数理解析研究所講究録
1517巻 2006年 69-77

Induction and restriction homomorphisms between Bak's $S(\Gamma/\Lambda)$

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1 Introduction

Let G be a finite group and Y a compact, connected, oriented, smooth G-manifold of even dimension $n=2k\geq 6$.

C. T. C. Wall ([8]) formulated surgery obstruction groups $L_{n}(\mathbb{Z}[G])$ in terms of quadratic modules and automorphisms. In Wall's surgery theory, it is assumed that the G-action is free. A. Bak ([1]) generalized the notion of quadratic module by introducing form parameters. A. Bak and M. Morimoto ([4], [2]) generalized surgery theory, and defined surgery obstruction groups (Bak groups) $\mathrm{WQ}_{n}(A,\Lambda)$. Wall calculated $L_{n}(\mathbb{Z}[G])$ $(=\mathrm{WQ}_{n}(A, \min))$, in the case where G is a finite cyclic group or a dihedral group ([7]). We will use the result to study the Bak groups $\mathrm{WQ}_{n}(A,\Lambda)$.

A morphism of form ring $f:(A,-,\lambda,\Lambda)\rightarrow(A,-,\lambda,\Gamma)$ induces a K-group's exact sequence (See §2)

$$\mathrm{KQ}_{1}(A,\Lambda)\longrightarrow \mathrm{KQ}_{1}(A,\Gamma)\longrightarrow \mathrm{KQ}_{0}(f)\longrightarrow \mathrm{KQ}_{0}(A,\Lambda)\longrightarrow \mathrm{KQ}_{0}(A,\Gamma)\longrightarrow 0.$$

Bak has shown that $\mathrm{KQ}_{0}(f)$ is isomorphic to Abelian group $S(\Gamma/\Lambda)$ calculated easily. Additionally, surgery obstruction groups $\mathrm{WQ}_{n}(A,\Lambda)$ are the quotient group of $\mathrm{KQ}_{n}(A,\Lambda)$. So we study the group $S(\Gamma/\Lambda)$ in this paper.

In the study of G-manifold, it is important to judge whether a given surgery obstruction is zero or not. Because of difficulty in calculation of K-groups, it is effective to evaluate the images of induction homomorphisms and restriction homomorphisms of the surgery obstruction. That is, the induction theory of Dress's type is effective. This motivates to prove the next theorem.

Theorem 1.1. *$H\mapsto S(\Gamma_{H}/\Lambda_{H})$ is a Mackey functor.*

We study the groups $S(\Gamma_{G}/\Lambda_{G})$ in the case $G=A_{5}$, the alternating group on the five letters $\{1,2,3,4,5\}$, and S_{5}, the symmetric group on the five letters.

A complete set of representatives for conjugacy classes of subgroups of $G=A_{5}$ is

$$\mathcal{R}_{A_{5}}=\{\{e\},\ C_{2},\ C_{3},\ D_{4},\ C_{5},\ D_{6},\ D_{10},\ A_{4},\ A_{5}\},$$

where $\{e\}$ is the unit group, C_{n} $(n=2,\ 3,\ 5)$ are the cyclic groups of order n, D_{2n} are the dihedral groups of order $2n$, and A_{4} is the alternating group on the four letters $\{1,2,3,4\}$.

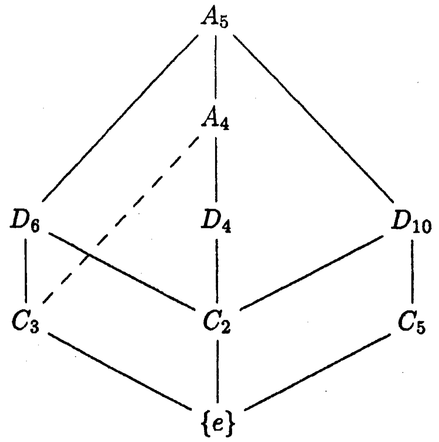

Figure 1: the representatives for conjugacy classes of subgroups of A_5

Let $\lambda = -1$ and let w be trivial. Then the following hold.

Theorem 1.2.

$$\begin{array}{lll} \mathrm{S}(\Omega_{D_4}/\min_{D_4}) & = \langle [a_1 \otimes b_1] \rangle_{\mathbb{Z}_2} & \cong \mathbb{Z}_2, \\ \mathrm{S}(\Omega_{D_6}/\min_{D_6}) & = \langle [a_2 \otimes b_2],\ [c_2 \otimes d_2] \rangle_{\mathbb{Z}_2} & \cong {\mathbb{Z}_2}^2, \\ \mathrm{S}(\Omega_{D_{10}}/\min_{D_{10}}) & = \langle [a_3 \otimes b_3],\ [e_3 \otimes f_3] \rangle_{\mathbb{Z}_2} & \cong {\mathbb{Z}_2}^2, \\ \mathrm{S}(\Omega_{A_5}/\min_{A_5}) & = \langle [a_4 \otimes b_4],\ [c_4 \otimes d_4],\ [e_4 \otimes f_4] \rangle_{\mathbb{Z}_2} & \cong {\mathbb{Z}_2}^3. \end{array}$$

where $\langle a_i, b_i \rangle \cong C_2, D_4$, $\langle c_i, d_i \rangle \cong D_6$, $\langle e_i, f_i \rangle \cong D_{10}$. *The natural homomorphism*

