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Non-existence of free S^1-actions on Kervaire spheres

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The purpose of this note is to prove that the Kervaire spheres Σ_K^{4k+1} where $4k+4$ is not a power of 2, does not admit any free S^1-action if k is not divisible by 4. Recall that the Kervaire sphere Σ_K^{4k+1} is a homotopy sphere that bounds a parallelizable $4k+2$ manifold W^{4k+2} with Kervaire invariant $c(W)$ one. The explicit description of the Kervaire sphere is well known. For example, it can be expressed as the subset of $\mathbb{C}^{2k+2}$ defined by the system of equations

$$z_1^d + z_2^2 + \cdots + z_{2k+2}^2 = 0$$
$$|z_1|^2 + |z_2|^2 + \cdots + |z_{2k+2}|^2 = 1,$$

where d is any positive integer such that $d \equiv \pm 3 \mod 8$. Homotopy spheres that bound parallelizable manifolds are considered to be "least" exotic among exotic spheres. From the standpoint of finite group actions, it is known that every homotopy sphere that bounds a parallelizable manifold admits a free cyclic group action of arbitrary order. As for compact Lie group actions, every odd dimensional standard spheres S^{2n+1} has free S^1-actions and it may seem natural to expect that Kervaire spheres also admit free S^1-actions. But the case is quite different as to the free action of Lie groups. More than thirty years ago, Brumfiel showed that 9-dimensional Kervaire sphere does not have any free S^1-actions ([1]). Since then, this problem has been left untouched. Brumfiel's calculation is essentially the calculation of index surgery obstruction and the relation given by the vanishing of the surgery obstruction is quite complicated. So it seemed quite hard to draw any meaningful conclusion by extending his result from furthe calculations.

In the workshop in 2004 at RIMS, we showed that every Kervaire sphere below dimension 130 does not admit any free S^1 actions based on computer calculation. However that was only an experiment and this time for the first time, we have obtained a partial general result on this problem.

Our main result is the following:

Theorem. The Kervaire sphere of dimension $4k+1$, where $k+1$ is not a power of 2, does not any free S^1-action if k is not divisible by 4.

We shall always assume that k is a positive integer such that $k+1$ is not a power of two. Under this assumption, it is known that the Kervaire sphere Σ_K^{4k+1} is not diffeomorphic to the standard sphere S^{4k+1}.

1. Surgery Obstruction

We shall translate the statement concerning group actions to the one about surgery obstructions.

Lemma 1. The following two statements are equivalent.
(a) The Kervaire sphere Σ_K^{4k+1} does not admit any free S^1-action.

(b) If the normal map

$$\begin{array}{ccc} \nu_M & \xrightarrow{b} & \xi \\ \downarrow & & \downarrow \\ M^{4k+2} & \xrightarrow{f} & \mathbb{C}P(2k+1) \end{array} \tag{1}$$

has zero $4k$-dimensional surgery obstruction $s_{4k} = 0$ for the surgery data

$$f|f^{-1}(\mathbb{C}P(2k)) : f^{-1}(\mathbb{C}P(2k)) \to \mathbb{C}P(2k)$$

obtained by restriction to the codimension 2 subspace, then the $(4k+2)$-dimensional surgery obstruction s_{4k+2} of f vanishes.

Proof. Let us prove that (a) implies (b). Suppose there exists a normal map $f : M^{4k+2} \to \mathbb{C}P(2k+1)$ such that the surgery obstruction s_{4k+2} of f is nonzero and the restricted surgery problem to $\mathbb{C}P(2k)$ has zero surgery obstruction $s_{4k} = 0$. Then we can perform surgery on $f^{-1}(\mathbb{C}P(2k))$ and within the normal cobordism class we may assume that $X = f^{-1}(\mathbb{C}P(2k)) \to \mathbb{C}P(2k)$ is a homotopy equivalence. The tubular neighborhood N of X is homotopy equivalent to $\mathbb{C}P(2k+1)_0 = \mathbb{C}P(2k+1) - \mathrm{int}D^{4k+2}$ and its boundary ∂N is homotopy equivalent to S^{4k+1}. But the remaining part $W = M - \mathrm{int}(N)$ is a parallelizable manifold and its surgery obstruction for the normal map $W \to D^{4k+2}$ rel. ∂W is nonzero. Therefore W has nonzero Kervaire obstruction and its boundary $\partial W = \partial N$ is the Kervaire sphere. Since ∂N is the total space of an S^1-bundle, this implies that the Kervaire sphere admits a free S^1-action.

