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MEASURE-PRESERVING HOMEOMORPHISMS OF NONCOMPACT MANIFOLDS AND MASS FLOW TOWARD ENDS

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1. Introduction

This article is concerned with groups of measure-preserving homeomorphisms of non-compact topological manifolds. Suppose M is a connected n-manifold and ω is a good Radon measure of M with $\omega(\partial M) = 0$. Let $\mathcal{H}(M)$ denote the group of homeomorphisms of M equipped with the compact-open topology, and by $\mathcal{H}(M;\omega) \subset \mathcal{H}(M;\omega\text{-end-reg})$ we denote the subgroups consisting of ω-preserving homeomorphisms and 'ω-end-biregular' homeomorphisms of M. (When M is compact, the conditions related to ends are redundant and are suppressed from the notations.) For any subgroup $\mathcal{G}$ of $\mathcal{H}(M)$, the symbol $\mathcal{G}_0$ denotes the connected component of id_M in $\mathcal{G}$.

Relations of these groups are studied in [6, 2, 3, 4, 8]. When M is compact, A. Fathi [6] showed that $\mathcal{H}(M;\omega)$ is a SDR (strong deformation retract) of $\mathcal{H}(M;\omega\text{-reg})$ and that $\mathcal{H}(M;\omega\text{-reg})$ is HD (homotopy dense) in finite dimension in $\mathcal{H}(M)$. In case $n = 2$, since $\mathcal{H}(M)$ is an ANR, this implies that $\mathcal{H}(M;\omega\text{-reg})$ is HD in $\mathcal{H}(M)$ and $\mathcal{H}(M;\omega)$ is a SDR of $\mathcal{H}(M)$. When M is non-compact, R. Berlanga [2, 3, 4] extended Fathi's arguments and showed that $\mathcal{H}(M;\omega)$ is a SDR of $\mathcal{H}(M;\omega\text{-end-reg})$. In case $n = 2$, we have shown that $\mathcal{H}(M;\omega\text{-end-reg})_0$ is HD in $\mathcal{H}(M)_0$ and thus $\mathcal{H}(M;\omega)_0$ is a SDR of $\mathcal{H}(M)_0$ [8]. However, we have no general results on relations between $\mathcal{H}(M;\omega\text{-end-reg})$ and $\mathcal{H}(M)$ in dimension $n \geq 3$.

A. Fathi [6] also studied the internal structure of $\mathcal{H}(M;\omega)$. When M is compact, he defined a mass flow homomorphism $\widetilde{\theta} : \widetilde{\mathcal{H}}_0(M,\omega) \to H_1(M,\mathbb{R})$ or $\theta : \mathcal{H}_0(M,\omega) \to H_1(M,\mathbb{R})/\Gamma$ and studied the existence of a section of $\widetilde{\theta}$ and the perfectness of $\mathrm{Ker}\,\theta$. In this article we consider the non-compact case and study a mass flow homomorphism toward ends [9]. Let $\mathcal{H}_E(M;\omega)$ denote the subgroup consisting of all $h \in \mathcal{H}(M;\omega)$ which fix the ends of M. There is a natural continuous homomorphism $J : \mathcal{H}_E(M;\omega) \to V_\omega$ which measures mass flow toward ends. This quantity has been introduced in [1] as the end charge c_h $(h \in \mathcal{H}_E(M;\omega))$, which are finitely additive signed measure on the ends of M. We use the following presentation of this notion: If $h \in \mathcal{H}_E(M;\omega)$ and C is a Borel subset of M such that $\mathrm{Fr}\,C$ is compact, then the mass transfered into C by h is counted by $J_h(C) = \omega(C - h(C)) - \omega(h(C) - C)$. The range V_ω is the topological vector space of functions $J_h : C \mapsto J_h(C)$, which parametrize mass flow toward ends.

We use deformation of measures by engulfing isotopy in M and show that the mass flow homomorphism J has a continuous (non-homomorphic) section.

Theorem 1.1. *There exists a continuous map* $s : V_\omega \to \mathcal{H}_\partial(M, \omega)_1$ *such that* $Js = id_{V_\omega}$ *and* $s(0) = id_M$.

The topological group $\mathcal{H}_E(M, \omega)$ acts continuously on V_ω by $h \cdot a = J_h + a$ ($h \in \mathcal{H}_E(M, \omega)$, $a \in V_\omega$). The mass flow homomorphism $J : \mathcal{H}_E(M, \omega) \to V_\omega$ coincides with the orbit map at $0 \in V_\omega$. The existence of section for this orbit map and the contractibility of the base space V_ω implies the following consequences.

Corollary 1.1. (1) $\mathcal{H}_E(M; \omega) \cong \mathrm{Ker}\, J \times V_\omega$.

(2) $\mathrm{Ker}\, J$ *is a strong deformation retract of* $\mathcal{H}_E(M; \omega)$.

In [10] we have obtained a version of Theorem 1.1 for smooth manifolds and volume-preserving diffeomorphisms. In the succeeding sections we explain definition of the mass flow homomorphism J toward ends (§§ 2 - 4) and give some details of arguments to deduce Theorem 1.1 (§§ 5 - 6).

2. End compactifications

2.1. **Conventions.** Throughout the paper, X denotes a connected, locally connected, locally compact, separable metrizable space, and the symbols $\mathcal{O}(X)$, $\mathcal{F}(X)$, $\mathcal{K}(X)$, and $\mathcal{C}(X)$ denote the sets of open subsets, closed subsets, compact subsets, and connected components of X respectively. When A is a subset of X, the symbols $\mathrm{Fr}_X A$, $\mathrm{cl}_X A$ and $\mathrm{Int}_X A$ denote the frontier, closure and interior of A relative to X.

