Jørgensen numbers of discrete groups

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Abstract

Let G be a non-elementary two-generator subgroup of the Möbius transformation group. The *Jørgensen number* $J(G)$ of G is defined by

$$J(G) := \inf\{|\mathrm{tr}^2(A) - 4| + |\mathrm{tr}(ABA^{-1}B^{-1}) - 2| \mid \langle A, B\rangle = G\}.$$

In this paper we announce the following two results: (1) For every positive integer r, there is a non-elementary Kleinian group G such that $J(G) = r$; (2) For every real number $r > 4$, there is a classical Schottky group G such that $J(G) = r$. The proofs will appear elsewhere.

0. INTRODUCTION.

0.1. It is one of the most important problems in the theory of Kleinian groups to decide whether or not a subgroup G of the Möbius transformation group is discrete. For this problem there are two important and useful theorems: One is Poincaré's

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polyhedron theorem, which gives a sufficient condition for G to be discrete. The other is Jørgensen's inequality theorem, which gives a necessary condition for a two-generator Möbius transformation group $G = \langle A, B\rangle$ to be discrete.

In 1976 Jørgensen gave the following important theorem called Jørgensen's inequality theorem.

THEOREM A (Jørgensen [4]). *Suppose that the Möbius transformations A and B generate a non-elementary discrete group. Then*

$$J(A,B) := |\mathrm{tr}^2(A) - 4| + |\mathrm{tr}(ABA^{-1}B^{-1}) - 2| \geq 1. \tag{$*$}$$

The lower bound 1 *is best possible.*

The inequality $(*)$ is called *Jørgensen's inequality*. A non-elementary discrete two-generator subgroup G of the Möbius transformation group is called a *Jørgensen group* if there exist generators A and B of G such that $J(A,B) = 1$.

There are some papers by Jørgensen [4], Jørgensen - Kiika [5], Jørgensen - Lascurain - Pignataro [6], Gehring - Martin [2], Sato - Yamada [13], Sato [11], Li - Oichi - Sato [7, 8, 9] and González-Acuña - Ramírez [3] on Jørgensen groups.

0.2. Let G be a non-elementary two-generator subgroup of the Möbius transformation group. The *Jørgensen number* $J(G)$ of G is defined by

$$J(G) := \inf\{|\mathrm{tr}^2(A) - 4| + |\mathrm{tr}(ABA^{-1}B^{-1}) - 2| \mid \langle A, B\rangle = G\}.$$

Now we have the following problem:

PROBLEM. Let r be a real number with $r \geq 1$. When is there a discrete group whose Jørgensen number is equal to r?

There are some papers by Sato [12] and González-Acuña - Ramírez [3] on Jørgensen numbers. In this paper we consider the problem on Jørgensen numbers.

1. DEFINITIONS AND EXAMPLES.

1.1. In this section we will state definitions and give some examples. Let Möb denote the set of all linear fractional transformations (Möbius transformations)

$$A(z) = \frac{az+b}{cz+d}$$

of the extended complex plane $\hat{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$, where a, b, c, d are complex numbers and the determinant $ad - bc = 1$. We call Möb the *Möbius transformation group.* There is an isomorphism between Möb and $PSL(2, \mathbf{C})$. Throughout this paper we will always write elements of Möb as matrices with determinant 1.

In this paper we use a Kleinian group in the same meaning as a discrete group of Möb. Namely, a *Kleinian group* is a discrete subgroup of Möb. A subgroup G of Möb is said to be *elementary* if there exists a finite G-orbit in $\mathbf{R}^3$ (see Beardon [**1**]). The trace of

$$A^* = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad (ad - bc = 1)$$

in $SL(2, \mathbf{C})$ is defined by $\mathrm{tr}(A^*) = a + d$. We remark that the traces of elements A, B of Möb ($= PSL(2, \mathbf{C})$) are not well-defined, but $\mathrm{tr}^2(A)$ and $\mathrm{tr}(ABA^{-1}B^{-1})$ are still well-defined after choosing matrix representatives.

