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EXISTENCE OF A SPECIAL $\mathcal{P}$-MATCHED PAIR

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1. INTRODUCTION

Let G be a finite group. Let V and W be complex (resp. real) G-modules. Let $\mathcal{L}$ be a family of subgroups of G. We define an $\mathcal{L}$-free complex (resp. self-conjugate, resp. real) $\mathcal{P}$-matched pair of type 1 of G-module. Let $\mathcal{P}(G)$ be the set of all subgroups of G of prime power order, possibly 1. A G-module W is called *$\mathcal{L}$-free* if $\dim W^H = 0$ for any $H \in \mathcal{L}$, and G-modules V and W are called *$\mathcal{P}$-matched* if $\mathrm{Res}^G_P V$ and $\mathrm{Res}^G_P W$ are isomorphic for any $P \in \mathcal{P}(G)$.

Definition 1.1. *A pair (V, W) is called an $\mathcal{L}$-free complex (resp. real) $\mathcal{P}$-matched pair of type 1 of G-modules if it holds that*

(1) $\dim V^G = 1$,
(2) *$V - V^G$ and W are $\mathcal{L}$-free, and*
(3) *V and W are $\mathcal{P}$-matched.*

A pair (V, W) is called an $\mathcal{L}$-free self-conjugate $\mathcal{P}$-matched pair of type 1 of complex G-modules if V and W satisfy the above three condition and their characters take real numbers.

In the case when $\mathcal{L} = \mathcal{L}(G)$, an $\mathcal{L}$-free complex (resp. self-conjugate, resp. real) $\mathcal{P}$-*matched pair of type* 1 of G-modules is also called a *special $\mathcal{P}$-matched pair* of complex (resp. self-conjugate, resp. real) G-modules. If there exists an $\mathcal{L}$-free complex (resp. self-conjugate, resp. real) $\mathcal{P}$-matched pair of type 1 of G-modules, we say that G has an $\mathcal{L}$-free complex (resp. self-conjugate, resp. real) $\mathcal{P}$-matched pair of type 1 or equivalently G has a special $\mathcal{P}$-matched pair of complex (resp. self-conjugate, resp. real) G-modules.

Oliver ([6]) defined classes $\mathcal{M}_{\mathbb{C}}$, $\mathcal{M}_{\mathbb{C}+}$, and $\mathcal{M}_{\mathbb{R}}$ as follows. A finite group G lies in the class $\mathcal{M}_{\mathbb{C}}$, (resp. $\mathcal{M}_{\mathbb{C}+}$, resp. $\mathcal{M}_{\mathbb{R}}$) if G has a $\{G\}$-free complex (resp. self-conjugate, resp. real) $\mathcal{P}$-matched pair of type 1. He used them to classify closed manifolds which are appeared as the fixed point set of a disk with smooth G-action. $\mathcal{P}$-matched G-modules are often appeared when we discuss two tangential G-modules V and W of G-disks or G-spheres with exactly two fixed points. Let $\mathcal{L}(G)$ be the set of all subgroups of G which contain some Dress subgroup. If we want to apply Morimoto's surgery theory, we expect the assumption $\mathcal{L}(G)$-freeness, since it is convenient for some construction. We say that G lies in the class $\mathcal{L}_{\mathbb{C}}$, (resp. $\mathcal{L}_{\mathbb{C}+}$, resp. $\mathcal{L}_{\mathbb{R}}$) if G has a $\mathcal{L}(G)$-free complex (resp. self-conjugate, resp. real) $\mathcal{P}$-matched pair of type 1. The class $\mathcal{L}_{\mathbb{C}}$, (resp. $\mathcal{L}_{\mathbb{C}+}$, resp. $\mathcal{L}_{\mathbb{R}}$) is a subclass of $\mathcal{M}_{\mathbb{C}}$, (resp. $\mathcal{M}_{\mathbb{C}+}$, resp. $\mathcal{M}_{\mathbb{R}}$). In [5], a $\mathcal{L}(G)$-free real $\mathcal{P}$-matched pair of type 1 is called a special pair.

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We consider the following problem: Classify finite groups into $\mathcal{L}^c_{\mathbb{C}}$, $\mathcal{L}_{\mathbb{C}}$, $\mathcal{L}_{\mathbb{C}+}$ and $\mathcal{L}_{\mathbb{R}}$. Recall that Oliver obtained the classes $\mathcal{M}_{\mathbb{C}}$, $\mathcal{M}_{\mathbb{C}+}$ and $\mathcal{M}_{\mathbb{R}}$ are determined by an existence of an element, a subgroup, a sub-quotient group. In the section 2, we give a sufficient condition to lies in the class $\mathcal{L}_{\mathbb{C}}$, $\mathcal{L}_{\mathbb{C}+}$ and $\mathcal{L}_{\mathbb{R}}$ and we classify all symmetric groups. In the section 3, we classify all simple groups by using Oliver's result. In the section 4, we classify all projective general linear groups and all general linear groups.

2. $\mathcal{P}$-MATCHED PAIRS

Let G be a finite group. First we recall Oliver's result about the classes $\mathcal{M}_{\mathbb{C}}$, $\mathcal{M}_{\mathbb{C}+}$ and $\mathcal{M}_{\mathbb{R}}$. An element x of G is called *self-conjugate* if it is conjugate to its inverse, and called *real* if it is conjugate to its inverse by an involution (an element of order 2).

Proposition 2.1 ([6]). *There is a $\{G\}$-free complex (resp. self-conjugate) $\mathcal{P}$-matched pair of type 1 of G-modules exactly when there is an element not of prime power order, (resp. a self-conjugate element not of prime power order) of G. There is a $\{G\}$-free real $\mathcal{P}$-matched pair of type 1 of G-modules only if there is a subquotient group of G which has a real element not of prime power order.*

A cyclic group C_n of order n lies in $\mathcal{M}_{\mathbb{C}}$ if and only if n is not a power of a prime, and lies in $\mathcal{L}_{\mathbb{C}}$ if and only if n is divisible by at least 3 distinct primes. This is shown as follows.

Observation 2.2 (cf. [4, Example 1.5]). *Let p, q and r be positive integers > 1 which are coprime each other. For an integer k, let C_k be a cyclic group of order k. Then $R(C_{pqr})$ is isomorphic to $R(C_p)\otimes R(C_q)\otimes R(C_r)$. Take a nontrivial irreducible module ξ_p, (resp. ξ_q, resp. ξ_r) over C_p, (resp. C_q, resp. C_r). Consider the element*

$$(\mathbb{C}-\xi_p)(\mathbb{C}-\xi_q)+(\mathbb{C}-\xi_q)(\mathbb{C}-\xi_r)+(\mathbb{C}-\xi_p)(\mathbb{C}-\xi_r)-2(\mathbb{C}-\xi_p)(\mathbb{C}-\xi_q)(\mathbb{C}-\xi_r)$$
$$=(\mathbb{C}+2\xi_p\xi_q\xi_r)-(\xi_p\xi_q+\xi_q\xi_r+\xi_p\xi_r)$$

of $R(C_p)\otimes R(C_q)\otimes R(C_r)$. Set $U=\mathbb{C}+2\xi_p\xi_q\xi_r$ and $V=\xi_p\xi_q+\xi_q\xi_r+\xi_p\xi_r$. It is easy to see that $\xi_p\xi_q\xi_r$ and V are both $\mathcal{L}(C_{pqr})$-free. Thus the pair (U,V) is a $\mathcal{L}(C_{pqr})$-free complex $\mathcal{P}$-matched pair of type 1 of C_{pqr}-modules and $C_{pqr}\in\mathcal{L}_{\mathbb{C}}$.