The symbol $\mathcal{H}_A(X)$ denotes the group of homeomorphisms h of X onto itself with $h|_A = id_A$, equipped with the compact-open topology. This group includes various subgroups. $\mathcal{H}_A^c(X)$ denotes the subgroup consisting of homeomorphisms with compact support. When X is a polyhedron, $\mathcal{H}^{\mathrm{PL}}(X)$ denotes the subgroup of PL-homeomorphisms of X. For any subgroup G of $\mathcal{H}(X)$, the symbols G_0 and G_1 denote the connected component and the path-component of id_M in G respectively. When $G \subset \mathcal{H}^c(X)$, by G_1^* we denote the subgroup of G_1 consisting of $h \in G$ which admits an isotopy $h_t \in G$ ($t \in [0, 1]$) such that $h_0 = id_X$, $h_1 = h$ and there exists $K \in \mathcal{K}(X)$ with $\mathrm{Supp}\, h_t \subset K$ ($t \in [0, 1]$).

2.2. **End compactifications.** (cf. [4])

Suppose X is a noncompact, connected, locally connected, locally compact, separable metrizable space. An end of X is a function e which assigns an $e(K) \in \mathcal{C}(X - K)$ to each $K \in \mathcal{K}(X)$ such that $e(K_1) \supset e(K_2)$ if $K_1 \subset K_2$. The set of ends of X is denoted by $E = E_X$. The end compactification of X is the space $\overline{X} = X \cup E$ equipped with the topology defined by the following conditions:

(i) X is an open subspace of $\overline{X}$,
(ii) the fundamental open neighborhoods of $e \in E$ are given by

$$N(e,K) = e(K) \cup \{e' \in E \mid e'(K) = e(K)\} \quad (K \in \mathcal{K}(X)).$$

Then, $\overline{X}$ is a connected, locally connected, compact, metrizable space, X is a dense open subset of $\overline{X}$ and E is a compact 0-dimensional subset of $\overline{X}$. We fix a metric d on $\overline{X}$. For any $\varepsilon > 0$ there exists a neighborhood U of E in $\overline{X}$ such that $\operatorname{diam}_d C < \varepsilon$ $(C \in \mathcal{C}(U))$.

Consider the family $\mathcal{S} = \mathcal{S}(X) = \{C \subset X \mid \operatorname{Fr}_X C : \text{compact}\}$. For each $C \in \mathcal{S}$ we set

$$\overline{C} = C \cup E_C, \qquad E_C = \{e \in E_X \mid e(K) \subset C \text{ for some } K \in \mathcal{K}(X)\}.$$

Then, E_C is open and closed in E_X and $\overline{C}$ is a neighborhood of E_C in $\overline{X}$.

Lemma 2.1. *Let $C, D \in \mathcal{S}(X)$.*

(1) (i) $C \cup D \in \mathcal{S}(X)$ *and* $E_{C\cup D} = E_C \cup E_D$.
(ii) $C \cap D \in \mathcal{S}(X)$ *and* $E_{C\cap D} = E_C \cap E_D$.
(iii) $X - C \in \mathcal{S}(X)$ *and* $E_{X-C} = E_X - E_C$.

(2) (i) $E_C \subset E_D$ *iff* $C - D$ *is relatively compact in* X *(i.e., has the compact closure in* X *).*
(ii) $E_C = E_D$ *iff the symmetric difference* $C \Delta D = (C - D) \cup (D - C)$ *is relatively compact in* X.

Each $h \in \mathcal{H}(X)$ has a unique extension $\overline{h} \in \mathcal{H}(\overline{X})$. The map $\mathcal{H}(X) \to \mathcal{H}(\overline{X}) : h \mapsto \overline{h}$ is a continuous group homomorphism. We set $\mathcal{H}_{A\cup E}(X) = \{h \in \mathcal{H}_A(X) \mid \overline{h}|_E = id_E\}$. Then $\mathcal{H}_{A\cup E}(X)_0 = \mathcal{H}_A(X)_0$, and if $C \in \mathcal{S}(X)$ and $h \in \mathcal{H}_E(X)$, then $h(C) \in \mathcal{S}(X)$ and $E_{h(C)} = E_C$.

3. Fundamental facts on Radon measures

Next we recall general facts on spaces of Radon measures cf. [4, 6]. Suppose X is a connected, locally connected, locally compact, separable metrizable space.

3.1. Spaces of Radon measures.

Let $\mathcal{B}(X)$ denote the σ-algebra of Borel subsets of X. A Radon measure on X is a measure μ on the measurable space $(X, \mathcal{B}(X))$ such that $\mu(K) < \infty$ for any compact subset K of X. Let $\mathcal{M}(X)$ denote the set of Radon measures on X. We say that $\mu \in \mathcal{M}(X)$ is good if $\mu(p) = 0$ for any point $p \in X$ and $\mu(U) > 0$ for any nonempty open subset U of X. For $A \in \mathcal{B}(X)$ let $\mathcal{M}_g^A(X)$ denote the set of good Radon measures μ on X with $\mu(A) = 0$.

The weak topology w on $\mathcal{M}(X)$ is the weakest topology such that the function

$$\Phi_f : \mathcal{M}(X) \to \mathbb{R} : \ \Phi_f(\mu) = \int_X f \, d\mu$$

is continuous for any continuous function $f : X \to \mathbb{R}$ with compact support. The notation $\mathcal{M}(X)_w$ denotes the space $\mathcal{M}(X)$ equipped with the weak topology w.

For $\mu \in \mathcal{M}(X)$ and $A \in \mathcal{B}(X)$ the restriction $\mu|_A \in \mathcal{M}(A)$ is defined by $(\mu|_A)(B) = \mu(B)$ $(B \in \mathcal{B}(A))$. For any $A \in \mathcal{F}(X)$ the restriction map $\mathcal{M}^{\mathrm{Fr}A}(X)_w \to \mathcal{M}(A)_w : \mu \mapsto \mu|_A$ is continuous, and for any $K \in \mathcal{K}(X)$ the map $\mathcal{M}^{\mathrm{Fr}K}(X)_w \to \mathbb{R} : \mu \mapsto \mu(K)$ is continuous ([4, Lemma 2.2]).