1.2. Here definitions of a Jørgensen number and a Jørgensen group are given.

Definition 1.1. Let A and B be Möbius transformations. The *Jørgensen number* $J(A, B)$ of the ordered pair (A, B) is defined by

$$J(A, B) := |\mathrm{tr}^2(A) - 4| + |\mathrm{tr}(ABA^{-1}B^{-1}) - 2|.$$

Definition 1.2. Let G be a non-elementary two-generator subgroup of Möb. The *Jørgensen number* $J(G)$ of G is defined by

$$J(G) := \inf\{J(A, B) \mid A \text{ and } B \text{ generate } G\}.$$

DEFINITION 1.3. A subgroup G of Möb is called a *Jørgensen group* if G satisfies the following four conditions: (1) G is a two-generator group. (2) G is a discrete group. (3) G is a non-elementary group. (4) There exist generators A and B of G such that $J(A, B) = 1$.

1.3. Here we will give some examples of Kleinian groups whose Jørgensen numbers are one and two.

(1) $J(G) = 1$.

Jørgensen groups, for example, the modular group, the Picard group and the figure-eight knot group (Jørgensen - Lascurain - Pignataro [**6**], Sato [**11**] and Li - Oichi - Sato [**7,8,9**]).

(2) $J(G) = 2$.

The Whitehead link group (Sato [**12**], González-Acuña - Ramírez [**3**]).

2. THEOREMS.

In this section we will state our main theorems.

THEOREM 1. *For every positive integer r, there is a non-elementary discrete group G whose Jørgensen number is r; $J(G) = r$.*

THEOREM 2. *For every real number $r > 4$, there is a classical Schottky group G whose Jørgensen number is r; $J(G) = r$.*

3. NORMALIZATION I.

In this section we consider the first normalization and present some lemmas.

LEMMA 3.1. *Let A be a parabolic transformation and let B be a loxodromic or an elliptic transformation such that A and B have no common fixed points. Then* *there uniquely exists a Möbius transformation T satisfying the following three conditions:*

(i) *The fixed point of* TAT^{-1} *is* ∞.

(ii) *The fixed points of* TBT^{-1} *are symmetric with respect to the origin.*

(iii) $TAT^{-1}(0) = 1$.

Then by easy calculations we have

$$TAT^{-1} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad TBT^{-1} = \begin{pmatrix} \mu\sigma & \mu^2\sigma - 1/\sigma \\ \sigma & \mu\sigma \end{pmatrix}$$

where $\sigma \in \mathbf{C} \setminus \{0\}$ and $\mu \in \mathbf{C}$.

LEMMA 3.2. *Let A and B be Möbius transformations. Then the Jørgensen number $J(A, B)$ is invariant under conjugation in* Möb, *that is,* $J(TAT^{-1}, TBT^{-1}) = J(A, B)$ *for* $T \in$ Möb.

3.3. Hereafter we consider the case of $\mu = ik$ $(k \in \mathbf{R})$ and $\sigma = -ire^{i\theta}$ $(r > 0,\ 0 \leq \theta \leq 2\pi)$. That is, we consider marked two-generator groups $G_{r,\theta,k} = \langle A, B_{r,\theta,k} \rangle$ generated by

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B := B_{r,\theta,k} = \begin{pmatrix} rke^{i\theta} & irk^2e^{i\theta} - ie^{-i\theta}/r \\ -ire^{i\theta} & rke^{i\theta} \end{pmatrix}.$$

4. PROOF OF THEOREM 1.

In this section we sketch the proof of Theorem 1. The complete proof will appear elsewhere. We consider the case of

$$r = \sqrt{n}\ (n \in \mathbf{N}),\ \ \theta = \pi/2,\ \ k = 0.$$

Then we have

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B_{\sqrt{n},\pi/2,0} = \begin{pmatrix} 0 & -1/\sqrt{n} \\ \sqrt{n} & 0 \end{pmatrix}. \qquad (**)$$

For simplicity we write B_n for $B_{\sqrt{n},\pi/2,0}$.

