数理解析研究所講究録
1518 巻 2006 年 132-143

HIGHER-ORDER ALEXANDER INVARIANTS FOR HOMOLOGY COBORDISMS OF A SURFACE

東京大学大学院数理科学研究科 逆井 卓也 (Takuya Sakasai)
Graduate School of Mathematical Sciences,
the University of Tokyo

1. Introduction

Let $\Sigma_{g,1}$ be a compact connected oriented surface of genus $g \geq 0$ with one boundary component. A *homology cylinder* (*over* $\Sigma_{g,1}$) consists of a homology cobordism from $\Sigma_{g,1}$ to itself with markings of its boundary. We denote by $C_{g,1}$ the set of all diffeomorphism classes of homology cylinders. Stacking two homology cylinders gives a new one, and by this, we can endow $C_{g,1}$ with a monoid structure. A systematic study of $C_{g,1}$ was initiated by Habiro in [4], where $C_{g,1}$ appeared as a nice collection of 3-manifolds to which his clasper surgery theory is applied. Later Garoufalidis-Levine [3] and Levine [9] introduced a group $\mathcal{H}_{g,1}$ by taking a quotient of $C_{g,1}$ with respect to homology cobordant of homology cylinders. A feature of the monoid $C_{g,1}$ and the group $\mathcal{H}_{g,1}$ is that they contain the mapping class group $\mathcal{M}_{g,1}$, which is the group of isotopy classes of orientation-preserving diffeomorphisms of $\Sigma_{g,1}$. Moreover some tools for studying $\mathcal{M}_{g,1}$ can be also used for $C_{g,1}$ and $\mathcal{H}_{g,1}$ after appropriate generalizations. From these facts, we can consider $C_{g,1}$ and $\mathcal{H}_{g,1}$ to be enlargements of $\mathcal{M}_{g,1}$.

Now we consider an application of higher-order Alexander invariants, which are numerical invariants of finitely presentable groups, to homology cylinders. Higher-order Alexander invariants were first defined by Cochran in [1] for knot groups, and then generalized for arbitrary finitely presentable groups by Harvey in [5, 6]. They are interpreted as degrees of "non-commutative Alexander polynomials", which have some unclear ambiguity except their degrees in difficulties of non-commutative rings. Using them, Harvey obtained various sharper results than those given by the ordinary Alexander invariants — lower bounds on the Thurston norm, necessary conditions for realizing a given group as the fundamental group of some compact oriented 3-manifold, and so on.

In the process of applying higher-order Alexander invariants to homology cylinders, we can see that the Magnus representation for homology cylinders [15] plays an important role. This representation generalizes not only the Magnus representation for $\mathcal{M}_{g,1}$ defined by Morita [11], but the Gassner representation for string links given by Le Dimet [8] and Kirk-Livingston-Wang [7]. In this paper, we begin by reviewing the definition and fundamental properties of the Magnus representation, and then study some relationships to higher-order Alexander invariants. Note that the paper [16] treats the same topics and complements the contents of this paper.

2. Homology cobordisms of surfaces

We proceed all our discussion in PL or smooth category.

Let $\Sigma_{g,1}$ be a compact connected oriented surface of genus $g \geq 0$ with one boundary component. We take a base point p on the boundary of $\Sigma_{g,1}$, and take $2g$ loops $\gamma_1, \ldots, \gamma_{2g}$ of $\Sigma_{g,1}$ as shown in Figure 1. We consider them to be an embedded bouquet R_{2g} of $2g$-circles tied at the base point $p \in \partial\Sigma_{g,1}$. Then R_{2g} and the boundary loop ζ of $\Sigma_{g,1}$ together with one 2-cell make up a standard CW-decomposition of $\Sigma_{g,1}$. It is well-known that the fundamental group $\pi_1\Sigma_{g,1}$ of $\Sigma_{g,1}$ is isomorphic to the free group F_{2g} of rank $2g$ generated by $\gamma_1, \ldots, \gamma_{2g}$, in which $\zeta = \prod_{i=1}^{g}[\gamma_i, \gamma_{g+i}]$.

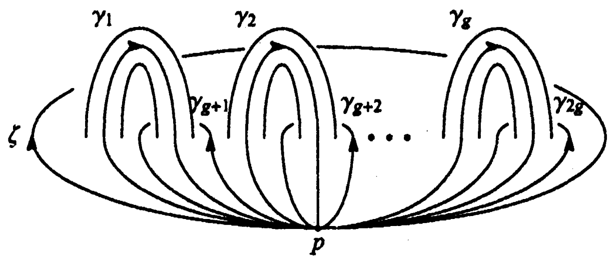

Figure 1

A *homology cylinder* (M, i_+, i_-) (*over* $\Sigma_{g,1}$), which has its origin in Habiro [4], Garoufalidis-Levine [3] and Levine [9], consists of a compact oriented 3-manifold M and two embeddings $i_+, i_- : \Sigma_{g,1} \to \partial M$ satisfying that

(1) i_+ is orientation-preserving and i_- is orientation-reversing,
(2) $\partial M = i_+(\Sigma_{g,1}) \cup i_-(\Sigma_{g,1})$ and $i_+(\Sigma_{g,1}) \cap i_-(\Sigma_{g,1}) = i_+(\partial\Sigma_{g,1}) = i_-(\partial\Sigma_{g,1})$,
(3) $i_+|_{\partial\Sigma_{g,1}} = i_-|_{\partial\Sigma_{g,1}}$,
(4) $i_+, i_- : H_*(\Sigma_{g,1}) \to H_*(M)$ are isomorphisms.

We denote $i_+(p) = i_-(p)$ by $p \in \partial M$ again and consider it to be the base point of M. We write a homology cylinder by (M, i_+, i_-) or simply by M.