Conversely, suppose that (b) holds, but (a) does not hold. If the Kervaire sphere Σ_K^{4k+1} admits a free S^1-action, the quotient space of the S^1-action $X^{4k} = \Sigma^{4k+1}/S^1$ is homotopy equivalent to the complex projective space $\mathbb{C}P(2k)$ and the associated D^2-bundle $N^{4k+2} = (\Sigma_K^{4k+1} \times D^2)/S^1$ is homotopy equivalent to $\mathbb{C}P(2k+1)_0 = (S^{4k+1} \times D^2)/S^1$ where the $S^1 \subset \mathbb{C}$ acts on $S^{4k+1} \subset \mathbb{C}^{2k+1}$ and on $D^2 \subset \mathbb{C}$ by complex number multiplication. Let W^{4k+2} be a smooth parallelizable manifold with $\partial W = \Sigma_K^{4k+1}$ and Kervaire invariant $c(W) = 1$. Then by gluing N and W along the common boundary Σ_K, we obtain a normal map $f : M^{4k+2} = N \cup_{\Sigma_K} W \to \mathbb{C}P(2k+1)$ with an appropriate vector bundle, and its surgery obstruction s_{4k+2} is equal to $c(W) = 1$. Hence we have a normal map f with target space $\mathbb{C}P(2k+1)$ with nonzero Kervaire surgery obstruction, but the codimension 2 surgery problem obtained by restricting the target manifold to $\mathbb{C}P(2k)$ has zero surgery obstruction $s_{4k} = 0$, since $f|X^{4k} : X^{4k} \to \mathbb{C}P(2k)$ is a homotopy equivalence. This contradicts the assumption (b). This completes the proof of Lemma 1.

Our objective of this note is to show that the statement (b) in Lemma 1 is true. To do so, we must deal with all possible vector bundles that appear in (1). We point out the following four items that needs consideration:

Bundle data: The stable bundle difference $\zeta = \nu_{\mathbb{C}P(2k+1)} - \xi$ is fiber homotopically trivial, namely it belongs to the kernel of the J-homomorphism $J : \widetilde{KO}(\mathbb{C}P(2k+1)) \to \tilde{J}(\mathbb{C}P(2k+1))$. The generators of the kernel can be expressed by Adams operations in KO-theory. The solution of the Adams conjecture imply that 2-local generators are given by the images of $\psi^3_{\mathbb{R}} - 1$ ([5]Theorem 11.4.1).

The surgery obstruction s_{4k} in dimension $4k$: In dimension $4k$, the surgery obstruction is given by the index obstruction, which can be computed using Hirzebruch's L classes. However, the exact form of the obstruction gets complicated and requires simplified treatment.

Surgery obstruction s_{4k+2} in dimension $4k+2$: The surgery obstruction s_{4k+2} in dimension $4k+2$ can be dealt with by the results of Stolz([4]) or [2],[3]. In fact, the obstruction s_{4k+2} is equal to the two dimensional obstruction s_2 for the surgery data s_2, which is essentially the 2-dimensional Kervaire class K_2.

Relation of K_2 and the first Pontrjagin class p_1: From the result originally due to Sullivan, the square of K_2 for the bundle data ζ is equal to $p_1(\zeta)/8 \mod 2$ (see [6], 14C). This fact gives the bridge connecting integral index obstruction and the mod 2 Kervaire obstruction.

2. Index obstruction in dimension $4k$

The kernel of the 2-local J-homomorphism $J : \widetilde{KO}(\mathbb{C}P(2k+1)) \to \tilde{J}(\mathbb{C}P(2k+1))$ is generated by Image$(\psi^3_{\mathbb{R}} - 1)$. The additive generators of $\widetilde{KO}(\mathbb{C}P(2k+1))$ are given by ω^j $(1 \le j \le k+1)$ where ω is the realification of the complex virtual vector bundle $\eta_{\mathbb{C}} - 1_{\mathbb{C}}$. The Adams operation $\psi^j_{\mathbb{R}}$ on ω is given by the formula

(2) $$\psi^j_{\mathbb{R}}(\omega) = T_j(\omega)$$

where $T_j(z)$ is a polynomial of degree j characterized by

(3) $$T_j(t + t^{-1} - 2) = t^j + t^{-j} - 2\ .$$

Since the coefficient of z^j in $T_j(z)$ is one, we may take $T_j(\omega)$ $(1 \le j \le k+1)$ as generators of $\widetilde{KO}(\mathbb{C}P(2k+1))$. However, when restricted on $\mathbb{C}P(2k)$, we have $\omega^{k+1} = 0$ and we may safely discard ω^{k+1} in the actual computation. In our argument, we have only to know the 2-local kernel of the J-map $J : \widetilde{KO}(\mathbb{C}P(2k+1)) \to \tilde{J}(\mathbb{C}P(2k+1))$. The 2-local generators of the kernel of J are

(4) $$\zeta_j = (\psi^3_{\mathbb{R}} - 1)\psi^j_{\mathbb{R}}(\omega) \qquad (j = 1, 2, \ldots, k)$$

and any element in the 2-local kernel of the J-homomorphism has the form

(5) $$\zeta = \sum_{j=1}^{k} m_j \zeta_j$$

where m_j are integers.