LEMMA 4.1. *Let A and B_n be the matrices in* (**). *Then the group $G_n = \langle A, B_n\rangle$ is a non-elementary Kleinian group for every positive integer n.*

LEMMA 4.2. *Let A and B_n be the matrices in* (**). *Let $G_n = \langle A, B_n\rangle$. Then $X \in G_n$ is either the following* (i) *type* 1 *or* (ii) *type* 2.

(i) Type 1.

$$X = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad (ad - bc = 1),$$

where $a = m_1 n \pm 1,\ b = m_2 n + \ell,\ c = m_3 n$ *and* $d = m_4 \pm 1$ $(m_j, \ell \in \mathbf{Z}\ (j = 1,2,3,4))$

(ii) Type 2.

$$X = \begin{pmatrix} a\sqrt{n} & b/\sqrt{n} \\ c\sqrt{n} & d\sqrt{n} \end{pmatrix} \quad (adn - bc = 1),$$

where $a = m_1 n + \ell_1,\ b = m_2 n \pm 1,\ c = m_3 n \pm 1$ *and* $d = m_4 n \pm \ell_2, \ell_k \in \mathbf{Z}$ $(m_j\ (j = 1,2,3,4;\ k = 1,2))$.

LEMMA 4.3. *Let A and B_n be the matrices in* (**). *Let $G_n = \langle A, B_n\rangle$. If $\langle X, Y\rangle$ be a non-elementary (discrete) subgroup of G_n, then*

$$|\mathrm{tr}(XYX^{-1}Y^{-1}) - 2| = \mathrm{n}|\mathrm{k}| \quad (\mathrm{k} \in \mathbf{Z}).$$

LEMMA 4.4. *Let A and B_n be the matrices in* (**). *Let $G_n = \langle A, B_n\rangle$. If $\langle X, Y\rangle$ be a non-elementary (discrete) subgroup of G_n, then*

$$J(X,Y) \geq n.$$

LEMMA 4.5. *Let A and B_n be the matrices in* (**). *Then* $J(A, B_n) = n$.

Theorem 1 follows from Lemmas 4.4 and 4.5. If A and B_n are the matrices in (**), then $J(G_n) = n$ for the group $G_n = \langle A, B_n\rangle$.

5. NORMALIZATION II.

In this section we consider the second normalization. Let A_1 and A_2 be loxiodromic transformations. For $j = 1, 2$, let λ_j ($|\lambda_j| > 1$), p_j and p_{2+j} be the multipliers, the repelling and the attracting fixed points of A_j, respectively. We define t_j by setting $t_j = 1/\lambda_j$. Thus $t_j \in D^* = \{z \mid 0 < |z| < 1\}$. We determine a Möbius transformation T by

$$T(p_1) = 0,\ T(p_3) = \infty,\ T(p_2) = 1$$

We define ρ by $\rho = T(p_4)$. Thus $\rho \in \mathbf{C} - \{0, 1\}$. Then by easy calculations we have

$$TA_1T^{-1} = \frac{1}{\sqrt{t_1}}\begin{pmatrix} 1 & 0 \\ 0 & t_1 \end{pmatrix} \quad \text{and} \quad TA_2T^{-1} = \frac{1}{\sqrt{t_2}(\rho-1)}\begin{pmatrix} \rho - t_2 & \rho(t_2-1) \\ 1-t_2 & t_2\rho - 1 \end{pmatrix}.$$

Hereafter let

$$A_1 = \frac{1}{\sqrt{t_1}}\begin{pmatrix} 1 & 0 \\ 0 & t_1 \end{pmatrix} \quad \text{and} \quad A_2 = \frac{1}{\sqrt{t_2}(\rho-1)}\begin{pmatrix} \rho - t_2 & \rho(t_2-1) \\ 1-t_2 & t_2\rho - 1 \end{pmatrix}.$$

We say that $\tau = (t_1, t_2, \rho)$ corresponds to the marked group $\langle A_1, A_2\rangle$.