Two homology cylinders are said to be *isomorphic* if there exists an orientation-preserving diffeomorphism between the underlying 3-manifolds which is compatible with the markings. We denote the set of isomorphism classes of homology cylinders by $C_{g,1}$. Given two homology cylinders $M = (M, i_+, i_-)$ and $N = (N, j_+, j_-)$, we can define a new homology cylinder $M \cdot N$ by

$$M \cdot N = (M \cup_{i_- \circ (j_+)^{-1}} N, i_+, j_-).$$

Then $C_{g,1}$ becomes a monoid with the identity element $1_{C_{g,1}} := (\Sigma_{g,1} \times I, \mathrm{id} \times 1, \mathrm{id} \times 0)$.

From the monoid $C_{g,1}$, we can construct the *homology cobordism group* $\mathcal{H}_{g,1}$ *of homology cylinders* as in the following way. Two homology cylinders $M = (M, i_+, i_-)$ and $N = (N, j_+, j_-)$ are *homology cobordant* if there exists a compact oriented 4-manifold W such that

(1) $\partial W = M \cup (-N)/(i_+(x) = j_+(x),\ i_-(x) = j_-(x)) \quad x \in \Sigma_{g,1}$,
(2) the inclusions $M \hookrightarrow W$, $N \hookrightarrow W$ induce isomorphisms on the homology,

where $-N$ is N with opposite orientation. We denote by $\mathcal{H}_{g,1}$ the quotient set of $\mathcal{C}_{g,1}$ with respect to the equivalence relation of homology cobordism. The monoid structure of $\mathcal{C}_{g,1}$ induces a group structure of $\mathcal{H}_{g,1}$. In the group $\mathcal{H}_{g,1}$, the inverse of (M, i_+, i_-) is given by $(-M, i_-, i_+)$.

Example 2.1. For each element φ of the mapping class group $\mathcal{M}_{g,1}$ of $\Sigma_{g,1}$, we can construct a homology cylinder $M_\varphi \in \mathcal{C}_{g,1}$ defined by

$$M_\varphi := (\Sigma_{g,1} \times I, \mathrm{id} \times 1, \varphi \times 0),$$

where collars of $i_+(\Sigma_{g,1})$ and $i_-(\Sigma_{g,1})$ are stretched half-way along $\partial\Sigma_{g,1} \times I$. This gives injective homomorphisms $\mathcal{M}_{g,1} \hookrightarrow \mathcal{C}_{g,1}$ and $\mathcal{M}_{g,1} \hookrightarrow \mathcal{H}_{g,1}$.

Let $N_k(G) := G/(\Gamma^k G)$ be the k-th nilpotent quotient of a group G, where we define $\Gamma^1 G = G$ and $\Gamma^{i+1} G = [\Gamma^i G, G]$ for $i \geq 1$. For simplicity, we write $N_k(X)$ for $N_k(\pi_1 X)$ where X is a CW-complex, and write N_k for $N_k(F_{2g}) = N_k(\Sigma_{g,1})$. It is known that N_k is a torsion-free nilpotent group for each $k \geq 2$.

Let (M, i_+, i_-) be a homology cylinder. By definition, $i_+, i_- : \pi_1\Sigma_{g,1} \to \pi_1 M$ are both 2-*connected*, namely they induce isomorphisms on H_1 and epimorphisms on H_2. Then, by Stallings' theorem [17], $i_+, i_- : N_k \xrightarrow{\cong} N_k(M)$ are isomorphisms for each $k \geq 2$. Using them, we obtain a monoid homomorphism

$$\sigma_k : \mathcal{C}_{g,1} \longrightarrow \mathrm{Aut} N_k \qquad ((M, i_+, i_-) \mapsto (i_+)^{-1} \circ i_-).$$

It can be easily checked that σ_k induces a group homomorphism $\sigma_k : \mathcal{H}_{g,1} \to \mathrm{Aut} N_k$. We define filtrations of $\mathcal{C}_{g,1}$ and $\mathcal{H}_{g,1}$ by

$$\mathcal{C}_{g,1}[1] := \mathcal{C}_{g,1}, \quad \mathcal{C}_{g,1}[k] := \mathrm{Ker}\left(\mathcal{C}_{g,1} \xrightarrow{\sigma_k} \mathrm{Aut} N_k\right) \text{ for } k \geq 2,$$
$$\mathcal{H}_{g,1}[1] := \mathcal{H}_{g,1}, \quad \mathcal{H}_{g,1}[k] := \mathrm{Ker}\left(\mathcal{H}_{g,1} \xrightarrow{\sigma_k} \mathrm{Aut} N_k\right) \text{ for } k \geq 2.$$

3. Magnus representations for homology cylinders

We first summarize our notation. For a matrix A with entries in a ring R, and a homomorphism $\varphi : R \to R'$, we denote by ${}^\varphi A$ the matrix obtained from A by applying φ to each entry. A^T denotes the transpose of A. When $R = \mathbb{Z}G$ for a group G or its right field of fractions (if exists), we denote by $\overline{A}$ the matrix obtained from A by applying the involution induced from $(x \mapsto x^{-1},\ x \in G)$ to each entry. For a module M, we write M^n (resp. M_n) for the module of column (resp. row) vectors with n entries.

For a finite CW-complex X and its regular covering X_Γ with respect to a homomorphism $\pi_1 X \to \Gamma$, Γ acts on X_Γ from the right through its deck transformation group. Therefore we regard the $\mathbb{Z}\Gamma$-cellular chain complex $C_*(X_\Gamma)$ of X_Γ as a collection of free right $\mathbb{Z}\Gamma$-modules consisting of column vectors together with differentials given by left multiplications of matrices. For each $\mathbb{Z}\Gamma$-bimodule A, the twisted chain complex $C_*(X; A)$ is given by the tensor product of the right $\mathbb{Z}\Gamma$-module $C_*(X_\Gamma)$ and the left $\mathbb{Z}\Gamma$-module A, so that $C_*(X; A)$ and $H_*(X; A)$ are right $\mathbb{Z}\Gamma$-modules.

Now we define and study the Magnus representation for homology cylinders. The following construction is based on Kirk-Livingston-Wang's paper [7].