Observation 2.3. *Let p, q be distinct prime powers which are coprime each other. We show that $C_{pq}\notin\mathcal{L}_{\mathbb{C}}$. Suppose there exists a complex $\mathcal{P}$-matched pair (U,V) of type 1 of C_{pq}-modules. In $R(C_{pq})$, we can express*

$$U-V=\mathbb{C}+\sum_{i=1}^{p-1}k_i\xi_p^i+\sum_{j=1}^{q-1}m_j\xi_q^j+\sum_{i=1}^{p-1}\sum_{j=1}^{q-1}n_{i,j}\xi_p^i\xi_q^j$$

for some integers k_i, m_j, $n_{i,j}$. Here ξ_p and ξ_q are nontrivial irreducible complex modules over C_p and C_q, respectively. If $\mathrm{Res}^{C_{pq}}_{C_p}(U-V)=0$, then $\sum_{j=1}^{q-1}m_j=-1$ and $\sum_{j=1}^{q-1}n_{i,j}=-k_i$ for each i. Thus that $\mathrm{Res}^{C_{pq}}_{P}(U-V)=0$ for each $P\in\mathcal{P}(C_{pq})$ yields that $\sum_{i=1}^{p-1}k_i=\sum_{j=1}^{q-1}m_j=-1$, $\sum_{i=1}^{p-1}n_{i,j}=-k_j$ for each j and $\sum_{j=1}^{q-1}n_{i,j}=-k_i$ for each i. In this case, we can express

$$U-V=\sum_{i=1}^{p-1}\sum_{j=1}^{q-1}n_{i,j}(\mathbb{C}-\xi_p^i)(\mathbb{C}-\xi_q^j).$$

On the other hand, if $(U-V)^{C_q} = \mathbb{C}$ as C_p-modules, then $k_i = 0$ for each i, and thus if (U, V) is $\mathcal{L}(C_{pq})$-free, then $k_i = m_j = 0$ for each i and each j. Therefore the above two conditions are contradiction. Thus C_{pq} has no $\mathcal{L}(C_{pq})$-free complex $\mathcal{P}$-matched pair of type 1.

We also show that there is no $\mathcal{L}(G)$-free complex $\mathcal{P}$-matched pair of type 1 if G is a dihedral group or a generalized quarternion group. Let $G = D_{2n}$ be a dihedral group of order $2n$ generated by a and b with relation $a^n = b^2 = 1$ and $bab = a^{-1}$. This group lies in $\mathcal{M}_{\mathbb{R}}$ if and only if n is not a power of a prime. The character table is as follows:

Character Table of D_{2n}, n odd

	1	$a^i (1 \le i \le \frac{n-1}{2})$	b
# of conj. cl.	1	2	n
ξ_0	1	1	1
ξ_1	1	1	-1
$\chi_j (1 \le j \le \frac{n-1}{2})$	2	$w^{ij} + w^{-ij}$	0

Character Table of D_{2n}, n even

	1	$a^i (1 \le i < \frac{n}{2})$	$a^{\frac{n}{2}}$	b	ab
# of conj. cl.	1	2	1	$\frac{n}{2}$	$\frac{n}{2}$
ξ_0	1	1	1	1	1
ξ_1	1	1	1	-1	-1
ξ_2	1	$(-1)^i$	$(-1)^{\frac{n}{2}}$	1	-1
ξ_3	1	$(-1)^i$	$(-1)^{\frac{n}{2}}$	-1	1
$\chi_j (1 \le j < \frac{n}{2})$	2	$w^{ij} + w^{-ij}$	$2(-1)^j$	0	0

where $w = exp(\pi\sqrt{-1}/n)$. Then χ_j is $\mathcal{L}(D_{2n})$-free. Since $\dim\chi_j$ is even, any $\mathcal{L}(D_{2n})$-free complex D_{2n}-module is of even dimension. Therefore there is no $\mathcal{L}(D_{2n})$-free complex $\mathcal{P}$-matched pair of type 1. Similarly we obtain that there is also no $\mathcal{L}(Q_{4n})$-free complex $\mathcal{P}$-matched pair of type 1 for a generalized quarternion group of order $4n$:

$$Q_{4n} = \langle x, y \mid x^{2n} = 1,\ y^2 = x^n,\ y^{-1}xy = x^{-1} \rangle$$

Character Table of Q_{4n}, n odd

	(1)	$(x^k)\ (1 \le k < n)$	(y)	(y^2)	(y^3)
# of conj.	1	2	n	1	n
$\tau_\ell\ (\ell = 0,1,2,3)$	1	$(-1)^{k\ell}$	$\sqrt{-1}^\ell$	$(-1)^\ell$	$(-\sqrt{-1})^\ell$
$\eta_\ell\ (1 \le \ell < n)$	2	$w^{k\ell} + w^{-k\ell}$	0	$2(-1)^\ell$	0

Character Table of Q_{4n}, n even

	(1)	(x^k) $(1 \le k < m)$	(y)	(y^2)	(yx)
# of conj.	1	2	n	1	n
$\tau_{\ell,j}$ $(\ell, j = 0, 1)$	1	$(-1)^{k\ell}$	$(-1)^j$	1	$(-1)^{\ell+j}$
η_ℓ $(1 \le \ell < n)$	2	$w^{k\ell} + w^{-k\ell}$	0	$2(-1)^\ell$	0

Here $w = \exp(\pi\sqrt{-1}/n)$. Note that $Q_{4n} \in \mathcal{M}_{\mathbb{R}}$ if n is not a power of a prime. Therefore $\mathcal{M}_{\mathbb{C}} \subset \mathcal{M}_{\mathbb{C}+} \subset \mathcal{M}_{\mathbb{R}}$ and $\mathcal{L}_{\mathbb{C}} \subset \mathcal{L}_{\mathbb{C}+} \subset \mathcal{L}_{\mathbb{R}}$ are different.

Let $\pi(G)$ be the set of all primes which divide the order of G.

$\lvert\pi(C_n)\rvert$	0, 1	≥ 2
D_{2n}	$\mathcal{M}_{\mathbb{C}}^c$	$\mathcal{M}_{\mathbb{R}} \setminus \mathcal{L}_{\mathbb{C}}$
Q_{4n}	$\mathcal{M}_{\mathbb{C}}^c$	$\mathcal{M}_{\mathbb{R}} \setminus \mathcal{L}_{\mathbb{C}}$

Now we discuss sufficient conditions to lie in the classes $\mathcal{L}_{\mathbb{C}}$, $\mathcal{L}_{\mathbb{C}+}$ and $\mathcal{L}_{\mathbb{R}}$. One elementary property for $\mathcal{M}_{\mathbb{C}}$, $\mathcal{M}_{\mathbb{C}+}$ and $\mathcal{M}_{\mathbb{R}}$ is that if K is a subgroup of G with $K \in \mathcal{M}_{\mathbb{C}}$ (resp. $\mathcal{M}_{\mathbb{C}+}$, resp. $\mathcal{M}_{\mathbb{R}}$), then G also lies in $\mathcal{M}_{\mathbb{C}}$ (resp. $\mathcal{M}_{\mathbb{C}+}$, resp. $\mathcal{M}_{\mathbb{R}}$). But this property does not hold for $\mathcal{L}_{\mathbb{C}}$, $\mathcal{L}_{\mathbb{C}+}$ and $\mathcal{L}_{\mathbb{R}}$.

The following two propositions are basic.

Proposition 2.4. *An $\mathcal{L}$-free real $\mathcal{P}$-matched pair of type* 1*is an $\mathcal{L}$-free self-conjugate $\mathcal{P}$-matched pair of type* 1 *and an $\mathcal{L}$-free self-conjugate $\mathcal{P}$-matched pair of type* 1*is an $\mathcal{L}$-free complex $\mathcal{P}$-matched pair of type* 1.

Proposition 2.5. *Let $f\colon K \to L$ be a group epimorphism. If there is a $\mathcal{L}(L)$-free complex (resp. self-conjugate, resp. real) $\mathcal{P}$-matched pair of type* 1 *of L-modules, then there is an $\mathcal{L}(G)$-free complex (resp. self-conjugate, resp. real) $\mathcal{P}$-matched pair of type* 1 *of G-modules.*

As a corollary, we obtain that the order of the abelianization $G/[G,G]$ of G is divisible by at least 3 primes, then there is an $\mathcal{L}(G)$-free complex $\mathcal{P}$-matched pair of type 1. Recall that if for a nontrivial element of G is conjugate to it's inverse, then the order of G is even. Furthermore, a nilpotent group G is classified by $|\pi(G)|$.

Theorem 2.6. *Let G be a nilpotent group. If $|\pi(G)| \le 1$, then $G \notin \mathcal{M}_{\mathbb{C}}$, if $|\pi(G)| = 2$, then $G \in \mathcal{M}_{\mathbb{C}} \setminus (\mathcal{M}_{\mathbb{C}+} \cup \mathcal{L}_{\mathbb{C}})$, and otherwise, $G \in \mathcal{L}_{\mathbb{C}} \setminus \mathcal{M}_{\mathbb{C}+}$.*

$\lvert\pi(G)\rvert$	0, 1	2	≥ 3
G	$\mathcal{M}_{\mathbb{C}}^c$	$\mathcal{M}_{\mathbb{C}} \setminus (\mathcal{M}_{\mathbb{C}+} \cup \mathcal{L}_{\mathbb{C}})$	$\mathcal{L}_{\mathbb{C}} \setminus \mathcal{M}_{\mathbb{C}+}$

For a prime p, we denote by $O^p(G)$ the smallest normal subgroup of G with index a power of p.

