	P.I.D.	$\bar{V}(n)/\pi$	P.I.D.
2	0.561004		0.625	
4	0.635799		0.65625	
8	0.657951	0.667271	0.664063	0.66667
16	0.664251	0.666754	0.666016	0.66667
32	0.666005	0.666682	0.666504	0.66667
64	0.666487	0.66667	0.666626	0.66667

As evident from this numerical calculation, $V(n)/\pi$ converges to the extreme value 2/3 sufficiently fast (if we use the Procedure of Incremental Divisor, this convergence becomes faster.) But $\bar{V}(n)/\pi$ converges much faster to the extreme value (if we use the Procedure of Incremental Divisor the third term gives an accurate approximation.) Although a geometrical meaning cannot be given to $\bar{V}(n)$, this gives a very accurate approximation. Takebe praised the latter approximation saying this was a "miraculous procedure".

Explanation of "miraculous procedure"



Approximating the hemisphere by the circumscriptive circular cylinders, we obtain

$$U(n) = \frac{\pi r}{n}\sum_{k=1}^{n} r_{k-1}^2.$$

This gives an excessive approximation of the volume of the hemisphere. Approximating the hemisphere by inscribed circular cylinders, we obtain

$$W(n) = \frac{\pi r}{n}\sum_{k=1}^{n} r_{k}^2.$$

This gives a deficient approximation of the volume of the hemisphere. By the inclusion relation of these objects, we have

$$W(n) < V(n) < U(n).$$

$\bar{V}(n)$ is nothing but the average of $W(n)$ and $U(n)$. By the Procedure of Piling (that is, the formulas $\sum_{k=1}^{n} k = n(n+1)/2$, $\sum_{k=1}^{n} k^2 = n(n+1)(2n+1)/6$, etc.), we can calculate exactly $U(n)$, $W(n)$, and consequently $\bar{V}(n)$:

$$U(n) = \frac{-1+3n+4n^2}{6n^2}, \quad W(n) = \frac{-1-3n+4n^2}{6n^2},$$

$$\bar{V}(n) = \frac{-1+4n^2}{6n^2} = \frac{2}{3} - \frac{1}{6n^2}.$$

Therefore, we have

$$\bar{V}(2^k) = \frac{2}{3} - \frac{1}{6}\left(\frac{1}{4}\right)^k.$$

In this case, the extreme value can be obtained exactly by the Procedure of Incremental Divisor. Therefore, only the first 3 terms of $\bar{V}(n)$ are necessary to obtain the extreme value 2/3.

This passage reveals that Takebe recognized this phenomenon through numerical calculation.

But it does not suggest that both Seki and Takebe had some notion of upper and lower integrals, used, for example, in today's Riemann integration.

2.4 Some Remarks

The last 4th part reads as follows:

> In the decomposition of the circle and related objects, we seek to conform with the **form and attribute** entirely, and never venture not to conform with it.
>
> If we cut into slices what should be whittled into shells, we are disobedient.
>
> When we cut according to the diameter what should be cut according the circular circumference, we are disobedient.
>
> When we cut horizontally what we should cut vertically, we are disobedient.
>
> When we do not obey the **form and attribute**, even when we can find the true number, we are slow in searching the extreme number and have difficulty in understanding the principle of [a solution] procedure. 33a
>
> In order to understand how to obey its **form and attribute**, we first discern the principle, determine numbers, and then, relying on the numbers, we investigate deeply and so attain understanding.
>
> Therefore, if we want to employ the [**procedure of**] **decomposition**, we should not concentrate only in seeking the true number nor lose sight of the principle which distinguishes obedience and disobedience.

[Closing Remark:] The above [**procedure of**] **decomposition** is the investigation of numbers by principles.

But once we start to investigate according to its **form and attribute**, we should recognize that numbers are to be investigated by numbers.

Takebe tried to understand the fast or slow convergences by the **form and attribute** of the decomposing method and that of the figure. If the **form and attributes** of these two are conformable, he said a good result could be expected.

Takebe also thought that, if the **form and attribute** of a mathematician is conformable to the attribute of the method of investigation, he could produce a good result. This kind of reasoning is stated in the concluding chapter of the *Tetsujutsu Sankei* dealing with the nature of mathematical investigation.

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