Conversely, λ_1, λ_2 and p_4 are uniquely determined from a given point $\tau = (t_1, t_2, \rho) \in (D^*)^2 \times (\mathbf{C} \setminus \{0, 1\})$ under the normalization condition $p_1 = 0$, $p_3 = \infty$ and $p_2 = 1$; we define λ_j ($j = 1, 2$) and p_4 by setting $\lambda_j = 1/t_j$ and $p_4 = \rho$. We determine $A_1(z), A_2(z) \in$ Möb from τ as follows: the multiplier, the repelling and the attracting fixed points of $A_j(z)$ are λ_j, p_j and p_{2+j}, respectively. We say that $G(\tau) = \langle A_1(\tau), A_2(\tau)\rangle$ is the marked group corresponding to $\tau = (t_1, t_2, \rho)$.

5. REAL SCHOTTKY SPACE.

In this section we consider the real classical Schottky space of type IV introduced by Sato ([**10**]). Hereafter let

$$A_1 = \frac{1}{\sqrt{t_1}}\begin{pmatrix} 1 & 0 \\ 0 & t_1 \end{pmatrix} \quad \text{and} \quad A_2 = \frac{1}{\sqrt{t_2(\rho-1)}}\begin{pmatrix} \rho - t_2 & \rho(t_2-1) \\ 1-t_2 & t_2\rho - 1 \end{pmatrix}.$$

with $0 < t_1 < 1$, $0 < t_2 < 1$ and $\rho < 0$.

We set

$$D_4 := \{\tau = (t_1, t_2, \rho) \in \mathbf{R}^3 \mid 0 < t_1 < 1,\ 0 < t_2 < 1,\ \rho < 0\}.$$

Let $G(\tau) = \langle A_1(\tau), A_2(\tau)\rangle$ be the group corresponding to $\tau = (t_1, t_2, \rho)$.

We set

$$R_{\mathrm{IV}}(S_2^0) := \{\tau = (t_1, t_2, \rho) \in D_4 \mid \langle A_1(\tau), A_2(\tau)\rangle : \text{classical Schottky group}\}.$$

We call $G(\tau) = \langle A_1(\tau), A_2(\tau)\rangle$ a *a real classical Schottky group of type* IV if $G(\tau) \in R_{\mathrm{IV}}(S_2^0)$.

Let $G = \langle A_1, A_2\rangle$ be a real classical Schottky group of type IV. Let $\tau = (t_1, t_2, \rho)$ correspond to the group $G = \langle A_1, A_2\rangle$. For given $0 < t_1 < 1$ and $\rho < 0$, let $t_2^*(t_1, \rho)$ be t_2 $(0 < t_2 < 1)$ satisfying

$$2\sqrt{t_1}\sqrt{t_2}(1-\rho) = \sqrt{(-\rho)}(1-t_1)(1-t_2).$$

Proposition 6.1 (Sato [**10**]).

$$R_{\mathrm{IV}}S_2^0 = \{(t_1, t_2, \rho) \in \mathbf{R}^3 \mid 0 < t_2 < t_2^*(t_1, \rho),\ 0 < t_1 < 1,\ \rho < 0\}.$$

7. A FUNDAMENTAL REGION.

7.1. Here we consider Nielsen transformations.

THEOREM B (Neumann). *Let $G = \langle A_1, A_2\rangle$ be a free group on two generators. The group Φ_2 of automorphisms of G have the following presentation:*

$$\Phi_2 = \langle N_1, N_2, N_3 \mid$$

$$(N_2N_1N_2N_3)^2 = 1,\ N_3^{-1}N_2N_3N_2N_1N_3N_1N_2N_1 = 1,\ N_1N_3N_1N_3 = N_3N_1N_3N_1\rangle,$$

where $N_1 : (A_1, A_2) \mapsto (A_1, A_2^{-1})$, $N_2 : (A_1, A_2) \mapsto (A_2, A_1)$, $N_3 : (A_1, A_2) \mapsto (A_1, A_1A_2)$.