$$\mathrm{Ind} : \mathrm{S}(\Omega_{D_4}/\min_{D_4}) \oplus \mathrm{S}(\Omega_{D_6}/\min_{D_6}) \oplus \mathrm{S}(\Omega_{D_{10}}/\min_{D_{10}}) \longrightarrow \mathrm{S}(\Omega_{A_5}/\min_{A_5})$$

is surjective. Moreover,

$$\mathrm{Res}^{A_5}_{D_4}([a \otimes b]) = \begin{cases} [a \otimes b] & \text{if } \langle a, b \rangle \cong C_2, D_4, \\ 0 & \text{otherwise.} \end{cases}$$

$$\mathrm{Res}^{A_5}_{D_6}([a \otimes b]) = \begin{cases} [a \otimes b] & \text{if } \langle a, b \rangle \cong D_6, \\ 0 & \text{otherwise.} \end{cases}$$

$$\mathrm{Res}^{A_5}_{D_{10}}([a \otimes b]) = \begin{cases} [a \otimes b] & \text{if } \langle a, b \rangle \cong D_{10}, \\ 0 & \text{otherwise.} \end{cases}$$

and the natural homomorphism

$$\mathrm{Res} : \mathrm{S}(\Omega_{A_5}/\min_{A_5}) \longrightarrow \mathrm{S}(\Omega_{D_4}/\min_{D_4}) \oplus \mathrm{S}(\Omega_{D_6}/\min_{D_6}) \oplus \mathrm{S}(\Omega_{D_{10}}/\min_{D_{10}})$$

is injective.

A complete set of representatives for conjugacy classes of subgroups of $G = S_5$ is

$$\begin{aligned} \mathcal{R}_{A_5} = \{ & \{e\},\ C_2(\cong \langle (1,2) \rangle),\ C_2(\cong \langle (1,2)(3,4) \rangle),\ C_3,\ C_4, \\ & D_4(\cong \langle (1,2),(3,4) \rangle),\ D_4(\cong \langle (1,2)(3,4),(1,4)(2,3) \rangle),\ C_5, \\ & C_2 \times C_3,\ D_6(\cong \langle (1,2),(1,2,3) \rangle),\ D_6(\cong \langle (1,2)(3,4),(1,2,5) \rangle), \\ & D_8,\ D_{10},\ D_{12},\ F_{20},\ A_4,\ S_4,\ A_5,\ S_5 \}, \end{aligned}$$

where F_{20} Frobenius group of order 20,

$$F_{20} = \langle s, t \mid s^4 = t^5 = e, ts = st^2 \rangle.$$

Let $\lambda = -1$ and let w be trivial. Then the following hold.

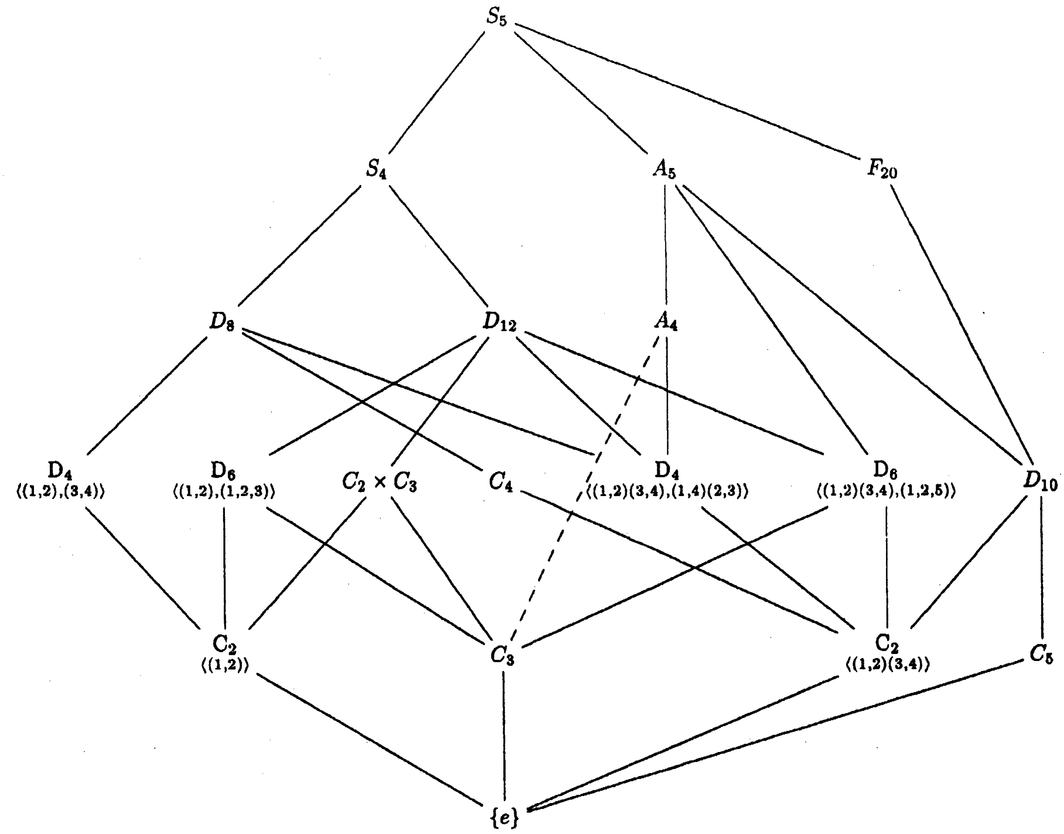

Figure 2: the representatives for conjugacy classes of subgroups of S_5

Theorem 1.3.