Lemma 2.7. *Assume that there exists a cyclic subgroup C of G such that $|\pi(C)| \ge 3$ and that $O^p(G)C = G$ for each $p \in \pi(G)$. Then G has an $\mathcal{L}(G)$-free complex $\mathcal{P}$-matched pair of type* 1*, namely $G \in \mathcal{L}_{\mathbb{C}}$.*

We set $G^{nil} = \bigcap_{r\in\pi(G)} O^r(G)$. This is a smallest normal subgroup N of G with G/N nilpotent.

Proposition 2.8. *Assume that there exists an element x of G such that*

(1) $CG^{nil} = G$,

(2) *the group $C \cap O^r(G)$ is not of prime power order for each $r \in \pi(G)$,*

where $C = \langle x\rangle$. Then there is an $\mathcal{L}(G)$-free complex $\mathcal{P}$-matched pair of type 1 of G-modules.

Let $x \in G$ and let $\pi(\langle x\rangle) = \{p_1, \dots, p_r\}$. Throughout the identification $R(C_1)\otimes\cdots\otimes R(C_r) \cong R(\langle x\rangle)$, where C_j is a Sylow p_j-subgroup of $\langle x\rangle$, take an irreducible complex C_j-module ξ_j such that the character of $\xi_1\cdots\xi_r$ at x is $\exp(2\pi\sqrt{-1}/|x|)$. We set $U_{\mathbb{C}}(x) = (\mathbb{C}-\xi_1)\cdots(\mathbb{C}-\xi_r)$. Let $\{g_j\}$ be the set of all representatives of conjugacy classes of elements not of prime power order. Then a $\mathcal{P}$-matched element of $R(G)$ is a linear combination of $U_{\mathbb{C}}(g_j)$'s with rational coefficients.

Proposition 2.9. *If there exists $p \in \pi(G)$ such that $\pi(C \cap O^p(G)) \neq \pi(C)$ for each cyclic subgroup C of G not of prime power order, then G has no $\mathcal{L}(G)$-free complex $\mathcal{P}$-matched pair of type 1 of G-modules.*

To show the existence of $\mathcal{L}(G)$-free complex (resp. self-conjugate, resp. real) $\mathcal{P}$-matched pair of type 1 for each prime p, the following lemma says that it suffices to see that the existence of $\{O^p(G)\}$-free complex (resp. self-conjugate, resp. real) $\mathcal{P}$-matched pair of type 1 for each prime p.

Lemma 2.10. *If there exists an $\{O^p(G)\}$-free complex (resp. self-conjugate, resp. real) $\mathcal{P}$-matched pair of type 1 for each prime p, then there exists an $\mathcal{L}(G)$-free complex (resp. self-conjugate, resp. real) $\mathcal{P}$-matched pair of type 1.*

Each character of G is a linear combination with integer coefficients of characters induced from characters of elementary subgroups. Namely, each character of G is a linear combination with integer coefficients of monomial characters. monomial means a character induced by a representation of degree 1 of a subgroup of G.

Lemma 2.11. *Let G be a finite group. Suppose that there is an epimorphism ρ from a subgroup K of G onto a cyclic group C such that $O^p(G)K = G$ and that $\rho(K \cap O^p(G)) \in \mathcal{M}_{\mathbb{C}}$ for each prime p. Then there is an $\mathcal{L}(G)$-free complex $\mathcal{P}$-matched pair of type 1 of G-modules.*

Now, we consider about the existence of $\mathcal{L}(G)$-free real $\mathcal{P}$-matched pair of type 1 of G-modules.

Lemma 2.12. *Let G be a finite group. Suppose that there exists a subgroup K of G for each prime $p \in \pi(G)$ such that there is an $\mathcal{L}(K)$-free real $\mathcal{P}$-matched pair of type 1 of K-modules and $O^p(G)K = G$. Then there is an $\mathcal{L}(G)$-free real $\mathcal{P}$-matched pair of type 1 of G-modules.*

Proposition 2.13. *For each prime p, assume that there are subgroups K of G and a group H satisfying as follows:*

(1) *There is a $\{H\}$-free real $\mathcal{P}$-matched pair of type 1 of H-modules,*

(2) *there is an epimorphism* $\rho: K \to H$ *such that* $\rho(K \cap O^p(G)) = H$, *and*
(3) $O^p(G)K = G$.

Then there is an $\mathcal{L}(G)$*-free real* $\mathcal{P}$*-matched pair of type* 1 *of G-modules.*

Note that Oliver showed that the condition (1) of Proposition 2.13 is equivalent to what there is a subquotient group D_{2pq} of H such that p and q are distinct primes.

These propositions answer the classes of symmetric groups. For $n \geq 5$, a symmetric group Σ_n on n letters has a subgroup generated by $(1,2)$, $(1,2,3)$ and $(4,5)$ which is isomorphic to D_{12}. By Proposition 2.1, there is a $\{\Sigma_n\}$-free real $\mathcal{P}$-matched pair of type 1 of Σ_n-modules and $\Sigma_n \in \mathcal{M}_{\mathbf{R}}$. A symmetric group Σ_n for $n \leq 4$ has no elements not of prime power order and then $\Sigma_n \notin \mathcal{M}_{\mathbf{C}}$.

Proposition 2.14. *The symmetric group* Σ_n *has an* $\mathcal{L}(\Sigma_n)$*-free real* $\mathcal{P}$*-matched pair of type* 1 *if and only if* $n \geq 7$.

Proof. Assume $n \geq 7$. Let K be a subgroup generated by four elements $(1,2,3)$, $(1,2)$, $(4,5)$ and $(6,7)$ and H a group generated by $(1,2,3)(4,5)$, $(1,2)(4,5)$. K and H are isomorphic to $D_6 \times C_2 \times C_2$ and D_{12}, respectively. In particular, there is an $\{H\}$-free real $\mathcal{P}$-matched pair of type 1 of H-modules. Recall that $D_6 \times C_2 \cong D_{12}$. Let $\varphi: K \to H$ be a homomorphism which sends $(1,2,3)(4,5)(6,7)$, $(1,2)(4,5)$, $(6,7)$ to $(1,2,3)(4,5)$, $(1,2)(4,5)$, $()$, respectively. It clearly holds $\varphi(K \cap O^p(\Sigma_n)) = H$ and $O^p(\Sigma_n)K = \Sigma_n$ for any prime p. Thus by Proposition 2.8, Σ_n has an $\mathcal{L}(\Sigma_n)$-free real $\mathcal{P}$-matched pair of type 1. □

n	2,3,4	5,6	≥ 7
Σ_n	$\mathcal{M}_{\mathbf{C}}^c$	$\mathcal{M}_{\mathbf{R}} \setminus \mathcal{L}_{\mathbf{C}}$	$\mathcal{L}_{\mathbf{R}}$

3. Simple Groups

For a perfect group G, it holds $\mathcal{L}(G) = \{G\}$ and then $G \in \mathcal{L}_{\mathbf{R}}$, $\mathcal{L}_{\mathbf{C}+}$ and $\mathcal{L}_{\mathbf{C}}$ respectively if and only if $G \in \mathcal{M}_{\mathbf{R}}$, $\mathcal{M}_{\mathbf{C}+}$ and $\mathcal{M}_{\mathbf{C}}$ respectively. In this section, we classify all finite simple groups as follows.

finite simple groups	class
C_q, $L_2(4) = L_2(5) = Alt_5$, $L_2(7) = L_3(2)$, $L_2(8)$, $L_2(9) = Alt_6$, $L_2(17)$, $L_3(4)$, $Sz(8)$, $Sz(32)$	$\mathcal{M}_{\mathbf{C}}^c$
$L_3(8)$, $U_3(4)$, $U_3(8)$	$\mathcal{M}_{\mathbf{C}} \setminus \mathcal{M}_{\mathbf{C}+}$
$U_3(3)$	$\mathcal{M}_{\mathbf{C}+} \setminus \mathcal{M}_{\mathbf{R}}$
others	$\mathcal{M}_{\mathbf{R}}$

Note that any cyclic simple group does not lie in $\mathcal{M}_{\mathbf{C}}$. We decide classes for nonabelian simple groups.