The surgery obstruction s_{4k} of the surgery data (1) when restricted on $\mathbb{C}P(2k)$ is given by

(6) $$8s_{4k} = (\mathrm{Index}(M) - \mathrm{Index}(\mathbb{C}P(2k))) = ((\mathcal{L}(\zeta) - 1)\mathcal{L}(\mathbb{C}P(2k)))\,[\mathbb{C}P(2k)]$$

where $\mathcal{L}$ is the multiplicative class defined by the power series

(7) $$h(x) = \frac{x}{\tanh x} = 1 + \sum_{i \ge 1} \frac{(-1)^{i+1} 2^{2i} B_i}{(2i)!} x^{2i}$$

where B_i is the i-th Bernoulli number. Remark that all the coefficients of $h(x)$ belong to $\mathbb{Z}_{(2)}$ the rational numbers with odd denominator because all the denominators of Bernoulli numbers are even but not divisible by four. If the total

Pontrjagin class of a bundle ξ is given by $p(\xi) = \prod_i(1+x_i^2)$, $\mathcal{L}(\xi)$ is given by $\prod_i h(x_i)$ and when M is a manifold, we define $\mathcal{L}(M) = \mathcal{L}(\tau_M)$.

It is not difficult to show that the total Pontrjagin class of $\psi^j_{\mathbb{R}}(\omega)$ is $1 + j^2x^2$, where x is the generator of $H^2(\mathbb{C}P(2k+1))$. For the virtual bundle ζ in (5), we have

$$\mathcal{L}(\zeta) = \prod_{j=1}^{k} \left(\frac{3jx}{\tanh 3jx} \frac{\tanh jx}{jx} \right)^{m_j}. \tag{8}$$

Given a power series $f(x)$ in x, let us express the the coefficient of x^n in $f(x)$ by $(f(x))_n$. The $4k$-dimensional obstruction s_{4k} is given by

$$\left((\mathcal{L}(\zeta) - 1) \left(\frac{x}{\tanh x} \right)^{2k+1} \right)_{2k} / \ 8. \tag{9}$$

To calculate this, we put

$$g(x) = \left(\frac{3x}{\tanh 3x} \frac{\tanh x}{x} \right) - 1. \tag{10}$$

Lemma 2. All the coefficients of $g(x)$ is divisible by 8 in $\mathbb{Z}_{(2)}$.

Proof. From the expansion (7), we have

$$\frac{3x}{\tanh 3x} \equiv \frac{x}{\tanh x} \mod 8 \quad \text{in} \quad \mathbb{Z}_{(2)}[[x]].$$

Noting that $x/\tanh x$ is invertible in $\mathbb{Z}_{(2)}[[x]]$, we have

$$\frac{3x}{\tanh 3x} \frac{\tanh x}{x} \equiv 1 \mod 8 \quad \text{in} \quad \mathbb{Z}_{(2)}[[x]].$$

and the assertion follows.

Let j be an integer. If j is even we have $g(jx) \equiv 0 \mod 32$, and if j is odd, we have $g(jx) \equiv g(x) \mod 64$ since the coefficiets of $g(x)$ are all divisible by 8. Hence we have

$$\begin{aligned} \mathcal{L}(\zeta) - 1 &= \prod_j (1 + g(jx))^{m_j} - 1 \\ &\equiv \prod_{j:\text{odd}} (1 + g(x))^{m_j} - 1 \qquad \mod 32 \\ &\equiv \left(\sum_{j:\text{odd}} m_j \right) g(x) \qquad \mod 64. \end{aligned}$$

From this, if we want to calculate the $4k$-dimensional surgery obstruction $8s_{4k}$, we can get it as the coefficient of x^{2k} in

$$\begin{aligned} (\mathcal{L}(\zeta) - 1)h(x)^{2k+1} &\equiv mg(x)(h(x))^{2k+1} \quad \mod 32 \\ &= m \left(\frac{3 \tanh x}{\tanh 3x} - 1 \right) \left(\frac{x}{\tanh x} \right)^{2k+1} \\ &= \frac{8m}{3} \sum_{i \geq 1} \left(\frac{-1}{3} \right)^{i-1} \tanh^{2i} x \left(\frac{x}{\tanh x} \right)^{2k+1} \end{aligned}$$