$$\begin{array}{lll} \mathrm{S}(\Omega_{D_8}/\min_{D_8}) & = \langle [a_1 \otimes b_1] \rangle_{\mathbb{Z}_2} & \cong \mathbb{Z}_2, \\ \mathrm{S}(\Omega_{D_{12}}/\min_{D_{12}}) & = \langle [a_2 \otimes b_2],\ [c_2 \otimes d_2] \rangle_{\mathbb{Z}_2} & \cong {\mathbb{Z}_2}^2, \\ \mathrm{S}(\Omega_{D_{10}}/\min_{D_{10}}) & = \langle [a_3 \otimes b_3],\ [e_3 \otimes f_3] \rangle_{\mathbb{Z}_2} & \cong {\mathbb{Z}_2}^2, \\ \mathrm{S}(\Omega_{S_5}/\min_{S_5}) & = \langle [a_4 \otimes b_4],\ [c_4 \otimes d_4],\ [e_4 \otimes f_4] \rangle_{\mathbb{Z}_2} & \cong {\mathbb{Z}_2}^3. \end{array}$$

where, $\langle a_i, b_i \rangle \cong C_2, D_4, D_8$*,* $\langle c_i, d_i \rangle \cong D_6, D_{12}$*,* $\langle e_i, f_i \rangle \cong D_{10}$. *The natural homomorphism*

$$\mathrm{Ind} : \mathrm{S}(\Omega_{D_8}/\min_{D_8}) \oplus \mathrm{S}(\Omega_{D_{12}}/\min_{D_{12}}) \oplus \mathrm{S}(\Omega_{D_{10}}/\min_{D_{10}}) \longrightarrow \mathrm{S}(\Omega_{S_5}/\min_{S_5})$$

is surjective. In addition,

$$\mathrm{Res}^{S_5}_{D_8}([a \otimes b]) = \begin{cases} [a \otimes b] & \text{if } \langle a, b \rangle \cong C_2, D_4, D_8, \\ 0 & \text{otherwise.} \end{cases}$$

$$\mathrm{Res}^{S_5}_{D_{12}}([a \otimes b]) = \begin{cases} [a \otimes b] & \text{if } \langle a, b \rangle \cong D_6, D_{12}, \\ 0 & \text{otherwise.} \end{cases}$$

$$\mathrm{Res}^{S_5}_{D_{10}}([a \otimes b]) = \begin{cases} [a \otimes b] & \text{if } \langle a, b \rangle \cong D_{10}, \\ 0 & \text{otherwise.} \end{cases}$$

and the natural homomorphism

$$\mathrm{Res} : \mathrm{S}(\Omega_{S_5}/\min_{S_5}) \longrightarrow \mathrm{S}(\Omega_{D_8}/\min_{D_8}) \oplus \mathrm{S}(\Omega_{D_{12}}/\min_{D_{12}}) \oplus \mathrm{S}(\Omega_{D_{10}}/\min_{D_{10}})$$

is injective.

2 Form parameter

We recall the definition of form parameter ([1, p.5]). Let A be a ring with identity and $- : A \to A$ an *involution* on A, namely $-$ is a bijection such that

(1) $\overline{\overline{a}} = a$,

(2) $\overline{a+b} = \overline{a} + \overline{b}$,

(3) $\overline{ab} = \overline{b}\overline{a}$,

(4) $\overline{1} = 1$

for all $a, b \in A$. Let λ be an element of Center(A) such that $\lambda\overline{\lambda} = 1$. This element λ is called a *symmetry* of A. Then a *form parameter* Λ on A is defined to be an additive subgroup of A such that

(Λ1) $\{a - \lambda\overline{a} \mid a \in A\} \subseteq \Lambda \subseteq \{a \in A \mid a = -\lambda\overline{a}\}$,

(Λ2) $a\Lambda\overline{a} \subseteq \Lambda \quad (\forall a \in A)$.

A quadruple $\boldsymbol{A} = (A, -, \lambda, \Lambda)$ is called a *form ring.*

Let Λ and Γ denote form parameters on A such that $\Lambda \subset \Gamma$. A morphism of form ring $f : (A, -, \lambda, \Lambda) \to (A, -, \lambda, \Gamma)$ induces an exact sequence

$$\mathrm{KQ}_1(A,\Lambda) \longrightarrow \mathrm{KQ}_1(A,\Gamma) \longrightarrow \mathrm{KQ}_0(f) \longrightarrow \mathrm{KQ}_0(A,\Lambda) \longrightarrow \mathrm{KQ}_0(A,\Gamma) \longrightarrow 0.$$

For the details, see [1, Theorem 6.20].

Let A be the integral group ring $\mathbb{Z}[G]$ of a finite group G, $w : G \to \{-1, 1\}$ a homomorphism, and $- : \mathbb{Z}[G] \to \mathbb{Z}[G]$ the involution associated with w, namely

$$\overline{\sum_{g\in G} r_g g} = \sum_{g\in G} w(g) r_g g^{-1},$$

for $r_g \in \mathbb{Z}$. Let $\lambda = 1$ or -1, and

$$X = \{g \in G \mid g^2 = 1, g = -\lambda\overline{g}\}.$$

Let $Y \subset X$ be a G-invariant subset with respect to conjugations. Let Λ be the form parameter of $R[G]$ given by

$$\Lambda = \Big\{\sum_{g\in Y} r_g g \mid r_g \in \mathbb{Z}\Big\} + \{x - \lambda\overline{x} \mid x \in \mathbb{Z}[G]\}.$$

We know the Burnside ring $\Omega(G)$ acts on the groups

$$\mathrm{KQ}_1(\mathbb{Z}[G],\Lambda),\ \mathrm{KQ}_1(\mathbb{Z}[G],\Gamma),\ \mathrm{KQ}_0(\mathbb{Z}[G],\Lambda),\ \text{and}\ \mathrm{KQ}_0(\mathbb{Z}[G],\Gamma).$$

In this article, we prove the Burnside ring $\Omega(G)$ also acts on $\mathrm{KQ}_0(f)$.