We call N_1, N_2 and N_3 in Theorem B the Nielsen transformations.

Let $\tau = (t_1, t_2, \rho)$ correspond to a marked group $\langle A_1, A_2\rangle$. Let $(t_1(j), t_2(j), \rho(j))$ be the images of (t_1, t_2, ρ) under the mappings N_j $(j = 1, 2, 3)$, that is, $(t_1(1), t_2(1), \rho(1))$, $(t_1(2), t_2(2), \rho(2))$ and $(t_1(3), t_2(3), \rho(3))$ correspond to marked Schottky groups $\langle A_1, A_2^{-1}\rangle$, $\langle A_2, A_1\rangle$ and $\langle A_1, A_1A_2\rangle$, respectively.

7.2. Let $G = \langle A_1, A_2\rangle$ be a marked Schottky group and Φ_2 the group of automorphisms of G. The *Schottky modular group* of genus 2 is the set of all equivalence classes of orientation preserving automorphisms in Φ_2.

PROPOSITION 7.1 (Sato [10]). *Let $S = N_1N_3N_1$ and $T = N_1N_2$, where N_1, N_2 and N_3 be the Nielsen transofrmations defined in Theorem B. The Schottky modular group* Mod(S_2^0)) *acting on $R_{\mathrm{IV}}S_2^0$ is generated by S and T.*

7.3. We set

$$\rho^*(t_1, t_2) = (1 - \sqrt{t_1}t_2)/(t_2 - \sqrt{t_1})$$

for $0 < t_1 < 1$ and $0 < t_2 < 1$.

PROPOSITION 7.2 (Sato [10]). *Let* Mod(S_2^0)) *be the Schottky modular group* *acting on* $R_{\mathrm{IV}}S_2^0$. *Set*

$$F_{\mathrm{IV}}(\mathrm{Mod}(S_2^0))$$

$$= \{(t_1, t_2, \rho) \in R_{\mathrm{IV}}S_2^0 \mid \rho^*(t_1, t_2) < \rho < 1/\rho^*(t_1, t_2),\ t_2 < t_1,\ 0 < t_2 < t_2^*(t_1, \rho), \{0 < t_1 < 1\}.$$

Then $F_{\mathrm{IV}}(\mathrm{Mod}(S_2^0))$ *is a fundamental region for* $\mathrm{Mod}(S_2^0)$ *acting on* $R_{\mathrm{IV}}S_2^0$.

8. JØRGENSEN NUMBERS.

8.1. Let A_1 and A_2 be loxodromic transformations. Let $\tau = (t_1, t_2, \rho)$ correspond to the marked group $\langle A_1, A_2 \rangle$. We set

$$J_1(A_1) := |\mathrm{tr}^2(A_1) - 4|$$

$$J_2(A_1, A_2) := |\mathrm{tr}(A_1 A_2 A_1^{-1} A_2^{-1}) - 2|$$

$$J_1(\tau) := \frac{|1 - t_1|^2}{|t_1|}$$

$$J_2(\tau) := \frac{|1 - t_1|^2 |1 - t_2|^2 |\rho|}{|t_1||t_2||\rho - 1|^2}.$$

Then $J(A_1, A_2) = J_1(A_1) + J_2(A_1, A_2)$, where $J(A_1, A_2)$ is the Jørgensen number of $\langle A_1, A_2 \rangle$. We set $J(\tau) := J_1(\tau) + J_2(\tau)$.

PROPOSITION 8.1.

(1) $J_1(A_1, A_2) = J_1(\tau),\ J_2(A_1, A_2) = J_2(\tau),\ J(A_1, A_2) = J(\tau)$.

