数理解析研究所講究録
1517巻 2006年 56-59

ON THE CLASSIFYING SPACES OF A PARTIAL ABELIAN MONOID ASSOCIATED TO $SU(2)$

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1. Introduction

Topological partial monoid is a generalization of the notion of topological monoid. It occurs naturally in the construction of configuration spaces [2] and known to be a suitable data to construct generalized homology theories [3]. However, the topology of partial monoids is not well-studied.

In this paper, we investigate an aspect of the topology of partial abelian monoids. More precisely, our point of view can be explained as follows : For a topological group G, we can associate to it a partial monoid M generated by commutative pairs in G. If we think of M as an abelian part of G, it is natural to ask if M can recover the data of G. We compute the homology group of the classifying space BM in low degrees and show that it is not homology equivalent to BG when $G = SU(2)$.

2. Classifying spaces of partial abelian monoids

Definition 1. A partial abelian monoid is a based space M equipped with a subspace $M_2 \subset M \times M$ and a map $m : M_2 \to M$ such that

(1) $M \vee M \subset M_2$ and $m(a, *_M) = m(*_M, a) = a$,
(2) $(a, b) \in M_2$ implies $(b, a) \in M_2$ and $m(a, b) = m(b, a)$, and
(3) If (a, b) and (b, c) are both in M_2 then $(m(a, b), c) \in M_2$ implies $(a, m(b, c)) \in M_2$, and $m(m(a, b), c) = m(a, m(b, c))$.

We write $m(a, b) = a + b$. Any element of M_2 is called a summable pair. Let M_k denote the subspace of M^k which consists of those k-tuples $(a_1, \ldots, a_k)$ such that $a_1 + \cdots + a_k$ is defined. A map between PAMs are called a PAM homomorphism if it sends summable pairs to summable pairs and preserves the sum.

Example 2.

(1) Obviously, any abelian monoid G is a partial abelian monoid by setting $G_2 = G \times G$.
(2) Any based space X can be considered as a partial abelian monoid by setting $X_2 = X \vee X$ and $m : X \vee X \to X$ a folding map. We call this structure a trivial partial abelian monoid.
(3) Let G be an abelian group. Then any subspace $A \subset G$ which contains 0 is a partial abelian monoid by setting
$$A_2 = \{(a, b) \mid a + b \in A\}.$$
(4) Let G be a (possibly non-commutative) topological group. We have a partial abelian monoid M as follows. Topologically $M = G$. Let $M_2 = \{(g, h) \in M \times M \mid gh = hg\}$ and $m : M_2 \to M$ be the multiplication of G.

Definition 3. For any partial abelian monoid M, we have a simplicial space denoted M_* as follows. Let M_n be the subspace of summable n-tuples of M^n. Its structure maps are given by $1 \times \cdots \times m \times \cdots \times 1 : M_n \to M_{n-1}$ and $i_k : M_{n-1} \to M_n$, where i_k

inserts 0 at the k-th entry. The geometric realization of this simplicial space is called the classifying space of M and is denoted by BM.

Example 4.

(1) If $M = G$ is an abelian monoid, then we have BG the usual classifying space of G in a usual sense.
(2) If $M = X$ is a trivial partial abelian monoid; $X_2 = X \vee X$, then we have $BX \simeq \Sigma X$, the reduced suspension of X.
(3) In Example 2 (4) we associate to any topological G a partial abelian monoid M. From a view point given in the first section, it is natural to ask how much BM approximates BG.

3. Homology of the space of commutative pairs in $SU(2)$

Using an isomorphism $SU(2) \cong Sp(1)$, we view $SU(2)$ as the unit sphere in the quarternions $\mathbb{H}$. By a direct calculation we see that $x = x_1 + ix_2 + jx_3 + kx_4$ and $y = y_1 + iy_2 + jy_3 + ky_4$ commute *iff* $x = \pm 1, y = \pm 1$, or $x, y \neq \pm 1$ and $[x_2 : x_3 : x_4] = [y_2 : y_3 : y_4]$. Thus the space of commutative pairs in $SU(2)$ can be constructed as follows : Let $E = \mathbb{RP}^2 \cup_\pi (S^2 \times I) \cup_\pi \mathbb{RP}^2$ be a space constructed from $S^2 \times I$ by taking a quotient of each of $S^2 \times \{0\}$ and $S^2 \times \{1\}$ to $\mathbb{RP}^2$ by the standard projection π. Then E can be considered as the total space of an S^1-bundle over $\mathbb{RP}^2$, with the projection $p : E \to \mathbb{RP}^2$ which maps two copies of $\mathbb{RP}^2$ identically and maps $S^2 \times I$ by the composition of the sequence