Let $(M, i_+, i_-) \in \mathcal{C}_{g,1}$ be a homology cylinder. By Stallings' theorem, N_k and $N_k(M)$ are isomorphic. Since N_k is a finitely generated torsion-free nilpotent group for each $k \geq 2$, we can embed $\mathbf{Z}N_k$ into the right field of fractions $\mathcal{K}_{N_k} := \mathbf{Z}N_k(\mathbf{Z}N_k - \{0\})^{-1}$. (See Section 5.) Similarly, we obtain $\mathbf{Z}N_k(M) \hookrightarrow \mathcal{K}_{N_k(M)} := \mathbf{Z}N_k(M)(\mathbf{Z}N_k(M) - \{0\})^{-1}$. We consider $\mathcal{K}_{N_k}$ (resp. $\mathcal{K}_{N_k(M)}$) to be a local coefficient system on $\Sigma_{g,1}$ (resp. M).

By a standard argument using covering spaces, we have the following.

Lemma 3.1. $i_\pm : H_*(\Sigma_{g,1}, p; i_\pm^*\mathcal{K}_{N_k(M)}) \to H_*(M, p; \mathcal{K}_{N_k(M)})$ *are isomorphisms as right* $\mathcal{K}_{N_k(M)}$*-vector spaces.*

Since $R_{2g} \subset \Sigma_{g,1}$ is a deformation retract, we have

$$H_1(\Sigma_{g,1}, p; i_\pm^*\mathcal{K}_{N_k(M)}) \cong H_1(R_{2g}, p; i_\pm^*\mathcal{K}_{N_k(M)}) = C_1(\widetilde{R_{2g}}) \otimes_{F_{2g}} i_\pm^*\mathcal{K}_{N_k(M)} \cong \mathcal{K}_{N_k(M)}^{2g}$$

with a basis

$$\{\overline{\gamma_1} \otimes 1, \ldots, \overline{\gamma_{2g}} \otimes 1\} \subset C_1(\widetilde{R_{2g}}) \otimes_{F_{2g}} i_\pm^*\mathcal{K}_{N_k(M)}$$

as a right $\mathcal{K}_{N_k(M)}$-vector space, where $\overline{\gamma_i}$ is a lift of γ_i on the universal covering $\widetilde{R_{2g}}$.

Definition 3.2. (1) For each $M = (M, i_+, i_-) \in \mathcal{C}_{g,1}$, we denote by $r_k'(M) \in GL(2g, \mathcal{K}_{N_k(M)})$ the representation matrix of the right $\mathcal{K}_{N_k(M)}$-isomorphism

$$\mathcal{K}_{N_k(M)}^{2g} \cong H_1(\Sigma_{g,1}, p; i_-^*\mathcal{K}_{N_k(M)}) \xrightarrow[i_-]{\cong} H_1(M, p; \mathcal{K}_{N_k(M)}) \xrightarrow[i_+^{-1}]{\cong} H_1(\Sigma_{g,1}, p; i_+^*\mathcal{K}_{N_k(M)}) \cong \mathcal{K}_{N_k(M)}^{2g}$$

(2) The *Magnus representation* for $\mathcal{C}_{g,1}$ is the map $r_k : \mathcal{C}_{g,1} \to GL(2g, \mathcal{K}_{N_k})$ which assigns to $M = (M, i_+, i_-) \in \mathcal{C}_{g,1}$ the matrix ${}^{i_+^{-1}}r_k'(M)$.

While we call $r_k(M)$ the Magnus "representation", it is actually a crossed homomorphism.

Theorem 3.3 ([14, Theorem 7.12]). *For* $M_1 = (M_1, i_+, i_-)$*,* $M_2 = (M_2, j_+, j_-) \in \mathcal{C}_{g,1}$*, we have*

$$r_k(M_1 \cdot M_2) = r_k(M_1) \cdot {}^{\sigma_k(M_1)}r_k(M_2).$$

Moreover, we can show the following.

Theorem 3.4 ([14, Theorem 7.13]). $r_k : \mathcal{C}_{g,1} \to GL(2g, \mathcal{K}_{N_k})$ *factors through* $\mathcal{H}_{g,1}$.

Consequently, we obtain the Magnus representation $r_k : \mathcal{H}_{g,1} \to GL(2g, \mathcal{K}_{N_k})$, which is a crossed homomorphism. Note that if we restrict r_k to $\mathcal{C}_{g,1}[k]$ (and $\mathcal{H}_{g,1}[k]$), it becomes a homomorphism.

Example 3.5. For $\varphi \in \mathcal{M}_{g,1} \hookrightarrow \mathrm{Aut}F_{2g}$, we can obtain

$$r_k(M_\varphi) = \overline{{}^{\rho_k}\left(\frac{\partial \varphi(\gamma_j)}{\partial \gamma_i}\right)_{i,j}},$$

where $\rho_k : \mathbf{Z}F_{2g} \to \mathbf{Z}N_k \subset \mathcal{K}_{N_k}$ is the natural map and $\partial/\partial\gamma_i$ are free differentials. From this, we see that r_k generalizes the original Magnus representation for $\mathcal{M}_{g,1}$ in [11].

In general, the Magnus matrix $r_k(M)$ of a homology cylinder M can be obtained from a finite presentation of the form

$$\pi_1 M \cong \left\langle \begin{array}{l} i_-(\gamma_1), \ldots, i_-(\gamma_{2g}), \\ z_1, \ldots, z_{2g+l}, \\ i_+(\gamma_1), \ldots, i_+(\gamma_{2g}) \end{array} \middle| \begin{array}{l} i_-(\gamma_1)s_1, \ldots, i_-(\gamma_{2g})s_{2g}, \\ r_1, \ldots r_l, \\ i_+(\gamma_1)u_1, \ldots, i_+(\gamma_{2g})u_{2g} \end{array} \right\rangle,$$

where s_i, r_i and u_i are words in $z_1, \ldots, z_{2g+l}$, by a purely algebraic calculation. Note that such a presentation does exist for each homology cylinder.

As in the case of $\mathcal{M}_{g,1}$ (see [11] and [18]), the Magnus representation for $\mathcal{H}_{g,1}$ satisfies the following "symplectic" property.