Let $a \mapsto ga\overline{g}$ (resp. $a \mapsto \overline{g}ag$) be a left (resp. right) action of A on Γ/Λ. A. Bak defined the group

$$\mathrm{S}(\Gamma/\Lambda) = (\Gamma/\Lambda \otimes_A \Gamma/\Lambda)/\{a \otimes b - b \otimes a, a \otimes b - a \otimes ba\overline{b}\}\ ([1, \text{p.191}]).$$

We quote the following lemmas from [1].

Lemma 2.1. $\mathrm{S}(\Gamma/\Lambda)$ *is a* $\mathbb{Z}_2$*-module.*

Lemma 2.2 ([1] Corollary 6.22, Throem 11.2). $\mathrm{KQ}_0(f) \cong \mathrm{S}(\Gamma/\Lambda)$.

Let H be a subgroup of G and $Y_H = Y \cap H$. We pay our attention to the form parameters on $\mathbb{Z}[H]$

$$\begin{aligned}
\Lambda_H &= \{x - \lambda\overline{x} \mid x \in \mathbb{Z}[H]\} + \Big\{\sum_{g\in Y_H} r_g g \mid r_g \in \mathbb{Z}\Big\},\\
\max\nolimits_H &= \{x - \lambda\overline{x} \mid x \in \mathbb{Z}[H]\} + \Big\{\sum_{g\in X_H} r_g g \mid r_g \in \mathbb{Z}\Big\},\\
\min\nolimits_H &= \{x - \lambda\overline{x} \mid x \in \mathbb{Z}[H]\},\\
\Omega_H &= \{x - \lambda\overline{x} \mid x \in \mathbb{Z}[H]\} + \Big\{\sum_{g\in X_H, g\neq e} r_g g \mid r_g \in \mathbb{Z}\Big\}.
\end{aligned}$$

3 w-Mackey Functor

We begin this section with recalling several concepts related to w-Mackey functor. Let G be a finite group and let $\mathcal{S}(G)$ denote the set of all subgroups of G. We define a category $\mathfrak{G}(=\mathfrak{G}(G))$ as follows. The set $\mathrm{Obj}(\mathfrak{G})$ is $\mathcal{S}(G)$. $\mathrm{Morph}_{\mathfrak{G}}(H,K)$ is the set of all triples (H,g,K) where $g\in G$ such that $gHg^{-1}\subseteq K$. The composition of morphisms is given by $(K,g',L)\circ(H,g,K)=(H,g'g,L)$. Let $\mathfrak{A}$ denote the category whose objects are Abelian groups and whose morphisms are group homomorphisms.

A *bifunctor* $M=(M^*,M_*):\mathfrak{G}\to\mathfrak{A}$ is a pair consisting of a contravariant functor $M^*:\mathfrak{G}\to\mathfrak{A}$ and covariant functor $M_*:\mathfrak{G}\to\mathfrak{A}$ such that $M^*(H)=M_*(H)$ for all $H\in\mathcal{S}(G)$.

Proposition 3.1 **([5] § 2).** *Let $M:\mathfrak{G}\to\mathfrak{A}$ be a bifunctor satisfying $M^*((gHg^{-1},g^{-1},H))=M_*((H,g,gHg^{-1}))$ for all $H\in\mathcal{S}(G)$ and $g\in G$. The Burnside ring $\Omega(G)$ canonically acts on $M(G)$ if and only if*

$$M^*((G,g,G))\circ M_*((H,e,G))\circ M^*((H,e,G))=M_*((H,e,G))\circ M^*((H,e,G))\circ M^*((G,g,G)).$$

Definition 3.2 **([3] Definition 2.3).** A bifunctor $M:\mathfrak{G}\to\mathfrak{A}$ is called a *w-Mac functor* if the following conditions are fulfilled.

(1) $M_*((H,g,gHg^{-1}))=M^*((gHg^{-1},g^{-1},H))$ for all $H\in\mathcal{S}(G)$ and $g\in G$,

(2) $M_*((H,h,H))=w(h)\,\mathrm{id}_{M(H)}$ (hence $M^*((H,h,H))=w(h)\,\mathrm{id}_{M(H)}$) for all $H\in\mathcal{S}(G)$ and $h\in H$.

Definition 3.3 **([3] Definition 2.4).** A w-Mac functor $M:\mathfrak{G}\to\mathfrak{A}$ is called a *w-Mackey functor* if the following conditions are fulfilled.

$$\begin{aligned}M^*((K,e,L))\circ M_*((H,e,L)) = \bigoplus_{KgH\in K\backslash L/H} & M_*((K\cap gHg^{-1},e,K))\circ \\ & (w(g)M_*((H\cap g^{-1}Kg,g,K\cap gHg^{-1})))\circ M^*((H\cap g^{-1}Kg,e,H))\end{aligned}$$

for any $H,K\in\mathcal{S}(G)$.