3.2. **Action of homeomorphism groups.**

Suppose $A \in \mathcal{B}(X)$ and $\omega \in \mathcal{M}(X)$.

Definition 3.1. $\mu \in \mathcal{M}(X)$ is said to be

(i) ω-biregular if μ and ω have same null sets (i.e., $\mu(B) = 0$ iff $\omega(B) = 0$ for any $B \in \mathcal{B}(X)$),

(ii) ω-mass-biregular if μ is ω-biregular and $\mu(X) = \omega(X)$,

(iii) ω-cpt-biregular if μ is ω-biregular and $\mu|_{X-K} = \omega|_{X-K}$ for some $K \in \mathcal{K}(X)$.

The corresponding subspaces are denoted by the following symbols respectively:

$$\mathcal{M}(X, \omega\text{-reg}), \quad \mathcal{M}(X, \omega\text{-mass-reg}), \quad \mathcal{M}(X, \omega\text{-cpt-reg}).$$

Definition 3.2. $h \in \mathcal{H}(X)$ is said to be

(i) ω-preserving if $h_*\omega = \omega$ (i.e., $\omega(h(B)) = \omega(B)$ for any $B \in \mathcal{B}(X)$),

(ii) ω-biregular if $h_*\omega$ and ω have the same null sets (i.e., $\omega(h(B)) = 0$ iff $\omega(B) = 0$ for any $B \in \mathcal{B}(X)$) ([6]).

The corresponding subgroups are denoted by the following symbols:

$$\mathcal{H}(X; \omega) = \{h \in \mathcal{H}(X) \mid h : \omega\text{-preserving}\}, \quad \mathcal{H}(X; \omega\text{-reg}) = \{h \in \mathcal{H}(X) \mid h : \omega\text{-biregular}\}.$$

The group $\mathcal{H}(X)$ acts continuously on $\mathcal{M}(X)_w$ by $h \cdot \mu = h_*\mu$ $(h \in \mathcal{H}(X),\ \mu \in \mathcal{M}(X))$. The orbit map at $\omega \in \mathcal{M}(X)$ is defined by $\pi_\omega : \mathcal{H}(X) \to \mathcal{M}(X)$, $\pi_\omega(h) = h_*\omega$. The subgroup $\mathcal{H}(X; \omega)$ coincides with the stabilizer of ω under this action.

Suppose M is a compact connected n-manifold. The von Neumann-Oxtoby-Ulam theorem [7] asserts that if $\mu, \nu \in \mathcal{M}_g^\partial(M)$ and $\mu(M) = \nu(M)$, then there exists $h \in \mathcal{H}_\partial(M)_0$ such that $h_*\mu = \nu$. A. Fathi [6] extended this theorem to a parametrized version.

Theorem 3.1. *Suppose $\mu, \nu : P \to \mathcal{M}_g^\partial(M; \omega\text{-reg})_w$ are continuous maps with $\mu_p(M) = \nu_p(M)$ $(p \in P)$. Then there exists a continuous map $h : P \to \mathcal{H}_\partial(M; \omega\text{-reg})_1$ such that $(h_p)_*\mu_p = \nu_p$ $(p \in P)$ and if $p \in P$ and $\mu_p = \nu_p$, then $h_p = id_M$.*

R. Berlanga [4] obtained a similar theorem in the case that M is noncompact.

3.3. **Spaces of Radon measures with direct limit topology.**

Suppose $A \in \mathcal{B}(X)$ and $\omega \in \mathcal{M}_g^A(X)$. Let $\mathcal{B}_\omega(X) = \{C \in \mathcal{B}(X) \mid \omega(\mathrm{Fr}_X C) = 0\}$ and $\mathcal{F}_\omega(X) = \mathcal{F}(X) \cap \mathcal{B}_\omega(X)$. For $\mu, \nu \in \mathcal{M}_g^A(X, \omega\text{-cpt-reg})$ and $C \in \mathcal{B}(X)$ we define $(\mu - \nu)(C) \in \mathbb{R}$ by

$(\mu - \nu)(C) = (\mu - \nu)(C \cap K)$, where K is any compact subset of X such that $\mu|_{X-K} = \nu|_{X-K}$.

For the sake of notational simplicity, we put $\mathcal{M} = \mathcal{M}_g^A(X, \omega\text{-cpt-reg})$ (as a set). Let $C \in \mathcal{B}_\omega(X)$. One can see that the function $\mathcal{M}_w \times \mathcal{M}_w \to \mathbb{R} : (\mu, \nu) \mapsto (\mu - \nu)(C)$ is not continuous (if X is noncompact). This forces us to introduce the direct limit topology *lim* instead of the weak topology w (cf. [4, p 244]). For each $K \in \mathcal{K}(X)$ consider the subspace $\mathcal{M}_{K,w} = \left\{\mu \in \mathcal{M}_w \;\middle|\; \mu|_{X-K} = \omega|_{X-K}\right\}$ of $\mathcal{M}_w$. The family $\{\mathcal{M}_K\}_{K \in \mathcal{K}(X)}$ is a closed cover of $\mathcal{M}_w$ (cf. [4, Lemma 3.1]). The topology *lim* on $\mathcal{M}$ is the finest topology on $\mathcal{M}$ such that the inclusion $i_K : \mathcal{M}_{K,w} \subset \mathcal{M}$ is continuous for each $K \in \mathcal{K}(X)$. The space $\mathcal{M}$ equipped with this topology is denoted by $\mathcal{M}_{lim} = \mathcal{M}_g^A(X, \omega\text{-cpt-reg})_{lim}$. Each $\mathcal{M}_{K,w}$ is a closed subspace of $\mathcal{M}_{lim}$ and a map $f : \mathcal{M}_{lim} \to Z$ is continuous iff the composition $f i_K$ is continuous for each $K \in \mathcal{K}(X)$.