Theorem 3.6. *For any homology cylinder M, we have the equality*

$$\overline{r_k(M)^T}\, \widetilde{J}\, r_k(M) = {}^{\sigma_k(M)}\widetilde{J},$$

where $\widetilde{J} = \begin{pmatrix} J_1 & J_2 \\ J_3 & J_4 \end{pmatrix} \in GL(2g, \mathbb{Z}N_k)$ *is defined by*

$$J_1 = \begin{pmatrix} 1-\gamma_1 & & & & 0 \\ (1-\gamma_2)(1-\gamma_1^{-1}) & 1-\gamma_2 & & & \\ (1-\gamma_3)(1-\gamma_1^{-1}) & (1-\gamma_3)(1-\gamma_2^{-1}) & 1-\gamma_3 & & \\ \vdots & \vdots & & \ddots & \\ (1-\gamma_g)(1-\gamma_1^{-1}) & (1-\gamma_g)(1-\gamma_2^{-1}) & \cdots & & 1-\gamma_g \end{pmatrix},$$

$$J_2 = \begin{pmatrix} \gamma_1\gamma_{g+1}^{-1} & & & & 0 \\ (1-\gamma_2)(1-\gamma_{g+1}^{-1}) & \gamma_2\gamma_{g+2}^{-1} & & & \\ (1-\gamma_3)(1-\gamma_{g+1}^{-1}) & (1-\gamma_3)(1-\gamma_{g+2}^{-1}) & \gamma_3\gamma_{g+3}^{-1} & & \\ \vdots & \vdots & & \ddots & \\ (1-\gamma_g)(1-\gamma_{g+1}^{-1}) & (1-\gamma_g)(1-\gamma_{g+2}^{-1}) & \cdots & & \gamma_g\gamma_{2g}^{-1} \end{pmatrix},$$

$$J_3 = \begin{pmatrix} 1-\gamma_1^{-1}-\gamma_{g+1} & & & & 0 \\ (1-\gamma_{g+2})(1-\gamma_1^{-1}) & 1-\gamma_2^{-1}-\gamma_{g+2} & & & \\ (1-\gamma_{g+3})(1-\gamma_1^{-1}) & (1-\gamma_{g+3})(1-\gamma_2^{-1}) & 1-\gamma_3^{-1}-\gamma_{g+3} & & \\ \vdots & \vdots & & \ddots & \\ (1-\gamma_{2g})(1-\gamma_1^{-1}) & (1-\gamma_{2g})(1-\gamma_2^{-1}) & \cdots & & 1-\gamma_g^{-1}-\gamma_{2g} \end{pmatrix},$$

$$J_4 = \begin{pmatrix} 1-\gamma_{g+1}^{-1} & & & & 0 \\ (1-\gamma_{g+2})(1-\gamma_{g+1}^{-1}) & 1-\gamma_{g+2}^{-1} & & & \\ (1-\gamma_{g+3})(1-\gamma_{g+1}^{-1}) & (1-\gamma_{g+3})(1-\gamma_{g+2}^{-1}) & 1-\gamma_{g+3}^{-1} & & \\ \vdots & \vdots & & \ddots & \\ (1-\gamma_{2g})(1-\gamma_{g+1}^{-1}) & (1-\gamma_{2g})(1-\gamma_{g+2}^{-1}) & \cdots & & 1-\gamma_{2g}^{-1} \end{pmatrix}.$$

Note that the matrix $\widetilde{J}$ appeared in Papakyriakopoulos' paper [12], and that it is mapped to the ordinary symplectic matrix by the augmentation map $\mathbb{Z}N_k \to \mathbb{Z}$.

Sketch of Proof. First we define a natural pairing

$$\langle\, ,\, \rangle : H_1(\Sigma_{g,1}, p; \mathcal{K}_{N_k}) \times H_1(\Sigma_{g,1}, p; \mathcal{K}_{N_k}) \longrightarrow \mathcal{K}_{N_k}$$

satisfying

$$\langle af, b\rangle = \overline{f}\langle a, b\rangle, \quad \langle a, bf\rangle = \langle a, b\rangle f$$

for all $f \in \mathcal{K}_{N_k}$. This generalizes Suzuki's higher intersection form in [18]. To construct it, we use the following type of the Poincaré-Lefschetz duality: *Let X be a compact oriented n-manifold whose boundary ∂M is decomposed as the union of two compact manifolds A and B*

with $\partial A = \partial B = A \cap B$, and let M be a local coefficient system on X. Then the cap product with a fundamental class gives isomorphisms $H^k(X, A; M) \xrightarrow{\cong} H_{n-k}(X, B; M)$ for all k.

The naturality of the Poincaré-Lefschetz duality shows the equality

$$\langle r_k(M)a, r_k(M)b \rangle = {}^{\sigma_k(M)}\langle a, b \rangle$$

for each homology cylinder M. By writing down this equality with respect to the basis $\{\overline{\gamma_1} \otimes 1, \ldots, \overline{\gamma_{2g}} \otimes 1\}$ of $H_1(\Sigma_{g,1}, p; \mathcal{K}_{N_k})$, where we use Papakyrirakopoulos' argument in [12], we obtain the desired equality. □

4. Example: Relationship to the Gassner representation for string links

In [9], Levine gave a method for constructing homology cylinders from pure string links. By this, we can obtain many homology cylinders not belonging to the subgroup $\mathcal{M}_{g,1}$. Also, we can see a relationship between the Gassner representation for string links and our representation.

For a g-component pure string link $L \subset D^2 \times I$, we now construct a homology cylinder $M_L \in \mathcal{C}_{g,1}$ as follows. Consider a closed tubular neighborhood of the loops $\gamma_{g+1}, \gamma_{g+2}, \ldots, \gamma_{2g}$ in Figure 1 to be the image of an embedding $\iota : D_g \hookrightarrow \Sigma_{g,1}$ of a g-holed disk D_g as in Figure 2.