$$S^2 \times I \xrightarrow{proj} S^2 \xrightarrow{\pi} \mathbb{RP}^2.$$

Let $E * E$ denote the fiber product of $E; E * E$ is a $S^1 \times S^1$-bundle over $\mathbb{RP}^2$. We heve four cross-sections

$$s_{00}, s_{01}, s_{10}, s_{11} : \mathbb{RP}^2 \to E * E,$$

where $s_{\varepsilon_1\varepsilon_2}([x])$ is the class represented by $((x, \varepsilon_1), (x, \varepsilon_2) \in (S^2 \times I)^2$. The space of commutative pairs in $M = S^3$, denoted M_2, is given by $M_2 = M * M / \sim$, where $\sim$ is the equivalence relation

$$(x, y) \sim (x', y') \iff (x, y) \text{ and } (x', y') \text{ both are in one of}$$
$$s_{00}(\mathbb{RP}^2), s_{01}(\mathbb{RP}^2), s_{10}(\mathbb{RP}^2), s_{11}(\mathbb{RP}^2).$$

The integral homology groups of M_2 can be computed as

$$H_*(M_2) = \begin{cases} \mathbb{Z} & (k=0) \\ 0 & (k=1) \\ \mathbb{Z} & (k=2) \\ \mathbb{Z}^2 \oplus \mathbb{Z}/2 & (k=3) \\ 0 & (k>3) \end{cases}$$

which coincides with the calculation of the integral cohomology groups of M_2 given in [1].

4. Homology of the space of commutative n-tuples in SU(2)

The construction of the previous section can be generalized to the space of commutative n-tuples in $SU(2)$ as follows : Let E be the fiberwise one point compactification of the canonical line bundle over $\mathbb{RP}^2$. We form a fiberwise direct product of n copies of E and get a $(S^1)^n$-bundle over $\mathbb{RP}^2$, denoted by E^{*n}.
For the purpose of the next section, we give a cell decomposition of E^{*n}. Let $p : E^{*n} \to \mathbb{RP}^2$ be the projection and $a^2 + a + 1$ denote the standard cell decomposition of $\mathbb{RP}^2$. We also denote the cell decomposition of $(S^1)^n$ by $(x_1 + 1) \cdots (x_n + 1)$, where

$x_k + 1$ denotes the cell decomposition of the k-th component of $(S^1)^n$. Then the cell decomposition of E^{*n} can be represented as

$$(a^2 + a + 1)(x_1 + 1) \cdots (x_n + 1).$$

Thus the k-cell of E^{*n} is represented by the monomial of degree k in the above polynomial and we have the chain complex with C_k generated freely by the monomials in $a^2\sigma_{k-2}$, $a\sigma_{k-1}$, and σ_k, where $\sigma_k = \sigma_k(x_1, \ldots, x_n)$ denotes the k-th fundamental symmetric polynomial in $x_1, \ldots, x_n$. Boundary homomorphisms are given by

$$\partial(a^2\sigma_{k-2}) = a(\sigma_{k-2}(-x_1, \ldots, -x_n) + \sigma_{k-2}(x_1, \ldots, x_n)) = \begin{cases} 0 & (k : \text{odd}) \\ 2a\sigma_{k-2} & (k : \text{even}) \end{cases},$$

$$\partial(a\sigma_{k-1}) = \sigma_{k-1}(-x_1, \ldots, -x_n) - \sigma_{k-1}(x_1, \ldots, x_n) = \begin{cases} 0 & (k : \text{odd}) \\ -2\sigma_{k-1} & (k : \text{even}) \end{cases},$$

and $\partial(\sigma_k) = 0$.
If n is odd, the integral homology of E^{*n} can be computed as