A w-Mackey functor for trivial w is called a *Mackey functor.*

Let $\varphi_{(H,g,K)}:H\to K$ be the map defined by $\varphi_{(H,g,K)}(h)=ghg^{-1}$. Let $\mathrm{S}=(\mathrm{S}_*,\mathrm{S}^*):H\mapsto \mathrm{S}(\Gamma_H/\Lambda_H)$ be the bifunctor given as follows. The map

$$\mathrm{S}_*(H,g,K):\mathrm{S}(\Gamma_H/\Lambda_H)\longrightarrow\mathrm{S}(\Gamma_K/\Lambda_K)$$

is defined by

$$[a\otimes b]\longmapsto[\widetilde{\varphi}(a)\otimes\widetilde{\varphi}(b)]$$

where $\widetilde{\varphi}(\sum_{h\in H}r_hh)=\sum_{h\in H}r_h\varphi_{(H,g,K)}(h)$, for $r_h\in\mathbb{Z}$. Let $\{k_1,\dots,k_m\}$, $m=|K|/|H|$, be a complete set of representatives of $K/\varphi_{(H,g,K)}(H)$. The map

$$\mathrm{S}^*(H,g,K):\mathrm{S}(\Gamma_K/\Lambda_K)\longrightarrow\mathrm{S}(\Gamma_H/\Lambda_H)$$

is defined by

$$[a\otimes b]\longmapsto\sum_{i=1}^{m}[(g^{-1}(k_i^{-1}ak_i)g)_H\otimes((g^{-1}(k_i^{-1}bk_i)g)_H]$$

where $(\sum_{g\in G}r_gg)_H=\sum_{g\in H}r_gg$ for $r_g\in\mathbb{Z}$. In particular,

$$\begin{aligned}&\mathrm{Ind}_H^K([a\otimes b])=\mathrm{S}_*(H,e,K)([a\otimes b])=[a\otimes b],\\&\mathrm{Res}_H^K([a\otimes b])=\mathrm{S}^*(H,e,K)([a\otimes b])=\sum_{i=1}^{m}[(k_i^{-1}ak_i)_H\otimes(k_i^{-1}bk_i)_H],\\&\mathrm{S}_*(H,g,gHg^{-1})([a\otimes b])=\mathrm{S}^*(gHg^{-1},g^{-1},H)([a\otimes b])=[gag^{-1}\otimes gbg^{-1}].\end{aligned}$$

Theorem 3.4. *The Burnside ring* $\Omega(G)$ *canonically acts on* $\mathrm{KQ}_0(f)$.

Proof. The theorem follows from Lemma 2.2 and Proposition 3.1. □

Proof of Theorem 1.1. This bifunctor clearly satisfies Conditions (1) and (2) in Definition 3.2. In order to check the equality in Definition 3.3, let $[a\otimes b] \in \mathrm{S}(\Gamma_H/\Lambda_H)$. Then

$$\begin{aligned}
&\mathrm{S}^*((K,e,L))\circ \mathrm{S}_*((H,e,L))([a\otimes b])\\
&=\mathrm{S}^*((K,e,L))([a\otimes b])\\
&=\sum_{i=1}^{m}[(l_i^{-1}al_i)_K\otimes (l_i^{-1}bl_i)_K],
\end{aligned}$$

where $\{l_1,\dots,l_m\}$, $m=|L/K|$, is a complete set of representatives of L/K. Thus

$$\begin{aligned}
&\bigoplus_{KgH\in K\backslash L/H} S_*((K\cap gHg^{-1},e,K))\circ S_*((H\cap g^{-1}Kg,g,K\cap gHg^{-1}))\\
&\qquad\circ S^*((H\cap g^{-1}Kg,e,H))([a\otimes b])\\
=&\bigoplus_{KgH\in K\backslash L/H} S_*((K\cap gHg^{-1},e,K))\circ S_*((H\cap g^{-1}Kg,g,K\cap gHg^{-1}))\\
&\qquad\Big(\sum_{j=1}^{n}[(h_j^{-1}ah_j)_{H\cap g^{-1}Kg}\otimes (h_j^{-1}bh_j)_{H\cap g^{-1}Kg}]\Big)=(*)
\end{aligned}$$

where $\{h_1,\dots,h_m\}$, $n=|H/H\cap g^{-1}Kg|$, is a complete set of representatives of $H/H\cap g^{-1}Kg$,

$$\begin{aligned}
(*)=&\bigoplus_{KgH\in K\backslash L/H} S_*((K\cap gHg^{-1},e,K))\Big(\sum_{j=1}^{n}[(gh_j^{-1}ah_jg^{-1})_{gHg^{-1}\cap K}\otimes (gh_j^{-1}bh_jg^{-1})_{gHg^{-1}\cap K}]\Big)\\
=&\bigoplus_{KgH\in K\backslash L/H}\Big(\sum_{j=1}^{n}[(gh_j^{-1}ah_jg^{-1})_{gHg^{-1}\cap K}\otimes (gh_j^{-1}bh_jg^{-1})_{gHg^{-1}\cap K}]\Big)\\
=&\sum_{i=1}^{m}[(l_i^{-1}al_i)_K\otimes (l_i^{-1}bl_i)_K]\quad\text{where } l_i=h_jg^{-1}.
\end{aligned}$$

□

4 The proof of Theorem 1.2

We can prove Theorem 1.3 completely analogous by to Theorem 1.2. So, we prove here only Theorem 1.2.

Proof of Theorem 1.2.

Part 1.