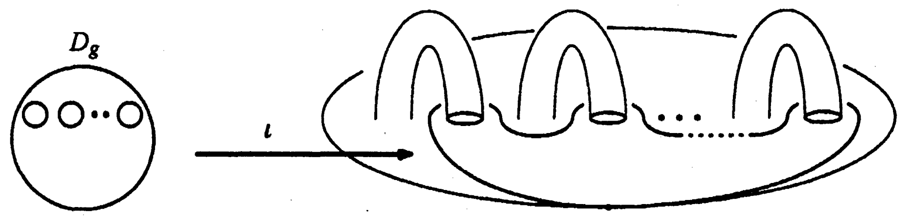

Figure 2

Let C be the complement of an open tubular neighborhood of L in $D^2 \times I$. For each choice a framing of L, a homeomorphism $h : \partial C \xrightarrow{\cong} \partial(\iota(D_g) \times I)$ is fixed. Then the manifold M_L given from $\Sigma_{g,1} \times I$ by removing $\iota(D_g) \times I$ and regluing C by h becomes a homology cylinder. This construction gives an injective monoid homomorphism $\mathcal{L}_g \to \mathcal{C}_{g,1}$ from the monoid $\mathcal{L}_g$ of (framed) pure string links to $\mathcal{C}_{g,1}$. Moreover it also induces an injective homomorphism $\mathcal{S}_g \to \mathcal{H}_{g,1}$ from the concordance group of (framed) pure string links to $\mathcal{H}_{g,1}$. In particular, the (smooth) knot concordance group, which coincides with $\mathcal{S}_1$, is embedded in $\mathcal{H}_{g,1}$. If we restrict these embeddings to the pure braid group, which is a subgroup of $\mathcal{L}_g$ and $\mathcal{S}_g$, their images are contained in $\mathcal{M}_{g,1}$.

We fix an integer $k \geq 2$. By the Gassner representation, we mean the crossed homomorphism $r_{G,k} : \mathcal{L}_g \to GL(g, \mathcal{K}_{N_k(D_g)})$ or $r_{G,k} : \mathcal{S}_g \to GL(g, \mathcal{K}_{N_k(D_g)})$ given by a construction similar to that in the previous section. (In [8] and [7], only $r_{G,2}$ is treated.) Comparing methods for calculating the Gassner and the Magnus representations, we obtain the following.

Theorem 4.1 ([14, Theorem 7.18]). *For any pure string link $L \in \mathcal{L}_g$, $r_k(M_L) = \begin{pmatrix} * & 0_g \\ * & r_{G,k}(L) \end{pmatrix}$.*

We mention two remarks about this theorem. First we identify $F_g = \pi_1 D_g$ with the subgroup of $F_{2g} = \pi_1 \Sigma_{g,1}$ generated by $\gamma_{g+1}, \dots, \gamma_{2g}$. Then the maps $F_g = \langle \gamma_{g+1}, \dots, \gamma_{2g} \rangle \hookrightarrow F_{2g} \twoheadrightarrow F_g$, where the second map sends $\gamma_1, \dots, \gamma_g$ to 1, show that $N_k(F_g) \subset N_k$ and $\mathcal{K}_{N_k(F_g)} \subset \mathcal{K}_{N_k}$. Second, the embeddings $\mathcal{L}_g \hookrightarrow C_{g,1}$ and $\mathcal{S}_g \hookrightarrow \mathcal{H}_{g,1}$ have ambiguity with respect to framings. However we can check that the lower right part of $r_k(M_L)$ does not depend on the choice of framings.

Corollary 4.2. *$\mathcal{M}_{g,1}$ is not a normal subgroup of $\mathcal{H}_{g,1}$ for $g \geq 3$.*

Proof. In [7], they gave 3-component pure string links denoted by L_5 and L_6 having the condition that L_5 is a pure braid, while the conjugate $L_6 L_5 L_6^{-1}$ is not. To show that $L_6 L_5 L_6^{-1}$ is not a pure braid, they use the fact that $r_{G,2}(L_6 L_5 L_6^{-1})$ has an entry not belonging to $\mathbb{Z}N_2(D_3)$. Then our claim follows from Theorem 4.1 with respect to this example. □

Example 4.3. Let L be a 2-component pure string link as depicted in Figure 3.

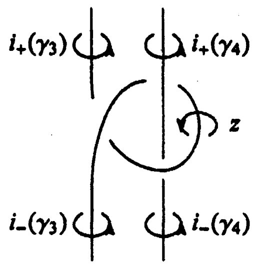

Figure 3

Then the presentation of $\pi_1 M_L$ is given by

$$\pi_1 M_L \cong \left\langle \begin{matrix} i_-(\gamma_1), \dots, i_-(\gamma_4) \\ z \\ i_+(\gamma_1), \dots, i_+(\gamma_4) \end{matrix} \;\middle|\; \begin{matrix} i_+(\gamma_1) i_-(\gamma_3)^{-1} i_+(\gamma_4) i_-(\gamma_1)^{-1}, \\ [i_+(\gamma_1), i_+(\gamma_3)] i_+(\gamma_2) z i_-(\gamma_2)^{-1} [i_-(\gamma_3), i_-(\gamma_1)], \\ i_+(\gamma_4) i_-(\gamma_3) i_+(\gamma_4)^{-1} z^{-1}, \quad i_-(\gamma_3) i_+(\gamma_3)^{-1} i_-(\gamma_3)^{-1} z, \\ i_-(\gamma_4) z^{-1} i_+(\gamma_4)^{-1} z, \end{matrix} \right\rangle,$$

where we use the blackboard framing. We identify N_2 and $N_2(M_L)$ by using i_+. Using the presentation, we have $z = i_-(\gamma_3) = \gamma_3$, $i_-(\gamma_4) = \gamma_4$, $i_-(\gamma_2) = \gamma_2 \gamma_3$ and $i_-(\gamma_1) = \gamma_1 \gamma_3^{-1} \gamma_4$ in N_2. Then we obtain