3.1. **Alternating Groups** $\boldsymbol{Alt_n}$**.** Note that $Alt_5 = L_2(5)$ and $Alt_6 = L_2(9)$. For $n \geq 7$, the alternating group Alt_n lies in $\mathcal{M}_{\mathbf{R}}$, since the subgroup of Alt_n generated by two elements $(1,2)(5,6)$ and $(1,2)(3,4)(5,6,7)$ is isomorphic to a dihedral group D_{12} of order 12.

3.2. **Simple Groups of type A_n.** Let $A_n(q) = L_{n+1}(q) = PSL(n+1,q)$, $n \geq 1$ denote a linear simple group.

Proposition 3.2.1. (cf. [2, Theorem 6.5.1]) *Let $d = \gcd(2, q-1)$. The maximal dihedral subgroups of $L_2(q)$ are $D_{2(q\pm1)/d}$.*

Then we decide prime power q such that $q \pm 1$ are prime power.

Lemma 3.2.2. (1) *Let q be 2-power. If $q-1$ and $q+1$ are both prime power, possibly 1, then $q = 2, 4, 8$.*
(2) *Let $q > 1$ be odd prime power. If $\frac{q-1}{2}$ and $\frac{q+1}{2}$ are both prime power, possibly 1, then $q = 3, 5, 7, 9, 17$.*

Thus, we obtain that $L_2(q)$ lies in $\mathcal{M}_{\mathbb{R}}$ if $q \neq 2, 3, 4, 5, 7, 8, 9, 17$, and does not lie in $\mathcal{M}_{\mathbb{C}}$ if otherwise.

The following proposition is checked directly.

Proposition 3.2.3. (1) *Let q be odd prime power. Set $x = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$, $y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}$. Then the subgroup of $L_3(q) = SL(3,q)$ generated by x and y is isomorphic to D_{12}. ($x^2 = y^6 = (xy)^2 = 1$)*
(2) *Let q be 2-power, E_2 the identity matrix and B a matrix of order q^2-1 in $GL_2(q)$. Set $x = \begin{pmatrix} 0 & E_2 \\ E_2 & 0 \end{pmatrix}$, and $y = \begin{pmatrix} B & 0 \\ 0 & B^{-1} \end{pmatrix}$. Then x, y lie in $L_4(q)$, and it holds that $x^2 = y^{q^2-1} = (xy)^2 = 1$. Thus $L_4(q)$ has a subgroup $D_{2(q^2-1)}$.*
(3) *Take $x = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$, and $y = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$ which are elements of $L_4(2) = SL(4,2)$. Then it holds that $x^2 = y^6 = (xy)^2 = 1$. Thus $L_4(2)$ has a subgroup D_{12}.*

Hartley and Mitchell has completely decided maximal subgroups of $L_3(q)$ (cf. [2, Theorem 6.5.3]). Since the order of an element of $L_3(2)$ is either 1, 2, 3, 4 or 7, $L_3(2) \notin \mathcal{M}_{\mathbb{C}}$. Since the order of an element of $L_3(4)$ is either 1, 2, 3, 4, 5 or 7, $L_3(4) \notin \mathcal{M}_{\mathbb{C}}$. The order of an element of $L_3(8)$ is either 1, 2, 3, 4, 7, 9, 14, 21, 63 or 73. If y is an element of order 14, then the normalizer $N(\langle y \rangle)$ of the cyclic group generated by y and the centralizer $Z(y)$ of y are same group of order 56. If y is an element of order 21 or 63, then $|N(\langle y \rangle)) = 126$, $|Z(y)| = 63$ and $x^{-1}yx = y^2$ for any $x \in N(\langle y \rangle) \setminus Z(y)$. Therefore $L_3(8) \in \mathcal{M}_{\mathbb{C}} \setminus \mathcal{M}_{\mathbb{C}+}$.

3.3. **Simple Groups of type B_n.** Let $B_n(q) = O_{2n+1}(q)$, $n > 1$. Since $O_{2n+1}(q)$ has subgroups which are isomorphic to $O_{2n}(\pm1, q)$ and $O_4(-1, q) = L_2(q^2)$, $O_5(q) \in \mathcal{M}_{\mathbb{R}}$ if $q \neq 2$, 3. Since $O_4(1,q) = L_2(q) \times L_2(q)$, we obtain that $O_4(1,2) = D_6 \times D_6 > C_2 \times D_6 = D_{12}$ and $O_4(1,3) = A_4 \times A_4 \in \mathcal{M}_{\mathbb{C}}$. (Note that $O_5(2) = \Sigma_6$.) Furthermore we obtain

Proposition 3.3.1. *For the elements x and y of $SO_5(q)$, it holds that $x^2 = y^6 = (xy)^2 = 1$.*
Here $x = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$ *and* $y = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 0 \end{pmatrix}$. *Thus $O_5(q)$ has a subgroup D_{12} for any odd prime power q.*

3.4. **Simple groups of type C_n.** Let $C_n(q) = S_{2n}(q) = PSp_n(q)$ for $n > 2$. The group $S_{2n}(q)$ is of order $q^{n^2}\prod_{j=1}^{n}(q^{2j}-1)/(2, q-1)$. It holds $S_{2n}(q) \in \mathcal{M}_{\mathbb{R}}$ since $S_{2n}(q) \geq S_4(q) = O_5(q)$.

3.5. **Suzuki Groups $Sz(2^{2n+1})$.** Let $q = 2^{2n+1}$ and $r = 2^{n+1}$. Let ${}^2B_2(q) = {}^2C_2(q) = Sz(q)$. The order of $Sz(q)$ is $q^2(q-1)(q^2+1) = q^2(q-1)(q-r+1)(q+r+1)$.

Lemma 3.5.1. (1) $5^{2^k} = 2^{k+2} + 1 \mod 2^{k+3}$ *for $k \geq 0$.*
(2) $2^{2n+1} + 2^{n+1} + 1$ *is not 5-power for any n.*
(3) *The equation $2^{2n+1} - 2^{n+1} + 1 = 5^m$ implies $(n, m) = (1, 1), (2, 2)$.*

Proof. We show clearly the first assertion by induction on k and omit the proof. We show the second assertion. Suppose $2^{2n+1} + 2^{n+1} + 1 = 5^m$. Let $m = 2^k a$, a odd, $5^{2^k} = 2^{k+2}(1+2x)+1$. Then $2^{n+1} + 2^{2n+1} = 5^m - 1 = (2^{k+2}(1+2x)+1)^a - 1 = a \cdot 2^{k+2}(1+2x) + \ldots + 2^{a(k+2)}(1+2x)^a$ and thus $n+1 = k+2$, $2n+1 \geq a(k+2)+a$. If $a \geq 3$, then $2n+1 = 2k+3 < 3k+6 \leq a(k+3)$ is contradiction. Then $a = 1$, $m = 2^{n-1}$. If $k \geq 2$, then $5^m > 2^{2^n} \geq 2^{2(n+1)} > 2^{2n+1} + 2^{n+1} + 1$, which is contradiction. However it clearly holds $5^m < 2^{2n+1} + 2^{n+1} + 1$ for $k = 0, 1$.

We show the last assertion. Suppose $2^{2n+1} - 2^{n+1} + 1 = 5^m$. Let $m = 2^k a$, (ℓ odd), $5^{2^k} = 2^{k+2}(1+2x)+1$. By comparing

$$5^m - 1 = a \cdot 2^{k+2}(1+2x) + \ldots + 2^{a(k+2)}(1+2x)^a$$

and

$$2^{2n+1} - 2^{n+1} = 2^{n+1} + 2^{n+2} + \ldots + 2^{2n},$$

it holds $n+1 = k+2$, $2n \geq a(k+2)+a$, namely $n = k+1$, $(2-a)k + 2 - 3a \geq 0$. Hence $a = 1$, $m = 2^{n-1}$. If $n \geq 3$, then $5^m > 2^{2 \cdot 2^{n-1}} \geq 2^{2n+1} > 2^{2n+1} - 2^{n+1} + 1$. If $n = 2$, then $m = 2$, and if $n = 1$ then $m = 1$. This completes the proof. □

We remark that maximal subgroups of $Sz(q)$ have already classified. (See Theorem 6.5.4 [2].)

