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NOTES ON DISCRETE SUBGROUPS OF PU(2,1) WITH SCREW PARABOLIC ELEMENTS

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Abstract

We give a version of Shimizu's lemma for groups of complex hyperbolic isometries one of whose generators is a parabolic screw motion. And we show relation to other results. This is a part of our joint work [9].

1 Complex hyperbolic versions of Shimizu's lemma

Let G be a discrete subgroup of $\mathrm{PSL}(2, R)$ containing the parabolic map $A(z) = z + t$ for some $t > 0$. Then Shimizu's lemma [14] says that for any $B(z) = (az + b)/(cz + d) \in G$ with $c \neq 0$ then $|c| \geq 1/t$. Geometrically this result says that the radius r_B of the isometric sphere of B satisfies $r_B \leq t$.

Viewing $\mathrm{PSL}(2, R)$ as the isometry group of complex hyperbolic 1-space, $\mathbf{H}_C^1$, we generalise Shimizu's lemma to 2-dimensional complex hyperbolic isometries. Let ρ_0 denote the Cygan metric on the Siegel domain $\mathbf{H}_{\mathbf{C}}^2$ and its boundary $\partial\mathbf{H}_{\mathbf{C}}^2$.

Theorem 1.1 (Theorem 3.2 of [5]) *Let A be vertical translation by $(0, t)$ in $\mathrm{PU}(2, 1)$ fixing ∞. Let B be any element of $\mathrm{PU}(2, 1)$ not projectively fixing ∞ and let r_B denote the radius of the isometric sphere of B. If*

$$\frac{t}{{r_B}^2} = \frac{\rho_0(B(\infty), AB(\infty))\rho_0(B^{-1}(\infty), AB^{-1}(\infty))}{{r_B}^2} < 1,$$

then the group $\langle A, B\rangle$ is not discrete.

For the case where the stabiliser of ∞ is a cyclic group of non-vertical translations it was shown in [11] that there is no uniform bound on the radii of isometric spheres. However, there is a bound on the radii of isometric spheres in terms of the translation lengths at their centres.

Theorem 1.2 (Theorem 2.1 of [12]) *Let A be a Heisenberg translation by (τ, t) in $\mathrm{PU}(2,1)$ fixing ∞. Let B be any element of $\mathrm{PU}(2,1)$ not projectively fixing ∞ and let r_B denote the radius of the isometric sphere of B. If*

$$\frac{\rho_0(B(\infty), AB(\infty))\rho_0(B^{-1}(\infty), AB^{-1}(\infty)) + 4|\tau|^2}{r_B{}^2} < 1,$$

then $\langle A,\ B\rangle$ is not discrete.

Next we consider parabolic screw motions A in $\mathbf{H}^2_C$.

Theorem 1.3 *Let A be a positively oriented screw parabolic element of $\mathrm{PU}(2,1)$ fixing ∞. Let $e^{i\theta} \in \mathrm{U}(1)$ denote the rotational part of A and suppose that $|e^{i\theta} - 1| < 1/4$. Let B be any element of $\mathrm{PU}(2,1)$ not projectively fixing ∞ and let r_B denote the radius of the isometric sphere of B. If*

$$\frac{\rho_0(B(\infty), AB(\infty))\rho_0(B^{-1}(\infty), AB^{-1}(\infty))}{r_B{}^2} < \left(\frac{1 + \sqrt{1 - 4|e^{i\theta} - 1|}}{2}\right)^2,$$

then $\langle A,\ B\rangle$ is not discrete.

Remark. If the rotational part of A has finite order then some power is a vertical translation and we can use Theorem 1.1. Thus we take a screw parabolic map A with infinite order rotational part. If such an A is in a discrete group then the only elements of this group sharing a fixed point with A are screw parabolic and boundary elliptic maps with the same axis as A. Our result above concerns screw motions where the angle of rotation is small and is positively oriented relative to the direction of translation. It is clear that any parabolic screw motion with infinite order rotational part has a power satisfying these conditions.

2 Relation to other results

Jiang and Parker gave the following theorem.