$$r_2(M_L) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{-\gamma_1^{-1}}{\gamma_3^{-1}+\gamma_4^{-1}-1} & \frac{\gamma_2^{-1}\gamma_3^{-1}\gamma_4^{-1}-\gamma_4^{-1}+1}{\gamma_3^{-1}+\gamma_4^{-1}-1} & \frac{\gamma_3^{-1}}{\gamma_3^{-1}+\gamma_4^{-1}-1} & \frac{\gamma_4^{-1}(\gamma_4^{-1}-1)}{\gamma_3^{-1}+\gamma_4^{-1}-1} \\ \frac{\gamma_1^{-1}\gamma_3\gamma_4^{-1}}{\gamma_3^{-1}+\gamma_4^{-1}-1} & \frac{(1-\gamma_3^{-1})(\gamma_2^{-1}\gamma_3^{-1}-\gamma_2^{-1}-1)}{\gamma_3^{-1}+\gamma_4^{-1}-1} & \frac{\gamma_3^{-1}-1}{\gamma_3^{-1}+\gamma_4^{-1}-1} & \frac{-\gamma_3^{-1}\gamma_4^{-1}+\gamma_3^{-1}+2\gamma_4^{-1}-1}{\gamma_3^{-1}+\gamma_4^{-1}-1} \end{pmatrix}.$$

Note that $\det r_2(M_L) = \dfrac{\gamma_3 + \gamma_4 - 1}{\gamma_3\gamma_4(\gamma_3^{-1} + \gamma_4^{-1} - 1)}$.

5. Higher-order Alexander invariants and torsion-degree functions

Here we summarize the theory of higher-order Alexander invariants along the lines of Harvey's papers [5, 6]. For our use, we generalize them to functions of matrices called *torsion-degree functions*.

A group Γ is *poly-torsion-free-abelian* (PTFA, for short) if Γ has a normal series of finite length whose successive quotients are all torsion-free abelian. In particular, free nilpotent quotients N_k are PTFA for all $k \geq 2$. Note that any subgroup of a PTFA group is also PTFA.

For each PTFA group Γ, the group ring $\mathbf{Z}\Gamma$ is known to be an Ore domain, so that it can be embedded in the *right field of fractions* $\mathcal{K}_\Gamma := \mathbf{Z}\Gamma(\mathbf{Z}\Gamma - \{0\})^{-1}$, which is a skew field. We refer to [2], [13] for localizations of non-commutative rings.

We will also use the following localizations of $\mathbf{Z}\Gamma$ placed between $\mathbf{Z}\Gamma$ and $\mathcal{K}_\Gamma$. Let $\psi \in H^1(\Gamma)$ be a primitive element. This means the corresponding homomorphism, which is denoted by ψ again, under $H^1(\Gamma) \cong \mathrm{Hom}(\Gamma, \mathbf{Z})$ is onto. Then we have an exact sequence

$$1 \longrightarrow (\Gamma^\psi := \mathrm{Ker}\,\psi) \longrightarrow \Gamma \xrightarrow{\psi} \mathbf{Z} \longrightarrow 1.$$

We take a splitting $\xi : \mathbf{Z} \to \Gamma$ of this sequence and put $t := \xi(1) \in \Gamma$. Since Γ^ψ is again a PTFA group, $\mathbf{Z}\Gamma^\psi$ can be embedded in its right field of fractions $\mathcal{K}_{\Gamma^\psi} = \mathbf{Z}\Gamma^\psi(\mathbf{Z}\Gamma^\psi - \{0\})^{-1}$. Moreover, we can construct a right quotient ring $\mathbf{Z}\Gamma(\mathbf{Z}\Gamma^\psi - \{0\})^{-1}$. Then the splitting ξ gives an isomorphism between $\mathbf{Z}\Gamma(\mathbf{Z}\Gamma^\psi - \{0\})^{-1}$ and the skew Laurent polynomial ring $\mathcal{K}_{\Gamma^\psi}[t^\pm]$, in which $at = t(t^{-1}at)$ holds for each $a \in \Gamma$. $\mathcal{K}_{\Gamma^\psi}[t^\pm]$ is known to be a non-commutative right and left principal ideal domain. By definition, we have inclusions

$$\mathbf{Z}\Gamma \hookrightarrow \mathcal{K}_{\Gamma^\psi}[t^\pm] \hookrightarrow \mathcal{K}_\Gamma.$$

$\mathcal{K}_{\Gamma^\psi}[t^\pm]$ and $\mathcal{K}_\Gamma$ are known to be flat $\mathbf{Z}\Gamma$-modules. On $\mathcal{K}_{\Gamma^\psi}[t^\pm]$, we have a map $\deg^\psi : \mathcal{K}_{\Gamma^\psi}[t^\pm] \to \mathbf{Z}_{\geq 0} \cup \{\infty\}$ assigning to each polynomial its degree. We put $\deg^\psi(0) := \infty$. Note that the composite $\mathbf{Z}\Gamma(\mathbf{Z}\Gamma^\psi - \{0\})^{-1} \xrightarrow{\cong} \mathcal{K}_{\Gamma^\psi}[t^\pm] \xrightarrow{\deg^\psi} \mathbf{Z}_{\geq 0} \cup \{\infty\}$ does not depend on the choice of the splitting ξ.

Harvey's higher-order Alexander invariants [6] are defined as follows. Let G be a finitely presentable group, and let $\varphi : G \twoheadrightarrow \mathbf{Z}$ be an epimorphism. For a PTFA group Γ and an epimorphism $\varphi_\Gamma : G \twoheadrightarrow \Gamma$, $(\varphi_\Gamma, \varphi)$ is called an *admissible pair* for G if there exists an epimorphism $\psi : \Gamma \twoheadrightarrow \mathbf{Z}$ satisfying $\varphi = \psi \circ \varphi_\Gamma$. For each admissible pair $(\varphi_\Gamma, \varphi)$ for G, we regard $\mathcal{K}_{\Gamma^\psi}[t^\pm] = \mathbf{Z}\Gamma(\mathbf{Z}\Gamma^\psi - \{0\})^{-1}$ as a $\mathbf{Z}G$-module, and we define the higher-order Alexander invariant for $(\varphi_\Gamma, \varphi)$ by

$$\overrightarrow{\delta}^\psi_\Gamma(G) = \dim_{\mathcal{K}_{\Gamma^\psi}}(H_1(G; \mathcal{K}_{\Gamma^\psi}[t^\pm])) \in \mathbf{Z}_{\geq 0} \cup \{\infty\}$$