Lemma 3.1. (1) *Suppose $\mu, \nu : P \to \mathcal{M}_g^A(X, \omega\text{-cpt-reg})_{lim}$ are continuous maps and $C \in \mathcal{B}_\omega(X)$. Then the map $P \to \mathbb{R} : p \mapsto (\mu_p - \nu_p)(C)$ is continuous.*

(2) *Suppose $F \in \mathcal{F}_\omega(X)$ and F is regular closed (i.e., $F = cl\, U$ for some $U \in \mathcal{O}(X)$). Then the restriction map $r : \mathcal{M}_g^A(X, \omega\text{-cpt-reg})_{lim} \to \mathcal{M}_g^{A\cap F}(F, \omega|_F\text{-cpt-reg})_{lim} : r(\mu) = \mu|_F$ is continuous.*

Definition 3.3. Suppose G is a subgroup of $\mathcal{H}(X)$. Consider the following condition $(*)$ on a map $h : P \to G$.

- $(*)_0$ h is continuous.
- $(*)_1$ For any $p \in P$ there exists an open neighborhood U of p in P and $K \in \mathcal{K}(X)$ such that $h(U) \subset \mathcal{H}_{X-K}(X)$.
- $(*)_2$ There exists a locally compact T_2 space Q and continuous maps $f : P \to Q$, $g : Q \to G$ such that $h = gf$.

Since G is a topological group, if $h, k : P \to G$ satisfy the condition $(*)$, then the inverse $h^{-1} : P \to G : (h^{-1})_p = (h_p)^{-1}$ and the composition $kh : P \to G : (kh)_p = k_p h_p$ satisfy the same condition.

Lemma 3.2. *Suppose $\mu, \nu : P \to \mathcal{M}_g^A(X, \omega\text{-cpt-reg})_{lim}$ are continuous maps and $h : P \to \mathcal{H}_A^c(X, \omega\text{-reg})$ satisies the condition $(*)$.*

(1) *For any $C \in \mathcal{B}_\omega(X)$ the map $\varphi : P \to \mathbb{R} : \varphi(p) = ((h_p)_*\mu_p - \nu_p)(C)$ is continuous.*

(2) *The map $\psi : P \to \mathcal{M}_g^A(X, \omega\text{-cpt-reg})_{lim} : \psi(p) = (h_p)_*\mu_p$ is continuous.*

4. Mass flow homomorphism toward ends

Suppose X is a connected, locally connected, locally compact separable, metrizable space and $\mu \in \mathcal{M}(X)$. Let $\mathcal{S}_b = \mathcal{S}_b(X) = \mathcal{S}(X) \cap \mathcal{B}(X)$.

Definition 4.1. For $h \in \mathcal{H}_E(X, \mu)$ we define a function $J_h = J_h^\mu : \mathcal{S}_b \to \mathbb{R}$ as follows: Since $\overline{h}|_E = id$, for $C \in \mathcal{S}_b$ it follows that $E_C = E_{h(C)}$ and that $C \Delta h(C) = (C - h(C)) \cup (h(C) - C)$ is relatively compact in X (Lemma 2.1 (2)(ii)). Thus $\mu(C - h(C)), \mu(h(C) - C) < \infty$ and we can set $J_h(C) = \mu(C - h(C)) - \mu(h(C) - C)$.

Lemma 4.1. *Let $C, D \in \mathcal{S}_b$.*

(1) (i) *If $D \subset C \cap h(C)$ and $cl_X(C - D)$ is compact, then $J_h(C) = \mu(C - D) - \mu(h(C) - D)$.*
(ii) *If $L \in \mathcal{K}(X)$ and $C \cup L = h(C) \cup L$, then $J_h(C) = \mu(C \cap L) - \mu(h(C) \cap L)$.*
(2) *If $cl_X(C \Delta D)$ is compact (i.e. $E_C = E_D$), then $J_h(C) = J_h(D)$.*
(3) *If $C \cap D = \emptyset$, then $J_h(C \cup D) = J_h(C) + J_h(D)$.*
(4) *If $\mu(C) < \infty$, then $J_h(C) = 0$.*
(5) *$J_h(X) = 0$.*

This lemma suggests the next definition of the mass flow homomorphism J.

Definition 4.2.

(1) $V_\mu = V_\mu(X) = \{a : \mathcal{S}_b \to \mathbb{R} \mid (*)_1, (*)_2, (*)_3, (*)_4\}$
$(*)_1$ If $C, D \in \mathcal{S}_b$ and $cl_X(C \Delta D)$ is compact (i.e., $E_C = E_D$), then $a(C) = a(D)$.
$(*)_2$ If $C, D \in \mathcal{S}_b$ and $C \cap D = \emptyset$, then $a(C \cup D) = a(C) + a(D)$.
$(*)_3$ If $C \in \mathcal{S}_b$ and $\mu(C) < \infty$, then $a(C) = 0$.
$(*)_4$ $a(X) = 0$
(2) $J : \mathcal{H}_E(X, \mu) \to V_\mu : h \mapsto J_h$.