$$H_k(E^{*n}) = \begin{cases} \mathbb{Z} & (k = 0) \\ (\mathbb{Z}/2)^{n+1} & (k = 1) \\ \mathbb{Z}^{n_k} & (2 \le k \le n, k : \text{even}) \\ (\mathbb{Z}/2)^{n_k + n_{k-1}} \oplus \mathbb{Z}^{n_{k-2}} & (3 \le k \le n, k : \text{odd}) \\ 0 & (k = n+1) \\ \mathbb{Z} & (k = n+2) \\ 0 & (k \ge n+3), \end{cases}$$

where $n_k = \begin{pmatrix} n \\ k \end{pmatrix}$ are the binomial coefficients. If n is even, the integral homology of E^{*n} differs from the above formula when $k = n+1$ and $k = n+2$. They are given by

$$H_{n+1}(E^{*n}) = \mathbb{Z}^n \oplus \mathbb{Z}/2$$

and

$$H_{n+2}(E^{*n}) = 0.$$

As is the case of $n = 2$, we have 2^n cross sections $s_{\varepsilon_1 \ldots \varepsilon_n}$ ($\varepsilon_k \in \{0, 1\}$) and M_n is given by $M_n = E^{*n}/\sim$, where "$/\sim$" indicates that we squeeze each of 2^n images of the cross sections to one point. It follows that $H_k(M_n) = H_k(E^{*n})$ when $k = 0$ and $k \ge 3$. For the purpose of the next section, we give a cell decomposition of M_n. This time we use the cell decomposition of S^1 into two 1-cells and two 0-cells represented by $x^+ + x^- + z^+ + z^-$, where $x^\pm$ denote the 1-cells and $z^\pm$ the 0-cells. As above, the cell decomposition of M_n can be represented as

$$(a^2 + a + 1)(x_1^+ + x_1^- + z_1^+ + z_1^-) \cdots (x_n^+ + x_n^- + z_n^+ + z_n^-),$$

but monomials in $a^2(z_1^+ + z_1^-) \cdots (z_n^+ + z_n^-)$, and $a(z_1^+ + z_1^-) \cdots (z_n^+ + z_n^-)$ should be identified with corresponding monomials in $(z_1^+ + z_1^-) \cdots (z_n^+ + z_n^-)$. Thus the k-cell is represented by the monomials of degree k in the above polynomial, where we consider $z_k^\pm$ to have degree 0. We have the chain complex with C_k generated freely by such monomials. Boundary homomorphisms are given inductively by

$$\partial(a^2 f) = a(f - \bar{f}) + a^2 \partial(f),$$
$$\partial(af) = -f - \bar{f} - a\partial(f),$$

$\partial(x_k^\varepsilon) = z_k^{-\varepsilon} - z_k\varepsilon$, and the graded chain rule on f, where f denotes a monomial in $(x_1^+ + x_1^- + z_1^+ + z_1^-) \cdots (x_n^+ + x_n^- + z_n^+ + z_n^-)$ and $\bar{f}$ denotes the monomial given by replacing each x_k^ε in f into $x_k^{-\varepsilon}$. From these formulae, we compute the homology to be $H_1(M_n) = 0$, $H_2(M_n) = \mathbb{Z}^{n_2} \oplus (\mathbb{Z}/2)^{2^n - (n_2 + n + 1)}$ for $n \le 4$.

5. Homolgy of BM in low degrees

Since BM is a geometric realization of a simplicial space, we have the skeletal filtration on BM, which leads to a spectral sequence with $E^2_{p,q} = H_p(\{H_q(M_*), \partial\})$ converging to $H_*(BM)$, where $\{H_q(M_*), \partial\}$ denotes the Moore complex of the simplicial group $H_q(M_*)$. The computation and the genuine data of cells in the previous section gives us the E^2-term of the spectral sequence as $E^2_{p,q} = 0$ for $0 \leq p+q \leq 4$ except for $E^2_{2,2} = \mathbb{Z}/2$. Thus we have

Theorem 5. Let M be a partial abelian monoid generated by the commutative pairs in $SU(2)$, then the integral homology of its classifying space in low degrees are given by

$$H_k(BM) = \begin{cases} \mathbb{Z} & (k=0) \\ 0 & (1 \leq k \leq 3) \\ \mathbb{Z}/2 & (k=4) \end{cases} .$$

Corolary 6. BM is not homology equivalent to $BSU(2)$.

References

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