(I) *Proof of* $\mathrm{S}(\Omega_{D_4}/\mathrm{min}_{D_4})\cong\mathbb{Z}_2$. Let $D_4=\langle\sigma,\tau\mid\sigma^2=\tau^2=(\tau\sigma)^2=e\rangle$. Then the elements of order 2 in D_4 are σ, τ, and $\tau\sigma$. Thus $\mathrm{S}(\Omega_{D_4}/\mathrm{min})=\langle\sigma\otimes\sigma,\sigma\otimes\tau,\sigma\otimes\tau\sigma,\tau\otimes\tau,\tau\otimes\tau\sigma,\tau\sigma\otimes\tau\sigma\rangle_{\mathbb{Z}_2}$. By definition of $\mathrm{S}(\Gamma/\Lambda)$, we have the following equalities

$$\begin{aligned}
\tau\otimes\sigma&=\tau\otimes\sigma\tau\sigma=\tau\otimes\tau,\\
\sigma\otimes\tau&=\sigma\otimes\tau\sigma\tau=\sigma\otimes\sigma,\\
\tau\otimes\tau\sigma&=\tau\otimes\tau\sigma\tau\tau\sigma=\tau\otimes\tau,\\
\tau\sigma\otimes\tau&=\tau\sigma\otimes\tau\tau\sigma\tau=\tau\sigma\otimes\tau\sigma,\\
\sigma\otimes\tau\sigma&=\sigma\otimes\tau\sigma\sigma\tau\sigma=\sigma\otimes\sigma.
\end{aligned}$$

$\mathrm{S}(\Omega_{D_4}/\mathrm{min}_{D_4})\neq 0$ can be easily verified. Thus, $\mathrm{S}(\Omega_{D_4}/\mathrm{min}_{D_4})\cong\mathbb{Z}_2$.

(II) *Proof of* $\mathrm{S}(\Omega_{D_6}/\min_{D_6}) \cong \mathbb{Z}_2{}^2$. Let $D_6 = \langle \sigma, \tau \mid \sigma^3 = \tau^2 = (\tau\sigma)^2 = e \rangle$. Then the elements of order 2 in D_6 are τ, $\tau\sigma$, and $\tau\sigma^2$. Thus, $\mathrm{S}(\Omega_{D_6}/\min_{D_6}) = \langle \tau\otimes\tau, \tau\otimes\tau\sigma, \tau\otimes\tau\sigma^2, \tau\sigma\otimes\tau\sigma, \tau\sigma\otimes\tau\sigma^2, \tau\sigma^2\otimes\tau\sigma^2\rangle_{\mathbb{Z}_2}$. By definition of $\mathrm{S}(\Gamma/\Lambda)$, we have the equalities

$$\tau\otimes\tau\sigma = \tau\otimes\tau\sigma\tau\tau\sigma = \tau\otimes\tau\sigma^2,$$
$$\tau\sigma\otimes\tau = \tau\sigma\otimes\tau\tau\sigma\tau = \tau\sigma\otimes\tau\sigma^2,$$

and

$$\tau\otimes\tau = \sigma^{-1}\tau\sigma\otimes\sigma^{-1}\tau\sigma = \tau\sigma^2\otimes\tau\sigma^2,$$
$$\tau\sigma\otimes\tau\sigma = \sigma^{-1}\tau\sigma\sigma\otimes\sigma^{-1}\tau\sigma\sigma = \tau\otimes\tau.$$

Thus, $\mathrm{Rank}_{\mathbb{Z}_2}(\mathrm{S}(\Omega_{D_4}/\min_{D_4})) \leq 2$.

Let $M = \langle \tau, \tau\sigma, \tau\sigma^2\rangle_{\mathbb{Z}_2}$. Since $X = \{\tau, \tau\sigma, \tau\sigma^2\}$, we define the homomorphism

$$\varphi : M\otimes_{\mathbb{Z}_2} M \longrightarrow \mathbb{Z}_2\oplus\mathbb{Z}_2,$$
$$\sum_{(x,y)\in X\times X} r_{(x,y)} x\otimes y \longmapsto \Big(\sum_{x=y,(x,y)\in X\times X} r_{(x,y)}, \sum_{x\neq y,(x,y)\in X\times X} r_{(x,y)} \Big).$$

Let $R = \langle a\otimes b - b\otimes a, a\otimes b - a\otimes ba\bar{b}, ga\bar{g}\otimes b - a\otimes \bar{g}bg\rangle_{\mathbb{Z}_2}$. By definition,

$$\mathrm{S}(\Omega_{D_6}/\min_{D_6}) = (M\otimes_{\mathbb{Z}_2} M)/R.$$

Since $R\subset \mathrm{Ker}\,\varphi$ holds, the homomorphism map

$$\overline{\varphi} : \mathrm{S}(\Omega_{D_6}/\min_{D_6}) \longrightarrow \mathbb{Z}_2\oplus\mathbb{Z}_2$$

is well-defined. Clearly φ is surjective, then $\overline{\varphi}$ is also surjective. Thus, $\mathrm{S}(\Omega_{D_6}/\min_{D_6}) \cong \mathbb{Z}_2{}^2$.