For $a, b \in V_\mu$ and $\alpha, \beta \in \mathbb{R}$, we define $\alpha a + \beta b \in V_\mu$ by $(\alpha a + \beta b)(C) = \alpha a(C) + \beta b(C)$ $(C \in \mathcal{S}_b)$. Then V_μ is a real vector space under these addition and scalar product. We equip V_μ the product topology, that is, the topology induced by the projections $\pi_C : V_\mu \to \mathbb{R}$: $\pi_C(a) = a(C)$ $(C \in \mathcal{S}_b)$. Thus, a map $f : Y \to V_\mu$ is continuous iff $\pi_C f : Y \to \mathbb{R}$ is continuous for each $C \in \mathcal{S}_b$. With this topology V_μ is a topological vector space.

Lemma 4.2. *J is a continuous group homomorphism.*

5. Deformation of measures by engulfing isotopy

Throughout this section we suppose M^n is a connected separable metrizable PL n-manifold, d is any metric on the end compactification $\overline{M}$, and $\omega \in \mathcal{M}_g^\partial(M)$. As a consistency condition between the PL-structure of M and the measure ω, we assume that

$\mathcal{H}_\partial^{\mathrm{PL}}(M) \subset \mathcal{H}(M;\omega\text{-reg})$. It follows that $\omega(K) = 0$ for any subpolyhedron K of M with $\dim K \le n-1$.

5.1. **Deformation of measures by engulfing isotopy.**

Consider a decomposition $M = L \cup_S N$ such that

(i) L and N are connected PL n-submanifolds of M with $S = L \cap N = \mathrm{Fr}_M L = \mathrm{Fr}_M N$,
(ii) S is a compact proper PL $(n-1)$-submanifold of M.

Lemma 5.1. *There exists a continuous map $f : (-\infty,\infty) \to \mathcal{H}_\partial^{\mathrm{PL,c}}(M)_1^*$ such that*

(i) (a) $f_0 = id$, (b) $f_s(L) \subsetneqq f_t(L)$ $(s < t)$,
 (c) *there exists a subpolyhedron F of M such that* $\dim F \le n-1$, $\partial M \subset F$, *and for any $K \in \mathcal{K}(M-F)$ there exist $-\infty < s < t < \infty$ with $K \subset f_t(L) - f_s(L)$,*
(ii) *f satisfies the condition $(*)$,*
(iii) *$\{f_t\}_{-\infty<t<\infty}$ is equi-continuous with respect to $d|_M$.*

This engulfing isotopy f_t can be used to deform measures. Let $h_t = f_t^{-1}$. Then, for any $\mu \in \mathcal{M}_g^\partial(M,\omega\text{-reg})$ the function $(-\infty,\infty) \longrightarrow (-\mu(L),\mu(N)) : t \longmapsto ((h_t)_*\mu - \mu)(L)$ is a monotonically increasing homeomorphism. Consider the map

$$\lambda : \mathcal{M}_g^\partial(M,\omega\text{-cpt-reg})_{lim}{}^2 \times \mathbb{R} \to \mathbb{R} : \ \lambda(\mu,\nu,t) = ((h_t)_*\mu - \nu)(L).$$

By Lemma 3.2 (1) λ is continuous. Since $((h_t)_*\mu - \nu)(L) = ((h_t)_*\mu - \mu)(L) + (\mu-\nu)(L)$, for any $\mu,\nu \in \mathcal{M}_g^\partial(M,\omega\text{-cpt-reg})$, the function $\lambda(\mu,\nu,*) : \mathbb{R} \to (-\mu(L),\mu(N)) + (\mu-\nu)(L)$ is a monotonically increasing homeomorphism. When $\mu(M) < \infty$, we have that $a \in (-\mu(L),\mu(N)) + (\mu-\nu)(L)$ iff $0 < a + \nu(L) < \mu(M)$. We need the inverse of the above homeomorphism.

Definition 5.1. We define a map $t : \mathcal{V} \to \mathbb{R}$ as follows:

(1)
$$\begin{aligned}\mathcal{V} &= \mathcal{V}_g^\partial(M,\omega\text{-cpt-reg})_{lim} \\ &= \left\{(\mu,\nu,a) \in \mathcal{M}_g^\partial(M,\omega\text{-cpt-reg})_{lim}{}^2 \times \mathbb{R} \;\middle|\; a \in (-\mu(L),\mu(N)) + (\mu-\nu)(L)\right\}\end{aligned}$$

(2) $t : \mathcal{V} \to \mathbb{R} : t(\mu,\nu,a) = \lambda(\mu,\nu,*)^{-1}(a)$ (i.e., $t = t(\mu,\nu,a)$ iff $a = \lambda(\mu,\nu,t)$)

Then (i) $t : \mathcal{V} \to \mathbb{R}$ is continuous, and (ii) $(\mu-\nu)(L) = a$ iff $t(\mu,\nu,a) = 0$.

Lemma 5.2. *The map $H : \mathcal{V} \to \mathcal{H}_\partial^{\mathrm{PL,c}}(M)_1^*$, $H_{(\mu,\nu,a)} = h_{t(\mu,\nu,a)}$ has the following properties:*

(i) *H satisfies the condition $(*)$.*
(ii) *$(H_{(\mu,\nu,a)})_*\mu : \mathcal{V} \to \mathcal{M}_g^\partial(M,\omega\text{-cpt-reg})_{lim} : (\mu,\nu,a) \mapsto (H(\mu,\nu,a))_*\mu$ is continuous.*
(iii) (a) $((H_{(\mu,\nu,a)})_*\mu - \nu)(L) = a$, (b) $(\mu-\nu)(L) = a$ *iff* $H_{(\mu,\nu,a)} = id$.
(iv) *$\{H_{(\mu,\nu,a)}{}^{-1}\}_{(\mu,\nu,a)\in\mathcal{V}}$ is equi-continuous with respect to $d|_M$.*