Proposition 3.5.2. *If n is either 1 or 2, then $Sz(2^{2n+1}) \notin \mathcal{M}_{\mathbb{C}}$ and $Sz(2^{2n+1}) \in \mathcal{M}_{\mathbb{R}}$ if otherwise.*

Proof. Let $q = 2^{2n+1}$ and $r = 2^{n+1}$. The character table of $Sz(q)$ is well-known. From it, an order of an element of $Sz(q)$ is a divisor of one of 4, $q-1$, $q \pm r + 1$ and a maximal dihedral subgroup not of 2 power order is one of $D_{2(q-1)}$, $D_{2(q+r+1)}$, $D_{2(q-r+1)}$. Note that $q^2 + 1 = 0 \mod 5$, since $2^4 = 1 \mod 5$. Assume any of $q-1$, $q \pm r + 1$ is prime power. By Lemma 3.5.1, $q - r + 1$ is divisible by 5 and $n = 1, 2$, $q = 2^3, 2^5$. It is clear that any element of $Sz(q)$ has prime power order for $q = 8, 32$. We complete the proof. □

3.6. **Simple groups of type D_n.** Let $n > 3$, $D_n(q) = O_{2n}(1, q)$ and ${}^2D_n(q, q^2) = O_{2n}(-1, q)$.
The group $O_{2n}(1, q)$ is of order order $q^{n(n-1)/2}(q^n - 1)\prod_{j=1}^{n-1}(q^{2j} - 1)/(4, q^n - 1)$. Since $O_6(1, q) = L_4(q)$, it holds $O_{2n}(1, q) \in \mathcal{M}_{\mathbb{R}}$.
The group $O_{2n}(-1, q)$ is of order $q^{n(n-1)/2}(q^n+1)\prod_{j=1}^{n-1}(q^{2j}-1)/(4, q^n+1)$. Since $O_4(-1, q) = L_2(q^2)$, it lies in $\mathcal{M}_{\mathbb{R}}$ if $q \neq 2, 3$; And $O_6(-1, 2) = S_4(3) \in \mathcal{M}_{\mathbb{R}}$.

Proposition 3.6.1. *We may assume the group $O_6(-1, 3)$ preserves 2-form $f = x_1^2 + x_1x_2 + x_2^2 + x_3x_4 + x_5x_6$. Let*

$$a = \begin{pmatrix} -1 & 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 1 & -1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix} \text{ and } b = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

be matrices of $SL(6, 3)$. It holds that both a and b preserve f, $|a| = 6$, $|b| = 2$ and $|ba| = 2$. Therefore the group generated by a and b is D_{12}, which gives a subgroup of $O_6(-1, 3)$ through the homomorphism to $O_6(-1, 3)$.

The twisted Chevalley group ${}^3D_4(q, q^3)$ is of order $q^{12}(q^8 + q^4 + 1)(q^6 - 1)(q^2 - 1)$. By Table 4.5.2 in [2], ${}^3D_4(q, q^3)$ contains a subgroup $A_1(q) \times A_1(q^3)$ and thus lies in $\mathcal{M}_{\mathbb{R}}$.

3.7. **Projective Unitary Groups $PSU(n, q)$.** Let ${}^2A_n(q, q^2) = U_{n+1}(q) = PSU(n + 1, q)$, $n \geq 2$. We assume $n \geq 3$ throughout this subsection. Since $U_2(q) = L_2(q)$, we consider the eight cases where $q = 2, 3, 4, 5, 7, 8, 9, 17$. First if $n \geq 4$, then it holds that $U_n(q) \geq U_4(q) = O_6(-1, q) \in \mathcal{M}_{\mathbb{R}}$.
By ATLAS [1] we obtain the following facts which are also confirmed by GAP [3].

Fact 3.7.1. (1) *The order of an element of $U_3(2)$ is either 1, 2, 3 or 4 and thus $U_3(2) \notin \mathcal{M}_{\mathbb{C}}$.*
(2) *The order of an element of $U_3(3)$ is either 1, 2, 3, 4, 6, 7, 8 or 12. If x is an element of order 6, it holds that $N(\langle x \rangle) = C_3 \rtimes C_8$, $C(\langle x \rangle) = C_{12}$, and $g^{-1}xg = x^{-1}$ for some g of order 8, but $g^{-1}xg = x$ for any $g \in N(\langle x \rangle)$ of order 2. Thus $U_3(3) \in \mathcal{M}_{\mathbb{C}+} \setminus \mathcal{M}_{\mathbb{R}}$.*
(3) *The order of an element of $U_3(4)$ is either 1, 2, 3, 4, 5, 10, 13 or 15. Let x be a nontrivial element which holds $g^{-1}xg = x^{-1}$ for some g. Then the order of x is either 3 or 5. If x is of order 5, then the order of g is 2 or 10. For any element x of order 3, there exists an element g such that $g^{-1}xg = x^{-1}$. (The order of g is also 2 or 10.) Thus $U_3(4) \in \mathcal{M}_{\mathbb{C}} \setminus \mathcal{M}_{\mathbb{C}+}$.*
(4) *$U_3(8) \in \mathcal{M}_{\mathbb{C}} \setminus \mathcal{M}_{\mathbb{C}+}$.*
(5) *$U_3(9) \in \mathcal{M}_{\mathbb{R}}$, since $U_3(9) > D_{20}$. Note $U_3(9) \not> D_{12}$.*

Proposition 3.7.2. *Let p be one of 5, 7, 17. Set* $x = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$, $z = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$, $y = \eta^{p-3}\begin{pmatrix} 1 & -2 & -2 \\ -2 & 1 & -2 \\ -2 & -2 & 1 \end{pmatrix}$. *Then x, z, y are in $U_3(p)$ such that* $x^2 = y^2 = z^3 = [x,y] = [y,z] = (xz)^2 = 1$. *Then the group generated by x, y, z is isomorphic to D_{12} which yields $U_3(p) \in \mathcal{M}_{\mathbb{R}}$.*

3.8. **Simple groups of type E_n, F_4.** Since $E_6(q) > L_6(q)$, ${}^2E_6(q,q^2) > U_6(q)$, $E_7(q) > L_7(q)$ and $E_8(q) > L_8(q)$, their groups lie in $\mathcal{M}_{\mathbb{R}}$. The group $F_4(q)$ is of order $q^{24}(q^{12}-1)(q^8-1)(q^6-1)(q^2-1)$. Since $F_4(q) > PSp_3(q)$, $F_4(q)$ lies in $\mathcal{M}_{\mathbb{R}}$.

3.9. **Simple groups of type G_2.** The group $G_2(q)$ is of order $q^6(q^6-1)(q^2-1)$. First, assume that q is odd. let $\mathfrak{C} = \sum_{i=0}^{7} \mathbb{R}e_i$ be the Cayley algebra. (We put e_i ($1 \le i \le 6$) along the triangle rotating anti-clockwise and e_7 the barycenter.) (Recall $e_3 = e_1e_2$, $e_4 = e_1e_7$, $e_5 = e_2e_7$, and $e_6 = e_1e_2e_7$. Let x and y be automorphisms over $\mathfrak{C}$ defined by $x(e_0) = e_0$, $x(e_1) = -e_2$, $x(e_2) = -e_7$, $x(e_3) = e_5$, $x(e_4) = -e_3$, $x(e_5) = -e_4$, $x(e_6) = -e_6$, $x(e_7) = -e_1$, $y(e_0) = e_0$, $y(e_1) = e_2$, $y(e_2) = e_1$, $y(e_3) = -e_3$, $y(e_4) = e_5$, $y(e_5) = e_4$, $y(e_6) = -e_6$, $y(e_7) = e_7$. Then these elements lie in G_2 and $|x| = 6$, $|y| = 2$, $yxy = x^{-1}$. Therefore $G_2(q) \in \mathcal{M}_{\mathbb{R}}$.

Next, let q is 2 power. Then $G_2(q) \ge G_2(2)$, and by ATLAS, $G_2(2) = U_3(3) : 2 > 4^2 : D_{12}$, in particular $G_2(2) > D_{12}$ by GAP.

By Table 4.5.2 in [2], it holds $G_2(q) > A_1(q)^2$ and thus lies in $\mathcal{M}_{\mathbb{R}}$, since $G_2(q) > \Sigma_3 \times \Sigma_3 > D_{12}$ for $q = 2, 4, 8$.