(2)

$$J(\tau) = \frac{|1 - t_1|^2}{|t_1|} + \frac{|1 - t_1|^2 |1 - t_2|^2 |\rho|}{|t_1||t_2||\rho - 1|^2}.$$

LEMMA 8.1. $J_2(\tau)$ *is* Φ_2*–invariant, that is,* $J_2(\phi_2(\tau)) = J_2(\tau)$ *for all* $\phi \in \Phi_2$.

LEMMA 8.2. $J_1(\tau)$ *and* $J(\tau)$ *are invariant under the Nielsen transformations* N_1 *and* N_3.

Proposition 8.2 (Sato [10]). *The boundary $\partial R_{\mathrm{IV}} S_2^0$ of the real classical Schottky space of type* IV *is invariant under Φ_2 and under* $\mathrm{Mod}(S_2^0)$.

9. PROOF OF THEOREM 2.

9.1. In this section we sketch the proof of Theorem 2. The complete proof will appear elsewhere. We consider the following surface in $\mathbf{R}^3$. For $k \geq 2$

$$S_k := \{\tau = (t_1, t_2, \rho) \in \mathbf{R}^3 | \\ \frac{1-t_1}{\sqrt{t_1}}\frac{1-t_2}{\sqrt{t_2}} = k\frac{1-\rho}{\sqrt{-\rho}},\ 0 < t_1 < 1,\ 0 < t_2 < 1, \rho < 0\}$$

Proposition 9.1.

(1) *The surface S_k $(k \geq 2)$ is contained in the real classical Schottky space of type* IV.

(2) *The surface S_k $(k \geq 2)$ is Φ_2–invariant.*

Lemma 9.1. *Let $\tau_0 = (t_{10}, t_{20}, -1) \in \partial R_{\mathrm{IV}} S_2^0$ and $t_{10} > t_{20}$. Then*

(1) $J(\tau_0) = \dfrac{(1-t_{10})^2}{t_{10}} + 4$

(2) $J(\phi(\tau_0)) \geq J(\tau_0)$ *for* $\phi \in \Phi_2$.

Proposition 9.2. *Let r be a real number with $4 < r < 8$. Let t_{10} $(0 < t_{10} < 1)$ be a real number with $(1-t_{10})^2/t_{10} = r - 4$. Set $\rho_0 = -1$. Let $G_0 = \langle A_{10}, A_{20}\rangle$ be the marked group corresponding to $(t_{10}, t_{20}, -1)$. Then $J(G_0) = r$.*

9.2. By a similar method to the above, we have the following.

Proposition 9.3.

(1) *Let* $r > 4$. *Let* $\tau_0 = (t_{10}, t_{20}, -1) \in S_k$ $(k > 2)$ *with* $t_{10} > t_{20}$. *Let* $G_0 = \langle A_{10}, A_{20} \rangle$ *be the corresponding to* τ_0. *Then*

$$J(G_0) = \frac{(1 - t_{10})^2}{t_{10}} + k^2.$$

(2) *Given* $r > 4$. *Then there exist* t_1 $(0 < t_1 < 1)$ *and* $k \geq 2$ *such that* $r = (1 - t_1)^2/t_1 + k^2$.

9.3. Theorem 2 follows from Propositions 9.2 and 9.3. That is, there exists a classical Schottky group G in $R_{\mathrm{IV}}S_2^0$ such that $J(G) = r$.

Remark. Given $r > 4$. Set $t_{10} = 4/5$. Then there is a real number $k > 2$ such that $k = \sqrt{r^2 - 1/20}$. That is, there exists a classical Schottky group G in $R_{\mathrm{IV}}S_2^0$ such that $J(G) = r$.

10. OPEN PROBLEM.

In the last section we will state an open problem.

Open Problem. For $1 < r < 4$ $(r \neq 2, 3)$ when is there a non-elementary discrete group whose Jørgensen number is equal to r ?

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