Lemma 5.3. *Suppose $\mu, \nu : P \to \mathcal{M}_g^\partial(M, \omega\text{-cpt-reg})_{lim}$ and $a : P \to \mathbb{R}$ are continuous maps such that $a_p \in (-\mu_p(L), \mu_p(N)) + (\mu_p - \nu_p)(L)$ $(p \in P)$. Then the map $h : P \to \mathcal{H}_\partial^{\mathrm{PL,c}}(M, \omega\text{-reg})_1$, $h_p = H_{(\mu_p, \nu_p, a_p)}$ has the following properties:*

(i) *h satisfies the condition $(*)$,*
(ii) *The map $h_*\mu : P \to \mathcal{M}_g^\partial(M, \omega\text{-cpt-reg})_{lim} : p \mapsto (h_p)_*\mu_p$ is continuous.*
(iii) (a) $((h_p)_*\mu_p - \nu_p)(L) = a_p$, (b) $(\mu_p - \nu_p)(L) = a_p$ *iff* $h_p = id_M$.
(iv) *$\{h_p^{-1}\}_p$ is equi-continuous with respect to $d|_M$.*

5.2. **Fundamental deformation lemma.**

Consider a decomposition $M = N \cup A$, $A = A_1 \cup \cdots \cup A_m$, such that

(i) N is a connected PL n-submanifold of M such that $\mathrm{Fr}_M N$ is a compact PL $(n-1)$-submanifold of M.
(ii) $A_1, \cdots, A_m \in \mathcal{C}(cl(M - N))$.

Since $\mathrm{Fr}_M N$ is assumed to be compact, we have $E_M = E_N \cup E_{A_1} \cup \cdots \cup E_{A_m}$.

Suppose $\mu, \nu : P \to \mathcal{M}_g^\partial(M, \omega\text{-cpt-reg})_{lim}$ and $a(i) : P \to \mathbb{R}$ $(i = 1, \cdots, m)$ are continuous maps which satisfies the following conditions: for any $p \in P$

$(\#)_1$ $a_p(i) > -\nu_p(A_i) = -\mu_p(A_i) + (\mu_p - \nu_p)(A_i) \quad (i = 1, \cdots, m)$,

$(\#)_2$ $\sum_{i=1}^m a_p(i) < (\mu_p - \nu_p)(M) + \nu_p(N)$.

Lemma 5.4. *There exists a map $\varphi : P \to \mathcal{H}_\partial^{\mathrm{PL,c}}(M)_1^*$ such that*

(1) *φ satisfies the condition $(*)$,*
(2) *$\varphi_*\mu : P \to \mathcal{M}_g^\partial(M, \omega\text{-cpt-reg})_{lim}$ is continuous,*
(3) $((\varphi_p)_*\mu_p - \nu_p)(A_i) = a_p(i)$ $(p \in P, i = 1, \cdots, m)$,
(4) *$\{\varphi_p^{-1}\}_p$ is equi-continuous with respect to $d|_M$,*
(5) (a) *For any $p \in P$, "if $(\mu_p - \nu_p)(A_i) = a_p(i)$ $(i = 1, \cdots, m)$, then $\varphi_p = id$",*
(b) *for any $i \in \{1, \cdots, m\}$, "if $(\mu_p - \nu_p)(A_i) = a_p(i)$ $(p \in P)$, then $\varphi_p|_{A_i} = id|_{A_i}$ $(p \in P)$",*
(6) *if $p \in P$ and $\sum_{i=1}^m a_p(i) = (\mu_p - \nu_p)(M)$, then $((\varphi_p)_*\mu_p - \nu_p)(N) = 0$.*

The condition $(\#)$ on $a(i)$ is necessary to achieve the condition (3). In the sequal we write as $(\mu - \nu)(A_i) = a(i)$ or $\varphi|_{A_i} = id|_{A_i}$ instead of $(\mu_p - \nu_p)(A_i) = a_p(i)$ $(p \in P)$ or $\varphi_p|_{A_i} = id|_{A_i}$ $(p \in P)$ (we regard them as the identity of functions in $p \in P$).

6. Realization of mass flow toward ends

6.1. **Topological manifold-case.**

Main Theorem 1.1 is a consequence of the following realization theorem.

Theorem 6.1. *Suppose M^n is a noncompact connected separable metrizable n-manifold, $\omega \in \mathcal{M}_g^\partial(M)$, $\mu, \nu : P \to \mathcal{M}_g^\partial(M, \omega\text{-cpt-reg})_{lim}$ and $a : P \to V_\omega$ are continuous maps with $(\mu_p - \nu_p)(M) = 0$ $(p \in P)$. Then there exists a continuous map $h : P \to \mathcal{H}_\partial(M, \omega\text{-reg})_1$ such that*

(1) $(h_p)_*\mu_p = \nu_p$ $(p \in P)$,
(2) *if* $p \in P$ *and* $\mu_p = \nu_p$, *then* $h_p \in \mathcal{H}_\partial(M, \mu_p)_1$ *and* $J_{h_p}^{\mu_p} = a_p$,
(3) *if* $p \in P$, $\mu_p = \nu_p$ *and* $a_p = 0$, *then* $h_p = id_M$.

According to the usual strategy, the proof of Theorem 6.1 can be reduced to the PL-manifold case by the next mapping theorem. We use the following notations: $I = [0, 1]$, I^n is the n-hold product of I and $I_1 = \{(t, 1/2, \cdots, 1/2, 1) \in I^n \mid t \in [1/3, 2/3]\}$. m denotes the Lebesgue measure on I^n.