The coefficient of x^{2k} can be calculated using the residue theory:

$$\left(mg(x)h(x)^{2k+1}\right)_{2k} = \left(\frac{8m}{3}\sum_{i\geq 1}\left(\frac{-1}{3}\right)^{i-1}\tanh^{2i}x\left(\frac{x}{\tanh x}\right)^{2k+1}\right)_{2k}$$
$$= \frac{8m}{3}\mathrm{Res}\left(\frac{1}{\tanh^{2k+1}x}\sum_{i\geq 1}\left(\frac{-1}{3}\right)^{i-1}\tanh^{2i}x;\ x=0\right).$$

By changing variables $y = \tanh x$, this residue value is equal to

$$\frac{8m}{3}\mathrm{Res}\left(\frac{1}{y^{2k+1}(1-y^2)}\sum_{i\geq 1}\left(\frac{-1}{3}\right)^{i-1}y^{2i};\ y=0\right)$$
$$= \frac{8m}{3}\sum_{i=1}^{k}\left(\frac{-1}{3}\right)^{i-1} = \frac{2m(3^k-(-1)^k)}{3^k}$$

Thus we have shown that the surgery obstruction s_{4k} satisfies

$$8s_{4k} \equiv \frac{2m(3^k-(-1)^k)}{3^k} \mod 32 \tag{11}$$

where $m = \sum_{j:\mathrm{odd}} m_j$.

Given an integer a, let us denote by $\nu_2(a)$ the 2-order of a. The following is an easy exercise in elementary number theory.

Lemma 3. We have

$$\nu_2(3^k-(-1)^k) = \nu_2(k)+2.$$

Lemma 4. If the $4k$-dimensional surgery obstruction s_{4k} vanishes and suppose that k is not divisible by 4, then $m = \sum_{j:\mathrm{odd}} m_j$ is even.

Proof. If $s_{4k} = 0$, then $\nu_2(2m(3^k-(-1)^k)) \geq 5$. Suppose $\nu_2(k) \leq 1$, then $\nu_2(3^k-(-1)^k) \leq 3$, and m must be even.

3. First Pontrjagin class and Kervaire surgery obstruction

In the normal map (1), let $\zeta = \nu_{\mathbf{C}P(2k+1)} - \xi$, then it can be written (2-locally) $\zeta = \sum_{j=1}^{k} m_j\zeta_j$ where $\zeta_j = (\psi^3_{\mathbf{R}}-1)\psi^j_{\mathbf{R}}(\omega)$. The total Pontrjagin class of $\psi^m_{\mathbf{R}}(\omega)$ is given by

$$p(\psi^m_{\mathbf{R}}(\omega)) = 1+m^2x^2$$

and we have

$$p(\zeta_j) = \frac{1+9j^2x^2}{1+j^2x^2}$$
$$p(\zeta) = \prod_j\left(\frac{1+9j^2x^2}{1+j^2x^2}\right)^{m_j}.$$

For the first Pontrjagin class, we have

$$p_1(\zeta)/8 = \left(\sum_j j^2 m_j\right)x^2. \tag{12}$$

We know that the 2-dimensional surgery obstruction s_2 for $f|f^{-1}(\mathbb{C}P(1))$ is equal to $\sum_j j^2 m_j \mod 2$ since in the complex projective space surgery theory, the mod 2 reduction of $p_1(\zeta)$ coincides with the square of the 2-dimensional Kervaire class for the given normal map (see Wall's book [6, Chap 13.]). And it is known that if $k+1$ is not a power of 2, then $(4k+2)$-dimensional surgery obstruction coincides with the 2-dimensional surgery obstruction ([4],[2],[3]). From these facts we have

Lemma 5. If $\sum_{j:odd} m_j$ is even, then the surgery obstruction s_{4k+2} vanishes.

4. Proof of the main theorem

Let us suppose that k is not divisible by 4. Take any normal map with target manifold $\mathbb{C}P(2k+1)$ and its vector bundle data $\zeta = \sum_j \zeta_j$. Suppose that its codimension 2 surgery obstruction s_{4k} vanishes. Then from Lemma 4, $m = \sum_{j:\mathrm{odd}} m_j$ is even. Then by Lemma 5, the surgery obstruction s_{4k+2} vanishes. In view of Lemma 1, this proves our theorem.

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