(III) *Proof of* $\mathrm{S}(\Omega_{D_{10}}/\min_{D_{10}}) \cong \mathbb{Z}_2{}^2$. Let $D_{10} = \langle \sigma, \tau \mid \sigma^5 = \tau^2 = (\tau\sigma)^2 = e\rangle$. Then the elements of order 2 in D_6 are τ, $\tau\sigma$, $\tau\sigma^2$, $\tau\sigma^3$, $\tau\sigma^4$. Thus, $\mathrm{S}(\Omega_{D_{10}}/\min_{D_{10}}) = \langle \tau\otimes\tau, \tau\otimes\tau\sigma, \tau\otimes\tau\sigma^2, \tau\otimes\tau\sigma^3, \tau\otimes\tau\sigma^4, \tau\sigma\otimes\tau\sigma, \tau\sigma\otimes\tau\sigma^2, \tau\sigma\otimes\tau\sigma^3, \tau\sigma\otimes\tau\sigma^4, \tau\sigma^2\otimes\tau\sigma^2, \tau\sigma^2\otimes\tau\sigma^3, \tau\sigma^2\otimes\tau\sigma^4, \tau\sigma^3\otimes\tau\sigma^3, \tau\sigma^3\otimes\tau\sigma^4, \tau\sigma^4\otimes\tau\sigma^4\rangle_{\mathbb{Z}_2}$. By definition of $\mathrm{S}(\Gamma/\Lambda)$, we have the equalities

$$\tau\otimes\tau\sigma = \tau\otimes\tau\sigma\tau\tau\sigma = \tau\otimes\tau\sigma^2 = \tau\otimes\tau\sigma^2\tau\tau\sigma^2 = \tau\otimes\tau\sigma^4 = \tau\otimes\tau\sigma^4\tau\tau\sigma^4 = \tau\otimes\tau\sigma^3,$$

$$\tau\sigma\otimes\tau = \tau\sigma\otimes\tau\tau\sigma\tau = \tau\sigma\otimes\tau\sigma^4 = \tau\sigma\otimes\tau\sigma^4\tau\sigma\tau\sigma^4 = \tau\sigma\otimes\tau\sigma^2 = \tau\sigma\otimes\tau\sigma^2\tau\sigma\tau\sigma^2 = \tau\sigma\otimes\tau\sigma^3,$$

$$\tau\sigma^2\otimes\tau = \tau\sigma^2\otimes\tau\tau\sigma^2\tau = \tau\sigma^2\otimes\tau\sigma^3 = \tau\sigma^2\otimes\tau\sigma^3\tau\sigma^2\tau\sigma^3 = \tau\sigma^2\otimes\tau\sigma^4 = \tau\sigma^2\otimes\tau\sigma^4\tau\sigma^2\tau\sigma^4 = \tau\sigma^2\otimes\tau\sigma,$$

$$\tau\sigma^3\otimes\tau = \tau\sigma^3\otimes\tau\tau\sigma^3\tau = \tau\sigma^3\otimes\tau\sigma^2 = \tau\sigma^3\otimes\tau\sigma^2\tau\sigma^3\tau\sigma^2 = \tau\sigma^3\otimes\tau\sigma = \tau\sigma^3\otimes\tau\sigma\tau\sigma^3\tau\sigma = \tau\sigma^3\otimes\tau\sigma^4,$$

and

$$\tau\otimes\tau = \sigma^{-1}\tau\sigma\otimes\sigma^{-1}\tau\sigma = \tau\sigma^2\otimes\tau\sigma^2,$$
$$\tau\sigma^2\otimes\tau\sigma^2 = \sigma^{-1}\tau\sigma^2\sigma\otimes\sigma^{-1}\tau\sigma^2\sigma = \tau\sigma^4\otimes\tau\sigma^4,$$
$$\tau\sigma^4\otimes\tau\sigma^4 = \sigma^{-1}\tau\sigma^4\sigma\otimes\sigma^{-1}\tau\sigma^4\sigma = \tau\sigma\otimes\tau\sigma,$$
$$\tau\sigma\otimes\tau\sigma = \sigma^{-1}\tau\sigma\sigma\otimes\sigma^{-1}\tau\sigma\sigma = \tau\sigma^3\otimes\tau\sigma^3.$$

Thus, $\mathrm{Rank}_{\mathbb{Z}_2}(\mathrm{S}(\Omega_{A_5}/\min_{A_5})) \leqq 2$.

Similarly to (II), we can show $\mathrm{S}(\Omega_{D_{10}}/\min_{D_{10}}) \cong \mathbb{Z}_2{}^2$.

(IV) *Proof of* $\mathrm{S}(\Omega_{A_5}/\min_{A_5}) \cong \mathbb{Z}_2{}^3$. By direct computation, we can show that $\mathrm{S}(\Omega_{A_5}/\min_{A_5})$ is generated by the three elements $[a\otimes b]$, $[c\otimes d]$, and $[f\otimes h]$, where $\langle a,b\rangle \cong C_2, D_4$, $\langle c,d\rangle \cong D_6$, $\langle f,h\rangle \cong D_{10}$. Thus, $\mathrm{Rank}_{\mathbb{Z}_2}(\mathrm{S}(\Omega_{A_5}/\min_{A_5})) \leqq 3$. Let M be the free $\mathbb{Z}_2$-module generated by all elements of order 2 in A_5. We define the linear map $\varphi : M\otimes_{\mathbb{Z}_2} M \to \mathbb{Z}_2\oplus\mathbb{Z}_2\oplus\mathbb{Z}_2$ by

$$a\otimes b \longmapsto (x, y, z)$$

where

$$x = \begin{cases} 1 & \text{if } \langle a, b\rangle \cong C_2, D_4, \\ 0 & \text{otherwise,} \end{cases}$$

$$y = \begin{cases} 1 & \text{if } \langle a, b\rangle \cong D_6, \\ 0 & \text{otherwise,} \end{cases}$$

$$z = \begin{cases} 1 & \text{if } \langle a, b\rangle \cong D_{10}, \\ 0 & \text{otherwise.} \end{cases}$$