Proposition 6.1. ([4, Proposition 4.2]) *There exists a compact 0-dimensional subset $E \subset \partial I^n$ ($E \subset I_1$ if $n \geq 2$) and a continuous proper surjection $\pi : I^n - E \to M$ which satisfies the following conditions:*

(i) $U \equiv \pi(\operatorname{Int} I^n)$ *is an open dense subset of* $\operatorname{Int} M$ *and* $\pi|_{\operatorname{Int} I^n} : \operatorname{Int} I^n \to U$ *is a homeomorphism.*
(ii) $F \equiv \pi(\partial I^n - E) = M - U$ *and* $\omega(F) = 0$.
(iii) *(Since I^n is the end compactification of $I^n - E$, the map π has the natural extension $\overline{\pi} : I^n \to \overline{M}$.) The restriction $\overline{\pi}|_E : E \to E_M$ is a homeomorphism.*
(v) $\widetilde{\omega} = \pi^*\omega$ *is* $m|_{I^n - E}$*-biregular.*

6.2. **PL-manifold case.**

By Proposition 6.1 we may assume that M^n is a noncompact connected PL n-manifold, $\omega \in \mathcal{M}_g^\partial(M)$ and $\mathcal{H}_\partial^{\mathrm{PL}}(M) \subset \mathcal{H}(M; \omega\text{-reg})$. Under this assumption Theorem 6.1 is proved in a series of lemmas. By $\mathcal{N}(E_M)$ we denote the set of PL n-submanifolds of M of the form $A = cl(M - N)$, where N is a compact, connected PL n-submanifold of M such that each $C \in \mathcal{C}(A)$ is noncompact. Let d be a fixed metric on $\overline{M}$. For any neighborhood U of E_M in $\overline{M}$ and any $\varepsilon > 0$ there exists $A \in \mathcal{N}(E_M)$ such that $A \subset U$ and $\operatorname{diam}_d C < \varepsilon$ $(C \in \mathcal{C}(A))$.

The next statement follows from Lemma 5.4.

Lemma 6.1. *Suppose $A, B \in \mathcal{N}(E_M)$, $B \subset \operatorname{Int}_M A$, $L = cl(M - A)$, $N = cl(M - B)$, and $\mathcal{C}(A) = \{A_1, \cdots, A_m\}$. We assume that $N_i = cl(A_i - B)$ is connected $(i = 1, \cdots, m)$ and $(\mu_p - \nu_p)(A_i) = a_p(A_i)$ $(p \in P,\ i = 1, \cdots, m)$. Then there exists a continuous map $\varphi : P \to \mathcal{H}_{\partial \cup L}^{\mathrm{PL,c}}(M)_1^*$ such that*

(1) φ *satisfies the condition* $(*)$,
(2) $\varphi_p(A_i) = A_i$ $(p \in P,\ i = 1, \cdots, m)$,

(3) (a) $((\varphi_p)_*\mu_p)(N_i) = \nu_p(N_i)$ $(p \in P,\ i = 1, \cdots, m)$,
(b) $(\varphi_{p*}\mu_p - \nu_p)(B_j) = a_p(B_j)$ *for each* $B_j \in \mathcal{C}(B)$,
(4) $\{\varphi_p^{-1}\}_{p\in P}$ *is equi-continuous with respect to* $d|_M$,
(5) (a) *for any* $A_i \in \mathcal{C}(A)$ *and* $p \in P$ *"if* $(\mu_p - \nu_p)(B_j) = a_p(B_j)$ *for each* $B_j \in \mathcal{C}(B \cap A_i)$, *then* $\varphi_p|_{A_i} = id_{A_i}$*"*
(b) *for any* $B_j \in \mathcal{C}(B)$ *"if* $(\mu_p - \nu_p)(B_j) = a_p(B_j)$ $(p \in P)$, *then* $\varphi_p|_{B_j} = id_{B_j}$ $(p \in P)$*"*

Let $\mu^0 = \mu$, $\nu^0 = \nu$, and $B^0 = M$. By the assumption we have $(\mu^0 - \nu^0)(B^0) = 0$. By the repeated application of Lemma 6.1 we obtain the following sequence of maps.

Lemma 6.2. *For* $k = 1, 2, \cdots$ *there exist*

$(k)_A$: $A^k \in \mathcal{N}(E)$, $\varphi^k : P \to \mathcal{H}^{\mathrm{PL,c}}_{\partial\cup N^{k-1}}(M)^*_1$, $\mu^k : P \to \mathcal{M}^\partial_g(M; \omega\text{-}cpt\text{-}reg)_{lim}$

such that

(0) φ^k *and* μ^k *are continuous,*
(1) (a) $A^k \subset \mathrm{Int}_M B^{k-1}$, $N^{k-1} \equiv cl(M - B^{k-1})$,
(b) $L^k_j \equiv cl(B^{k-1}_j - A^k)$ *is connected* $(B^{k-1}_j \in \mathcal{C}(B^{k-1}))$,
(c) $\mathrm{diam}\, A^k_i \le \dfrac{1}{2^k}$, $\mathrm{diam}\,(\psi^{k-1}_p \cdots \psi^1_p)^{-1}(A^k_i) \le \dfrac{1}{2^k}$ $(A^k_i \in \mathcal{C}(A^k))$,
(2) $\varphi^k_p(B^{k-1}_j) = B^{k-1}_j$ $(B^{k-1}_j \in \mathcal{C}(B^{k-1}))$,
(3) (a) $\mu^k_p \equiv (\varphi^k_p)_*\mu^{k-1}_p$, (b) $\mu^k_p(L^k_j) = \nu^{k-1}_p(L^k_j)$, $(\mu^k_p - \nu^{k-1}_p)(A^k_i) = a_p(A^k_i)$ $(A^k_i \in \mathcal{C}(A^k))$,
(4) $\left\{(\varphi^k_p)^{-1}\right\}_p$ *is equi-continuous with respect to* $d|_M$,
(5) (a) *for any* $B^{k-1}_j \in \mathcal{C}(B^{k-1})$ *and any* $p \in P$
"if $(\mu^{k-1}_p - \nu^{k-1}_p)(A^k_i) = a_p(A^k_i)$ $(A^k_i \in \mathcal{C}(A^k \cap B^{k-1}_j))$, *then* $\varphi^k_p|_{B^{k-1}_j} = id_{B^{k-1}_j}$*"*,
(b) *for any* $A^k_i \in \mathcal{C}(A^k)$
"if $(\mu^{k-1}_p - \nu^{k-1}_p)(A^k_i) = a_p(A^k_i)$ $(p \in P)$, *then* $\varphi^k_p|_{A^k_i} = id_{A^k_i}$ $(p \in P)$*"*,