Fact 3.9.1. Let $q = 3^{2n+1}$. The Ree group ${}^2G_2(q) = Re(q)$ is of order $q^3(q^3+1)(q-1)$. Since $Re(q) > L_2(q)$, it holds $Re(q)$ lies in $\mathcal{M}_{\mathbb{R}}$ for $q \ge 3^3$. The nonsimple group $Re(3) = L_2(8).3$ lies in $\mathcal{M}_{\mathbb{C}} \setminus \mathcal{M}_{\mathbb{C}+}$ by GAP.

Notice that the maximal subgroups of $Re(q)$ have already classified (cf. [2, Theorem 6.5.5]) and $Re(q)$ contains no copy of D_{12} ([2, p.334]).

3.10. **Sporadic simple groups.** Twenty-six sporadic groups exist. By ATLAS table [1] we compute the class algebra constants and we obtain a sporadic group has a subgroup D_{2n} for some n, not a power of a prime. (This is also computed by using the software GAP.) Therefore each sporadic group lies in $\mathcal{M}_{\mathbb{R}}$.

4. Projective General Linear Groups

In this section we show the following results.

Projective general linear groups $PGL(n,q)$ lie in the following classes.

$n \setminus q$	2	4	8	3	5, 7, 9, 17	others
2	$\mathcal{M}_{\mathbb{C}}^{c}$	$\mathcal{M}_{\mathbb{C}}^{c}$	$\mathcal{M}_{\mathbb{C}}^{c}$	$\mathcal{M}_{\mathbb{C}}^{c}$	$\mathcal{M}_{\mathbb{R}} \setminus \mathcal{L}_{\mathbb{C}}$	$\mathcal{L}_{\mathbb{R}}$
3	$\mathcal{M}_{\mathbb{C}}^{c}$	$\mathcal{M}_{\mathbb{C}} \setminus (\mathcal{M}_{\mathbb{C}+} \cup \mathcal{L}_{\mathbb{C}})$	$\mathcal{L}_{\mathbb{C}} \setminus \mathcal{M}_{\mathbb{C}+}$	$\mathcal{L}_{\mathbb{R}}$	$\mathcal{L}_{\mathbb{R}}$	$\mathcal{L}_{\mathbb{R}}$
≥ 4	$\mathcal{L}_{\mathbb{R}}$	$\mathcal{L}_{\mathbb{R}}$	$\mathcal{L}_{\mathbb{R}}$	$\mathcal{L}_{\mathbb{R}}$	$\mathcal{L}_{\mathbb{R}}$	$\mathcal{L}_{\mathbb{R}}$

General linear groups $GL(n,q)$ lie in the following classes.

$n \setminus q$	2	4	8	3	5, 7, 9, 17	others
2	$\mathcal{M}_{\mathbb{C}}^c$	$\mathcal{L}_{\mathbb{C}} \setminus \mathcal{M}_{\mathbb{C}+}$	$\mathcal{M}_{\mathbb{C}}$	$\mathcal{M}_{\mathbb{R}} \cap \mathcal{L}_{\mathbb{C}} \setminus \mathcal{L}_{\mathbb{C}+}$	$\mathcal{M}_{\mathbb{R}} \setminus \mathcal{L}_{\mathbb{C}}$	$\mathcal{L}_{\mathbb{R}}$
3	$\mathcal{M}_{\mathbb{C}}^c$	$\mathcal{L}_{\mathbb{C}} \setminus \mathcal{M}_{\mathbb{C}+}$	$\mathcal{L}_{\mathbb{C}} \setminus \mathcal{M}_{\mathbb{C}+}$	$\mathcal{L}_{\mathbb{R}}$	$\mathcal{L}_{\mathbb{R}}$	$\mathcal{L}_{\mathbb{R}}$
≥ 4	$\mathcal{L}_{\mathbb{R}}$	$\mathcal{L}_{\mathbb{R}}$	$\mathcal{L}_{\mathbb{R}}$	$\mathcal{L}_{\mathbb{R}}$	$\mathcal{L}_{\mathbb{R}}$	$\mathcal{L}_{\mathbb{R}}$

4.1. **Special $\mathcal{P}$-matched pairs.** The classes $\mathcal{L}_{\mathbb{C}}$, $\mathcal{L}_{\mathbb{C}+}$, $\mathcal{L}_{\mathbb{R}}$ are subclasses of $\mathcal{M}_{\mathbb{C}}$, $\mathcal{M}_{\mathbb{C}+}$, $\mathcal{M}_{\mathbb{R}}$, respectively.

Note that $GL(n,2) = PSL(n,2)$ for any n.

Lemma 4.1.1. *If $GL(n,q) \in \mathcal{M}_{\mathbb{R}}$, then $GL(m,q) \in \mathcal{L}_{\mathbb{R}}$ for any $m > n$. If $PGL(n,q) \in \mathcal{M}_{\mathbb{R}}$, then $PGL(m,q) \in \mathcal{L}_{\mathbb{R}}$ for any $m > n$.*

Proof. It holds that $PGL(n,q) \in \mathcal{M}_{\mathbb{R}}$ implies $GL(n,q) \in \mathcal{M}_{\mathbb{R}}$. The condition $GL(n,q) \in \mathcal{M}_{\mathbb{R}}$ requires $n \geq 2$. Suppose that $GL(n,q) \in \mathcal{M}_{\mathbb{R}}$.

Take a subgroup $K = GL(n,q) \times C_{q-1}$ of $GL(m,q)$. In fact it is a subgroup of $GL(n+1,q)$. Let $H = GL(n,q)$ and $\tau\colon K \to H$ be a canonical epimorphism. For $A \in GL(n,q)$, the element of $(A, \det A^{-1})$ lies in $SL(n+1,q)$ and its image to H through τ is A. Thus $\tau(K \cap SL(n+1,q)) = H$. By Proposition 2.13, we obtain that $GL(m,q) \in \mathcal{L}_{\mathbb{R}}$. Similarly let $\phi\colon GL(n+1,q) \to PGL(n+1,q)$ be a canonical epimorphism, $K' = \phi(K)$, $H' = \phi(H)$ and let $\phi(\tau)\colon K' \to H'$ be the epimorphism induced by τ. Then $\phi(\tau)(K' \cap PSL(n+1,q)) = H'$. By applying Proposition 2.13 for $(G,K,H) = (PGL(m,q), K', H')$, if $PGL(n,q) \in \mathcal{M}_{\mathbb{R}}$, then $PGL(m,q) \in \mathcal{L}_{\mathbb{R}}$. □

Lemma 4.1.2. *Suppose that* $\gcd(n, q-1) = 1$. *If $PSL(n,q) \in \mathcal{M}_{\mathbb{C}}$ (resp. $\mathcal{M}_{\mathbb{C}+}$, resp. $\mathcal{M}_{\mathbb{R}}$), then $GL(n,q) \in \mathcal{L}_{\mathbb{C}}$ (resp. $\mathcal{L}_{\mathbb{C}+}$, resp. $\mathcal{L}_{\mathbb{R}}$.*

Proof. Let $\mathbb{F} = \mathbb{C}$ (resp. $\mathbb{R}$, resp. $\mathbb{R}$). The condition $\gcd(n, q-1) = 1$ implies that $PGL(n,q) \cong PSL(n,q)$ and then there exists a canonical epimorphism f from $GL(n,q)$ to $PSL(n,q)$. Let $(\mathbb{F} \oplus V, W)$ be a complex (resp. self-conjugate, resp. real) $\mathcal{P}$-matched pair of type 1 of $PSL(n,q)$-modules. Since $PSL(n,q)$ is simple, $GL(n,q)$ is perfect, and V and W are $\mathcal{L}(PSL(n,q))$-free. Then $(\mathbb{F} \oplus f^*V, f^*W)$ is an $\mathcal{L}(G)$-free complex (resp. real) $\mathcal{P}$-matched pair of type 1 of $GL(n,q)$. □

4.2. ***GL*(n,q).** We classify $GL(n,q)$ into $\mathcal{M}_{\mathbb{C}}^c$, $\mathcal{M}_{\mathbb{C}}$, $\mathcal{M}_{\mathbb{C}+}$, or $\mathcal{M}_{\mathbb{R}}$. Since $GL(2,2)$ is isomorphic to D_6, it holds $GL(2,2) \in \mathcal{M}_{\mathbb{C}}^c$. Considering a subgroup of $GL(2,3)$ which is generated by elements $\begin{pmatrix} \rho & 0 \\ 1 & \rho \end{pmatrix} \begin{pmatrix} \rho & 0 \\ 0 & 1 \end{pmatrix}$, it is isomorphic to D_{12}. Thus $GL(2,3)$ lies in $\mathcal{M}_{\mathbb{R}}$.