$\overrightarrow{\delta}^\psi_\Gamma(G)$ is also called the Γ-*degree*[1]. Note that the right $\mathcal{K}_{\Gamma^\psi}[t^\pm]$-module $H_1(G; \mathcal{K}_{\Gamma^\psi}[t^\pm])$ are decomposed into

$$H_1(G; \mathcal{K}_{\Gamma^\psi}[t^\pm]) = (\mathcal{K}_{\Gamma^\psi}[t^\pm])^r \oplus \left(\bigoplus_{i=1}^{l} \frac{\mathcal{K}_{\Gamma^\psi}[t^\pm]}{p_i(t)\mathcal{K}_{\Gamma^\psi}[t^\pm]} \right)$$

[1] Our definition is slightly different from that in [6].

for some $r \in \mathbf{Z}_{\geq 0}$ and $p_i(t) \in \mathcal{K}_{\Gamma\psi}[t^{\pm}]$, and then

$$\overline{\delta}_\Gamma^\psi(G) = \begin{cases} \sum_{i=1}^{l} \deg^\psi(p_i(t)) & (r = 0), \\ \infty & (r > 0) \end{cases}$$

For a space X and an admissible pair for $\pi_1 X$, we define $\overline{\delta}_\Gamma^\psi(X) := \overline{\delta}_\Gamma^\psi(\pi_1 X)$.

For a finitely presentable group G and an admissible pair $(\varphi_\Gamma, \varphi)$ for G. The Γ-degree can be computed from any presentation matrix of the right $\mathcal{K}_{\Gamma\psi}[t^{\pm}]$-module $H_1(G; \mathcal{K}_{\Gamma\psi}[t^{\pm}])$. Therefore we can consider it to be a $\mathbf{Z}_{\geq 0}$-valued function on the set $M(\mathcal{K}_{\Gamma\psi}[t^{\pm}])$ of all matrices with entries in $\mathcal{K}_{\Gamma\psi}[t^{\pm}]$. In [14] (see also [16]), we extended this function to

$$\overline{d}_\Gamma^\psi : M(\mathcal{K}_\Gamma) \to \mathbf{Z} \cup \{\infty\}$$

called the (*truncated*) *torsion-degree function* by using Reidemeister torsions and the Dieudonné determinant $\det : GL(\mathcal{K}_\Gamma) \to (\mathcal{K}_\Gamma^\times)_{\text{ab}}$, where $(\mathcal{K}_\Gamma^\times)_{\text{ab}}$ is the abelianization of the multiplicative group $\mathcal{K}_\Gamma^\times = \mathcal{K}_\Gamma - \{0\}$. The torsion-degree function is defined for each pair of a PTFA group Γ and an epimorphism $\psi : \Gamma \twoheadrightarrow \mathbf{Z}$. It can be regarded as a generalization of the extension of $\deg^\psi : \mathcal{K}_{\Gamma\psi}[t^{\pm}] \to \mathbf{Z}_{\geq 0} \cup \{\infty\}$ to $\deg^\psi : \mathcal{K}_\Gamma \to \mathbf{Z} \cup \{\infty\}$ by setting $\deg^\psi(fg^{-1}) = \deg^\psi(f) - \deg^\psi(g)$ for $f \in \mathbf{Z}\Gamma, g \in \mathbf{Z}\Gamma - \{0\}$ (see Proposition 9.1.1 in [2], for example). It induces a group homomorphism $\deg^\psi : (\mathcal{K}_\Gamma^\times)_{\text{ab}} \to \mathbf{Z}$.

Torsion-degree functions have the following properties.

Proposition 5.1. (1) *For $A \in GL(\mathcal{K}_\Gamma)$, we have $\overline{d}_\Gamma^\psi(A) = \deg^\psi(\det A)$. In particular, $\overline{d}_\Gamma^\psi(A) = 0$ for any $A \in GL(\mathcal{K}_{\Gamma\psi}[t^{\pm}])$.*
(2) *Let M be a finitely generated right $\mathcal{K}_{\Gamma\psi}[t^{\pm}]$-module presented by a matrix $A \in M(\mathcal{K}_{\Gamma\psi}[t^{\pm}])$. Then*

$$\overline{d}_\Gamma^\psi(A) = \begin{cases} \dim_{\mathcal{K}_{\Gamma\psi}}(T_{\mathcal{K}_{\Gamma\psi}[t^{\pm}]} M) & (\operatorname{rank}_{\mathcal{K}_{\Gamma\psi}[t^{\pm}]}(F_{\mathcal{K}_{\Gamma\psi}[t^{\pm}]} M) \leq 1) \\ \infty & (\text{otherwise}) \end{cases},$$

where $T_{\mathcal{K}_{\Gamma\psi}[t^{\pm}]} M$ (resp. $F_{\mathcal{K}_{\Gamma\psi}[t^{\pm}]} M$) is the $\mathcal{K}_{\Gamma\psi}[t^{\pm}]$-torsion (resp. $\mathcal{K}_{\Gamma\psi}[t^{\pm}]$-free) part of M.

Let G be a finitely presentable group and we take a presentation $\langle x_1, \ldots, x_l \mid r_1, \ldots, r_m \rangle$ of G. For each admissible pair $(\varphi_\Gamma, \varphi)$ for G, the Jacobi matrix $A := \overline{\left(\dfrac{\partial r_j}{\partial x_i}\right)}^{\varphi_\Gamma}_{\substack{1 \leq i \leq l \\ 1 \leq j \leq m}}$ at $\mathcal{K}_{\Gamma\psi}[t^{\pm}]$ gives a presentation matrix of $H_1(G, \{1\}; \mathcal{K}_{\Gamma\psi}[t^{\pm}])$. Then the Γ-degree is given by

$$\overline{\delta}_\Gamma^\psi(G) = \dim_{\mathcal{K}_{\Gamma\psi}}(H_1(G; \mathcal{K}_{\Gamma\psi}[t^{\pm}])) = \overline{d}_\Gamma^\psi(A),$$

where the second equality follows from the direct sum decomposition

$$H_1(G, \{1\}; \mathcal{K}_{\Gamma\psi}[t^{\pm}]) \cong H_1(G; \mathcal{K}_{\Gamma\psi}[t^{\pm}]) \oplus \mathcal{K}_{\Gamma\psi}[t^{\pm}]$$

given by Harvey in [5].