$(k)_B$: $B^k \in \mathcal{N}(E)$, $\psi^k : P \to \mathcal{H}^{\mathrm{PL,c}}_{\partial\cup L^k}(M)^*_1$, $\nu^k : P \to \mathcal{M}^\partial_g(M; \omega\text{-}cpt\text{-}reg)_{lim}$

such that

(0) ψ^k *and* ν^k *are continuous,*
(1) (a) $B^k \subset \mathrm{Int}_M A^k$, $L^k \equiv cl(M - A^k)$,
(b) $N^k_i = cl(A^k_i - B^k)$ *is connected* $(A^k_i \in \mathcal{C}(A^k))$
(c) $\mathrm{diam}\, B^k_j \le \dfrac{1}{2^k}$, $\mathrm{diam}\,(\varphi^k_p \cdots \varphi^1_p)^{-1}(B^k_j) \le \dfrac{1}{2^k}$ $(B^k_j \in \mathcal{C}(B^k))$,
(2) $\psi^k_p(A^k_i) = A^k_i$ $(A^k_i \in \mathcal{C}(A^k))$,
(3) (a) $\nu^k_p \equiv (\psi^k_p)_*\nu^{k-1}_p$, (b) $\nu^k_p(N^k_i) = \mu^k_p(N^k_i)$, $(\mu^k_p - \nu^k_p)(B^k_j) = a(B^k_j)$ $(B^k_j \in \mathcal{C}(B^k))$,

(4) $\left\{(\psi_p^k)^{-1}\right\}_p$ *is equi-continuous with respect to* $d|_M$,

(5) (a) *for any* $A_i^k \in \mathcal{C}(A^k)$ *and any* $p \in P$
"*if* $(\mu_p^k - \nu_p^{k-1})(B_j^k) = a_p(B_j^k)$ $(B_j^k \in \mathcal{C}(B^k \cap A_i^k))$, *then* $\psi_p^k|_{A_i^k} = id_{A_i^k}$",
(b) *for any* $B_j^k \in \mathcal{C}(B^k)$
"*if* $(\mu_p^k - \nu_p^{k-1})(B_j^k) = a_p(B_j^k)$ $(p \in P)$, *then* $\psi_p^k|_{B_j^k} = id_{B_j^k}$ $(p \in P)$".

The next assertions follow from the conditions $(k)_A$ (0) $\sim$ (5) and $(k)_B$ (0) $\sim$ (5).

Lemma 6.3.

(1) (i) *For any* $p \in P$ *the sequence* $\varphi_p^k \cdots \varphi_p^1$ $(k = 1, 2, \cdots)$ *converges* $d|_M$*-uniformly to some* φ_p *in* $\mathcal{H}_\partial^{\mathrm{PL}}(M)_1$.
(ii) *The map* $\varphi : P \to \mathcal{H}_\partial^{\mathrm{PL}}(M)_1 : p \mapsto \varphi_p$ *is continuous.*
(iii) $\varphi_p^{-1}|_{N^k} = (\varphi_p^k \cdots \varphi_p^1)^{-1}|_{N^k}$ *and* $((\varphi_p)_*\mu_p)|_{N^k} = \mu_p^k|_{N^k}$ $(k = 1, 2, \cdots)$.

(2) (i) *For any* $p \in P$ *the sequence* $\psi_p^k \cdots \psi_p^1$ $(k = 1, 2, \cdots)$ *converges* $d|_M$*-uniformly to some* ψ_p *in* $\mathcal{H}_\partial^{\mathrm{PL}}(M)_1$.
(ii) *The map* $\psi : P \to \mathcal{H}_\partial^{\mathrm{PL}}(M)_1 : p \mapsto \psi_p$ *is continuous.*
(iii) $\psi_p^{-1}|_{L^k} = (\psi_p^{k-1} \cdots \psi_p^1)^{-1}|_{L^k}$ *and* $((\psi_p)_*\nu_p)|_{L^k} = \nu_p^{k-1}|_{L^k}$ $(k = 1, 2, \cdots)$.

(3) *For any* $C = L_j^k \in \mathcal{C}(cl(B^{k-1} - A^k))$ *and* $N_i^k \in \mathcal{C}(cl(A^k - B^k))$ *we have* $((\varphi_p)_*\mu_p)(C) = ((\psi_p)_*\nu_p)(C)$ $(p \in P)$.

The next lemma follows from Theorem 3.1.

Lemma 6.4. *There exists a continuous map* $\chi : P \to \mathcal{H}_\partial(M, \omega\text{-reg})_1$ *such that*

(i) $(\chi\varphi)_*\mu = \psi_*\nu$
(ii) $\chi(C) = C$ *for any* $C = L_j^k \in \mathcal{C}(cl(B^{k-1} - A^k))$ *and* $N_i^k \in \mathcal{C}(cl(A^k - B^k))$
(iii) *if* $p \in P$ *and* $(\varphi_p)_*\mu_p = (\psi_p)_*\nu_p$, *then* $\chi_p = id_M$.

Proof of Theorem 6.1. The required map $h : P \to \mathcal{H}_\partial(M, \omega\text{-reg})_1$ is defined by $h_p = \psi_p^{-1}\chi_p\varphi_p$ $(p \in P)$. This completes the proof of Theorem 6.1 and Theorem 1.1. □

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