Theorem 2.1 (Theorem 5.1 of [4]) *Let A be a screw parabolic element of $\mathrm{PU}(2,1)$ fixing ∞. Let L_A denote the axis of A and $e^{i\theta} \in \mathrm{U}(1)$ denote the rotational part of A. Suppose that $|e^{i\theta} - 1| < 1$. Let $\sqrt{t}$ denote the Cygan* *translation length of A on L_A. Suppose that G is a discrete subgroup of* $\mathrm{PU}(2,1)$ *containing A. Let B be any element of G not projectively fixing ∞ and denote the radius of the isometric sphere of B by r_B. Let*

$$R = \max\Big\{\rho_0\big(L_A, B(\infty)\big),\ \rho_0\big(L_A, B^{-1}(\infty)\big)\Big\}.$$

Then

$$r_B{}^2 \le \frac{2R^2|e^{i\theta}-1|}{\big(1-|e^{i\theta}-1|\big)} + \frac{t}{\big(1-|e^{i\theta}-1|^{1/2}\big)^2}.$$

Observe that

$$\big(1-|e^{i\theta}-1|^{1/2}\big)^2 \le \left(\frac{1+\sqrt{1-4|e^{i\theta}-1|}}{2}\right)^2 \le 1-|e^{i\theta}-1|.$$

This enables us to show:

Theorem 2.2 *Let A be a positively oriented screw parabolic element of* $\mathrm{PU}(2,1)$ *and let B be any element of* $\mathrm{PU}(2,1)$ *not fixing ∞. If $\rho_0\big(L_A, B(\infty)\big)$ and $\rho_0\big(L_A, B^{-1}(\infty)\big)$ are both small enough then Theorem 2.1 follows from Theorem 1.3. On the other hand, if $\rho_0\big(L_A, B(\infty)\big)$ equals $\rho_0\big(L_A, B^{-1}(\infty)\big)$ and is sufficiently large then Theorem 1.3 follows from Theorem 2.1.*

Jørgensen's inequality is a generalisation of Shimizu's lemma. Complex hyperbolic versions of Jørgensen's inequality were given in [1], [3] and [13]. In particular, in [1] Basmajian and Miner give a version of Jørgensen's inequality for groups with a loxodromic generator, Theorem 9.1 of [1]. As a corollary, they give a generalisation of Shimizu's lemma, Theorem 9.11 of [1]. It was shown in Theorem 6.1 of [3] that the hypotheses of Basmajian and Miner's main theorem could be weakened somewhat (see also [13] for discussion of this result and Basmajian and Miner's stable basin theorem). With this change, their corollary is:

Theorem 2.3 (Theorem 9.11 of [1]) *Fix positive numbers r and ϵ so that $r^2+2\epsilon<1$. Let $A \in \mathrm{PU}(2,1)$ be a parabolic map fixing ∞. Let $B \in \mathrm{PU}(2,1)$ be a loxodromic map with attractive fixed point p and repulsive fixed point q. Suppose that neither p nor q is ∞. Suppose that B has complex dilation factor λ with $|\lambda|>1$ and $|\lambda-1|<\epsilon$. If*

$$\rho_0\big(A(p),p\big)\frac{1+r^2+\sqrt{1+r^2}}{r^2} \le \rho_0(p,q),$$

then the group generated by A and B is not discrete.

We now show that when $|e^{i\theta}-1|<3/16$ Theorem 2.3 follows from Theorem 1.3. This should be compared to [7] and [8] where a similar comparison was made between Theorem 2.3 and Theorem 1.2.

Theorem 2.4 *Fix positive numbers r and ϵ satisfying $r^2+2\epsilon<1$. Let $A\in \mathrm{PU}(2,1)$ be the screw parabolic map $A:(\zeta,v,u)\longmapsto(e^{i\theta}\zeta,v+t,u)$ where $|e^{i\theta}-1|<3/16$ and $t\sin(\theta)>0$. Let $B\in\mathrm{PU}(2,1)$ be loxodromic with attractive fixed point p and repelling fixed point q. Suppose that neither p nor q is ∞. Suppose that B has complex dilation factor λ with $|\lambda|>1$ and $|\lambda-1|<\epsilon$. Suppose that the isometric spheres of B and B^{-1} have radius r_B. If*

$$\rho_0(A(p),p)\frac{1+r^2+\sqrt{1+r^2}}{r^2}\leq\rho_0(p,q),$$

then

$$\frac{\rho_0(AB^{-1}(\infty),B^{-1}(\infty))\rho_0(AB(\infty),B(\infty))}{{r_B}^2}<\left(\frac{1+\sqrt{1-4|e^{i\theta}-1|}}{2}\right)^2.$$

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