Lemma 4.2.1. *For $q > 4$, $q \neq 8$, $GL(2,q) \in \mathcal{M}_{\mathbb{R}}$. For $q = 4, 8$, $GL(2,q) \in \mathcal{M}_{\mathbb{C}} \setminus \mathcal{M}_{\mathbb{C}+}$.*

Proof. We show $D_{2(q\pm1)}$ are (maximal) dihedral subgroups of $GL(2,q)$. The group generated by $\begin{pmatrix} \rho & 0 \\ 0 & \rho^{-1} \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is $D_{2(q-1)}$ which is one of maximal dihedral subgroups of $GL(2,q)$. Let x_2 be an element of order q^2-1. Then there is an element t of order 2 such that $tx_2t = x_2{}^q$. Therefore $GL(2,q)$ has a subgroup $D_{2(q+1)}$, since $tx_2{}^{q-1}t = x_2{}^{-q+1}$. Thus $GL(2,q) \in \mathcal{M}_{\mathbb{R}}$ if $q \neq 2,3,4,8$. Let $q = 4,8$. Then it holds $GL(2,q) \in \mathcal{M}_{\mathbb{C}}$, since x_2 is of composite order q^2-1. The order of an element of $GL(2,q)$ is divisor of one of $2(q-1)$, q and q^2-1. However any element of a composite order is not self-conjugate. (To show this, it suffices to calculate the class algebra constants by its character table.) □

Since $GL(3,2) = SL(3,2)$ is a simple group of order 168, we have already classified: $GL(3,2) \in \mathcal{M}_{\mathbb{C}}^{c}$. By ATLAS [1], $PGL(3,q)$ and then $GL(3,q)$ for $q = 4,8$ lies in $\mathcal{M}_{\mathbb{C}}$. Recall that any $PSL(4,q)$ for $q \geq 2$ lies in $\mathcal{M}_{\mathbb{R}}$. Thus any $GL(4,q)$ for $q \geq 2$ also lies in $\mathcal{M}_{\mathbb{R}}$.

$n \setminus q$	2	4	8	others
2	$\mathcal{M}_{\mathbb{C}}^{c}$	$\mathcal{M}_{\mathbb{C}} \setminus \mathcal{M}_{\mathbb{C}+}$	$\mathcal{M}_{\mathbb{C}}$	$\mathcal{L}_{\mathbb{R}}$
3	$\mathcal{M}_{\mathbb{C}}^{c}$	$\mathcal{M}_{\mathbb{C}} \setminus \mathcal{M}_{\mathbb{C}+}$	$\mathcal{M}_{\mathbb{C}} \setminus \mathcal{M}_{\mathbb{C}+}$	$\mathcal{M}_{\mathbb{R}}$
≥ 4	$\mathcal{M}_{\mathbb{R}}$	$\mathcal{M}_{\mathbb{R}}$	$\mathcal{M}_{\mathbb{R}}$	$\mathcal{M}_{\mathbb{R}}$

4.3. ***PGL(n, q).*** Note that $PGL(2,q)$ is isomorphic to the simple group $PSL(2,q)$ for q even and thus $PGL(2,q)$ for q even have already classified: The groups $PGL(2,q)$ for $q = 2,4,8$ do not lie in $\mathcal{M}_{\mathbb{C}}$ and $PGL(2,q)$ for other even q lies in $\mathcal{M}_{\mathbb{R}}$.

From now on, throughout this subsection we assume that q is odd. Let $\phi\colon GL(n,q) \to PGL(n,q)$ be a canonical projection. The group generated by $\phi\begin{pmatrix}\rho & 0\\ 0 & 1\end{pmatrix}$ and $\phi\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$ is isomorphic to $D_{2(q-1)}$. The group generated by $\phi\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}$ and $\phi\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$ is isomorphic to D_{2q}. Let x_2 be an element of order $q^2 - 1$ of $GL(2,q)$. Then there is an element t of $GL(2,q)$ of order 2 such that $tx_2t = x_2{}^q$. Thus the group generated by $\phi(x_2)$ and $\phi(t)$ is isomorphic to $D_{2(q+1)}$. Therefore $D_{2(q\pm1)}$, D_{2q} are maximal dihedral subgroups of $PGL(2,q)$ for q odd. In particular, any $PGL(2,q)$-module is real. Furthermore, the group $PGL(2,3)$ of order 24, which is isomorphic to S_4, does not lie in $\mathcal{M}_{\mathbb{C}}$, $PGL(2,q) \in \mathcal{M}_{\mathbb{R}}$ otherwise.

Since $PGL(3,2) \cong SL(3,2)$ and $PGL(3,3) \cong SL(3,3)$ are simple groups, we have already classified: $PGL(3,2) \notin \mathcal{M}_{\mathbb{C}}$, $PGL(3,3) \in \mathcal{M}_{\mathbb{R}}$. It holds that $PGL(3,8) \in \mathcal{M}_{\mathbb{C}} \setminus \mathcal{M}_{\mathbb{C}+}$, since $PGL(3,8) = PSL(3,8)$. Conjugacy classes of $PGL(3,q)$ are well-known. Then we get $PGL(3,4)$ lies in $\mathcal{M}_{\mathbb{C}} \setminus \mathcal{M}_{\mathbb{C}+}$. As any $PSL(4,q)$ for $q \geq 2$ lies in $\mathcal{M}_{\mathbb{R}}$, any $PGL(4,q)$ for $q \geq 2$ also lies in $\mathcal{M}_{\mathbb{R}}$. Therefore, we have classified $PGL(n,q)$ into $\mathcal{M}_{\mathbb{C}}^{c}$, $\mathcal{M}_{\mathbb{C}}$, $\mathcal{M}_{\mathbb{C}+}$, $\mathcal{M}_{\mathbb{R}}$.

$n \setminus q$	2	4	8	3	others
2	$\mathcal{M}_{\mathbb{C}}^{c}$	$\mathcal{M}_{\mathbb{C}}^{c}$	$\mathcal{M}_{\mathbb{C}}^{c}$	$\mathcal{M}_{\mathbb{C}}^{c}$	$\mathcal{M}_{\mathbb{R}}$
3	$\mathcal{M}_{\mathbb{C}}^{c}$	$\mathcal{M}_{\mathbb{C}} \setminus \mathcal{M}_{\mathbb{C}+}$	$\mathcal{M}_{\mathbb{C}} \setminus \mathcal{M}_{\mathbb{C}+}$	$\mathcal{M}_{\mathbb{R}}$	$\mathcal{M}_{\mathbb{R}}$
≥ 4	$\mathcal{M}_{\mathbb{R}}$	$\mathcal{M}_{\mathbb{R}}$	$\mathcal{M}_{\mathbb{R}}$	$\mathcal{M}_{\mathbb{R}}$	$\mathcal{M}_{\mathbb{R}}$

Now, we classify $PGL(n,q)$ into $\mathcal{L}_{\mathbb{C}}^{c}$, $\mathcal{L}_{\mathbb{C}}$, $\mathcal{L}_{\mathbb{C}+}$, and $\mathcal{L}_{\mathbb{R}}$.

Lemma 4.3.1. *Suppose that $q \neq 3,5,7,9,17$. If either $(q-1)^{[2]}$ or $(q+1)^{[2]}$ is a composite number, then $PGL(2,q) \in \mathcal{L}_{\mathbb{R}}$.*

Proof. There exists a subgroup $K = D_{2(q\pm1)}$ generated by a and b with relations $a^{q\pm1} = b^2 = (ab)^2 = 1$ of $PGL(2,q)$. Then $K \cdot PSL(2,q) = PGL(2,q)$. We can take the element b from $PSL(2,q)$. Let $\phi\colon K \to K/\langle a^{(q\pm1)^{[2]}}\rangle \cong D_{2(q\pm1)^{[2]}} = H$ be a canonical epimorphism. Clearly it holds that $\phi(K \cap PSL(2,q)) = \phi(D_{q\pm1}) = H$. Suppose that $(q \pm 1)^{[2]}$ is a composite number. Then $H \in \mathcal{M}_{\mathbb{R}}$ and $PGL(2,q) \in \mathcal{L}_{\mathbb{R}}$ by Proposition 2.13. □

Next we consider the case where each $(q \pm 1)^{[2]}$ is power of an odd prime.