Let $R = \langle a\otimes b - b\otimes a, a\otimes b - a\otimes ba\bar{b}, ga\bar{g}\otimes b - a\otimes \bar{g}bg\rangle_{\mathbb{Z}_2} \subset M \otimes_{\mathbb{Z}_2} M$. By definition,

$$\mathrm{S}(\Omega_{A_5}/\min_{A_5}) = (M \otimes_{\mathbb{Z}_2} M)/R.$$

Since $R \subset \operatorname{Ker}\varphi$ holds, the homomorphism map

$$\overline{\varphi} : \mathrm{S}(\Omega_{A_5}/\min) \longrightarrow \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$$

is well-defined. Clearly φ is surjective, hence $\overline{\varphi}$ is also surjective. Thus, $\mathrm{S}(\Omega_{A_5}/\min_{A_5}) \cong {\mathbb{Z}_2}^3$.

By the arguments above, the map

$$\mathrm{Ind} : \mathrm{S}(\Omega_{D_4}/\min_{D_4}) \oplus \mathrm{S}(\Omega_{D_6}/\min_{D_6}) \oplus \mathrm{S}(\Omega_{D_{10}}/\min_{D_{10}}) \longrightarrow \mathrm{S}(\Omega_{A_5}/\min_{A_5})$$

is surjective.

Part 2.

(I) *Proof of* $\mathrm{Res}^{A_5}_{D_4}([a\otimes b]) = \begin{cases} [a\otimes b] & \textit{if } \langle a, b\rangle \cong C_2, D_4, \\ 0 & \textit{otherwise} \end{cases}$.

The map $\mathrm{Res}^{A_5}_{D_4} : \mathrm{S}(\Omega_{A_5}/\min_{A_5}) \to \mathrm{S}(\Omega_{D_4}/\min_{D_4})$ is defined by

$$[a\otimes b] \longmapsto \sum_{i=1}^{m} [(g_i^{-1}ag_i)_{D_4} \otimes (g_i^{-1}bg_i)_{D_4}].$$

Direct computation shows that $\mathrm{S}(\Omega_{A_5}/\min_{A_5})$ is generated by the three elements $[a\otimes b]$, $[c\otimes d]$, $[f\otimes h]$ where $\langle a, b\rangle \cong C_2, D_4$, $\langle c, d\rangle \cong D_6$, $\langle f, h\rangle \cong D_{10}$.

(i) In the case $\langle a, b\rangle \cong D_4$, there exists an element $k \in A_5$ such that $k\langle a, b\rangle k^{-1} = D_4$. Let $a' = kak^{-1}$, $b' = kbk^{-1}$. Since $[a\otimes b] = [a'\otimes b']$, it follows that $\mathrm{Res}^{A_5}_{D_4}([a'\otimes b']) = \mathrm{Res}^{A_5}_{D_4}([a\otimes b])$. If $[(g_i^{-1}ag_i)_{D_4} \otimes (g_i^{-1}bg_i)_{D_4}] \neq 0$, then $g_i^{-1}a'g_i, g_i^{-1}b'g_i \in D_4$. Thus $a', b' \in g_iD_4g_i^{-1}$. Besides, $a', b' \in D_4$, so $g_i \in N_{A_5}(D_4) = A_4$. Therefore,

$$\mathrm{Res}^{A_5}_{D_4}([a\otimes b]) = (|A_4|/|D_4|)[a\otimes b] = 3[a\otimes b] = [a\otimes b].$$

(ii) In the case $\langle c, d\rangle \cong D_6$, if $[(g_i^{-1}cg_i)_{D_4} \otimes (g_i^{-1}dg_i)_{D_4}] \neq 0$, then $c, d \in g_iD_4g_i^{-1}$. Therefore,

$$\mathrm{Res}^{A_5}_{D_4}([c\otimes d]) = 0.$$

(iii) In the case $\langle f, h\rangle \cong D_{10}$, if $[(g_i^{-1}fg_i)_{D_4} \otimes (g_i^{-1}hg_i)_{D_4}] \neq 0$, then $f, h \in g_iD_4g_i^{-1}$. Therefore,

$$\mathrm{Res}^{A_5}_{D_4}([f\otimes h]) = 0.$$

(II) *Proof of* $\mathrm{Res}^{A_5}_{D_6}([a\otimes b]) = \begin{cases} [a\otimes b] & \textit{if } \langle a, b\rangle \cong D_6, \\ 0 & \textit{otherwise} \end{cases}$. We can prove it similarly to (I).

(III) *Proof of* $\mathrm{Res}^{A_5}_{D_{10}}([a\otimes b]) = \begin{cases} [a\otimes b] & \textit{if } \langle a, b\rangle \cong D_{10}, \\ 0 & \textit{otherwise} \end{cases}$. We can prove it similarly to (I).

By the arguments above, the map

$$\mathrm{S}(\Omega_{A_5}/\min_{A_5}) \xrightarrow{\mathrm{Res}} \mathrm{S}(\Omega_{D_4}/\min_{D_4}) \oplus \mathrm{S}(\Omega_{D_6}/\min_{D_6}) \oplus \mathrm{S}(\Omega_{D_{10}}/\min_{D_{10}})$$

is injective. □

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