Lemma 4.3.2. *Let q be power of an odd prime such that $q-1=2^{a+1}p^b$, where p is an odd prime, $a>0$ and $b>0$. Then $PGL(2,q)$ lies in $\mathcal{L}_{\mathbb{R}}$.*

Proof. Recall that any $PGL(2,q)$-module is real. Set $s=2^a(p^b-1)$, $t=(2^a-1)p^b$, $u=2^a+p^b$. Let $\chi=\chi_1+\chi_q-\chi_{q+1}^{(s)}-\chi_{q+1}^{(t)}+\chi_{q+1}^{(u)}$ be the element of $RO(PGL(2,q))$. Note that s, t and u are different numbers less than $\frac{q-1}{2}$. Let τ_2 (resp. τ_p) be a primitive 2^{a+1}-th (resp. p^b-th) root. Then it holds that $\tau_p^s=\tau_p^{-u}$, $\tau_p^t=1$, $\tau_2^t=\tau_2^{-u}$, and $\tau_2^s=1$. It follows that $\chi(g)=0$ for any element g of $PGL(2,q)$ of prime power order. Taking $V=\chi_1+\chi_q+\chi_{q+1}^{(u)}$ and $W=\chi_{q+1}^{(s)}+\chi_{q+1}^{(t)}$, the pair (V,W) is a real $\mathcal{L}(PGL(2,q))$-free $\mathcal{P}$-matched pair of type 1. □

Similarly we obtain

Lemma 4.3.3. *Let q be power of an odd prime such that $q+1=2^{a+1}p^b$, where p is an odd prime, $a>0$ and $b>0$. Then $PGL(2,q)$ lies in $\mathcal{L}_{\mathbb{R}}$.*

Proof. Set $s=2^a(p^b-1)$, $t=(2^a-1)p^b$, $u=2^a+p^b$. Let $\chi=\chi_1-\chi_q+\chi_{q-1}^{(s)}+\chi_{q-1}^{(t)}-\chi_{q-1}^{(u)}$ be the element of $RO(PGL(2,q))$. Note that s, t and u are different numbers less than $\frac{q+1}{2}$. Let τ_2 (resp. τ_p) be a primitive 2^{a+1}-th (resp. p^b-th) root. Then it holds that $\tau_p^s=\tau_p^{-u}$, $\tau_p^t=1$, $\tau_2^t=\tau_2^{-u}$, and $\tau_2^s=1$. It is easy to show that $\chi(g)=0$ for any element g of $PGL(2,q)$ of prime power order. Taking $V=\chi_1+\chi_{q-1}^{(s)}+\chi_{q-1}^{(t)}$ and $W=\chi_q+\chi_{q-1}^{(u)}$, the pair (V,W) is a real $\mathcal{L}(PGL(2,q))$-free $\mathcal{P}$-matched pair of type 1. □

By combining Lemmas 4.3.1, 4.3.2 and 4.3.3 we obtain

Theorem 4.3.4. *$PGL(2,q)$ for $q\neq 1,2,3,4,5,6,7,8,9,17$ lies in $\mathcal{L}_{\mathbb{R}}$.*

By Lemma 2.12, the groups $PGL(n,q)$ for $n>2$ lies in $\mathcal{L}_{\mathbb{R}}$ if $PGL(2,q)$ lies in $\mathcal{L}_{\mathbb{R}}$. By using Proposition 2.9, we obtain that $PGL(2,q)$ for $q=3,5,7,9,17$ does not lie in $\mathcal{L}_{\mathbb{C}}$. Recall that $PGL(n,q)=PSL(n,q)$ if $\gcd(n,q-1)=1$. Then $PGL(3,3)$, $PGL(3,5)$, $PGL(3,9)$, $PGL(3,17)$, $PGL(4,4)$, $PGL(4,8)$ lie in $\mathcal{L}_{\mathbb{R}}$ and $PGL(3,8)\in\mathcal{L}_{\mathbb{C}}\smallsetminus\mathcal{M}_{\mathbb{C}+}$.

Since $GL(3,4)$ does not lie in $\mathcal{M}_{\mathbb{C}+}$, it holds $GL(3,4)\notin\mathcal{L}_{\mathbb{C}+}$. Let $K=D_{12}\times C_3$ be a subgroup of $GL(3,4)$ and $\pi\colon K\to H=D_{12}$ be a first projection. Then $K\cap SL(3,4)$ is isomorphic to H through π. By Proposition 2.13, it holds $GL(3,4)\in\mathcal{L}_{\mathbb{C}}$.

Let $G=GL(2,5)$. Let $\xi_3=\mathbb{R}[C_3]-\mathbb{R}$ be the nontrivial irreducible real $G_{\{3\}}$-module. There exists an irreducible real $G_{\{2\}}$-module ξ_2 such that the number $\langle 2\mathbb{R}\oplus\xi_3,\eta\rangle_{G_{\{3\}}}+\langle 2\xi_2,\eta\rangle_{G_{\{2\}}}$ is divisible by 4 for any nontrivial irreducible real G-module and is not divisible by 4 for $\eta=\mathbb{R}$. Thus G does not lies in $\mathcal{L}_{\mathbb{C}+}$. Next let $G=GL(2,7)$. There are an irreducible real $G_{\{3\}}$-module ξ_3 and a real $G_{\{2\}}$-module ξ_2 such that the number $\langle 2\mathbb{R}\oplus\xi_3,\eta\rangle_{G_{\{3\}}}+\langle 2\xi_2,\eta\rangle_{G_{\{2\}}}$ is divisible by 4 for any nontrivial irreducible real G-module and is not divisible by 4 for $\eta=\mathbb{R}$. (We can take ξ_2 as the direct sum of some three nontrivial irreducible modules.) Let $G=GL(2,9)$. There are the real $G_{\{5\}}$-module $\xi_5=\mathbb{R}[C_5]-\mathbb{R}$ and a real $G_{\{2\}}$-module ξ_2 such that the number $\langle\mathbb{R},\eta\rangle_{G_{\{3\}}}+\langle\xi_5,\eta\rangle_{G_{\{5\}}}+\langle\xi_2,\eta\rangle_{G_{\{2\}}}$ is even for any nontrivial irreducible real G-module and is odd for $\eta=\mathbb{R}$. (We can take ξ_2 as the direct sum of some five nontrivial irreducible modules.) Finally let $G=GL(2,17)$ and $\xi_3=\mathbb{R}[C_9]-\mathbb{R}$ the nontrivial irreducible real $G_{\{3\}}$-module. There are an irreducible real $G_{\{17\}}$-module ξ_{17} and a real $G_{\{2\}}$-module ξ_2 such that the number $\langle 2\xi_3,\eta\rangle_{G_{\{3\}}}+\langle 3\xi_{17},\eta\rangle_{G_{\{17\}}}+\langle 2\xi_2,\eta\rangle_{G_{\{2\}}}$ is divisible by 4 for any nontrivial irreducible real G-module and is not divisible by 4 for $\eta = \mathbb{R}$. (We can take ξ_2 as the direct sum of some eight nontrivial irreducible modules and one trivial module.) Therefore we obtain that $GL(2,q)$ for $q = 5,7,9,17$ do not lie in $\mathcal{L}_{C+}$. We finish the classification.

REFERENCES

[1] Conway, J. H., Curtis, R. T., Norton, S. P., Parker, R. A. and Wilson, R. A., ATLAS of finite groups, Maximal Subgroups and Ordinary Characters for Simple Groups, Clarendon press, (1985), Oxford.

[2] Gorenstein, D., Lyons, R. and Solomon, R., The Classification of the Finite Simple Groups, Number 3, Mathematical Surveys and Monographs **40** (1998), AMS.

[3] The GAP Group, *GAP – Groups, Algorithms, and Programming, Version 4.4*; 2006, (`http://www.gap-system.org`).

[4] Morimoto, M. and Pawałowski, K. *The Equivariant Bundle Subtraction Theorem and its applications*, Fundam. Math. **161** (1999) 279–303.

[5] Morimoto, M. and Pawałowski, K. *Smooth actions of finite Oliver groups on spheres*, Topology **42** (2003) 395–421.

[6] Oliver, B. *Fixed point sets and tangent bundles of actions on disks and euclidian spaces*, Topology **35** (1996) 583–615.

[7] Serre, J. P. Linear Representations of Finite Groups, GTM **42** (1977), Springer-Verlag.

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