6. Applications of torsion-degree functions to homology cylinders

In this section, we study some invariants of homology cylinders arising from the Magnus representation, twisted homology groups of related manifolds and torsion-degree functions. In [14], we can see other applications.

6.1. **Torsion-degrees of Magnus matrices.** First, we consider torsion-degree functions associated to nilpotent quotients N_k of $\pi_1\Sigma_{g,1}$, and apply them to Magnus matrices. Since $H_1(N_k) = H_1(N_2) = H_1(\Sigma_{g,1})$ and $H^1(N_k) = H^1(N_2) = H^1(\Sigma_{g,1})$, taking an epimorphism $N_k \twoheadrightarrow \mathbf{Z}$, which is needed in the definition of a torsion-degree function, is done by choosing a primitive element of $H^1(\Sigma_{g,1})$.

Theorem 6.1. *Let M be a homology cylinder. For any $k \geq 2$ and any primitive element $\psi \in H^1(\Sigma_{g,1})$, the torsion-degree $\overrightarrow{d}^{\psi}_{N_k}(r_k(M))$ is always zero.*

Proof. Proposition 5.1 (1) shows that torsion-degrees are additive for products of invertible matrices and vanish for those in $GL(\mathbf{Z}N_k)$. It can be also checked that they are invariant under taking the transpose and operating the involution. Hence, by applying the torsion-degree function to the equality $\overline{r_k(M)^T}\, \overline{J} r_k(M) = {}^{\sigma_k(M)}\overline{J}$ in Theorem 3.6, we obtain $2\overrightarrow{d}^{\psi}_{N_k}(r_k(M)) = 0$. This completes the proof. □

Example 6.2. Consider the homology cylinder M_L in Example 4.3. $\overrightarrow{d}^{\psi}_{N_2}(r_2(M_L))$ is given by the degree of $\det r_2(M_L) = \frac{\gamma_3+\gamma_4-1}{\gamma_3\gamma_4(\gamma_3^{-1}+\gamma_4^{-1}-1)}$ with respect to ψ. It can be easily checked that it is zero.

Remark 6.3. In [14], we defined the Magnus representation $r_k : \mathrm{Aut}F_n^{\mathrm{acy}} \to GL(n, \mathcal{K}_{N_k(F_n)})$ for $\mathrm{Aut}F_n^{\mathrm{acy}}$, where F_n^{acy} is a completion of F_n in a certain sense and is called the *acyclic closure* of F_n. The natural map $F_n \to F_n^{\mathrm{acy}}$ is known to be injective and 2-connected. In particular, $N_k(F_n) = N_k(F_n^{\mathrm{acy}})$. $\mathrm{Aut}F_n^{\mathrm{acy}}$ can be regarded as an enlargement of $\mathrm{Aut}F_n$, and we have the enlarged Dehn-Nielsen homomorphism $\sigma^{\mathrm{acy}} : \mathcal{H}_{g,1} \to \mathrm{Aut}F_{2g}^{\mathrm{acy}}$ extending the classical one $\sigma : \mathcal{M}_{g,1} \hookrightarrow \mathrm{Aut}F_{2g}$. (Note that σ^{acy} is not injective.) That is, we have the following commutative diagram.

$$\begin{array}{ccc} \mathrm{Aut}F_{2g} & \hookrightarrow & \mathrm{Aut}F_{2g}^{\mathrm{acy}} \\ \uparrow \sigma & & \uparrow \sigma^{\mathrm{acy}} \\ \mathcal{M}_{g,1} & \hookrightarrow & \mathcal{H}_{g,1} \end{array}$$

The Magnus representation for homology cylinders is nothing other than the composite $\mathcal{H}_{g,1} \xrightarrow{\sigma^{\mathrm{acy}}} \mathrm{Aut}F_{2g}^{\mathrm{acy}} \xrightarrow{r_k} GL(2g, \mathcal{K}_{N_k})$. We can easily check that $\overrightarrow{d}^{\psi}_{N_k} \circ r_k : \mathrm{Aut}F_{2g}^{\mathrm{acy}} \xrightarrow{r_k} GL(2g, \mathcal{K}_{N_k})$ is non-trivial. Therefore $\overrightarrow{d}^{\psi}_{N_k} \circ r_k$ gives an invariant of $\mathrm{Aut}F_n^{\mathrm{acy}}$ which vanishes on $\mathcal{M}_{g,1}$, $\mathrm{Aut}F_n$ and $\mathcal{H}_{g,1}$ for each $k \geq 2$ and each primitive element $\psi \in H^1(N_k)$.

6.2. **Factorization formula of $N_{k,T}$-degree for the mapping torus of a homology cylinder.** For each homology cylinder $M = (M, i_+, i_-)$, we can construct a closed 3-manifold T_M as follows. First we attach a 2-handle $I \times D^2$ along $I \times i_\pm(\partial\Sigma_{g,1})$, so that we obtain a homology cylinder (M', i'_+, i'_-) over a closed surface Σ_g, which corresponds to the embedding $\Sigma_{g,1} \hookrightarrow \Sigma_g$. Then we put

$$T_M := M'/(i'_+(x) = i'_-(x)), \quad x \in \Sigma_g.$$

We call T_M the *mapping torus* of M. Indeed, for $M_\varphi \in \mathcal{M}_{g,1} \subset \mathcal{C}_{g,1}$, the resulting manifold T_{M_φ} is nothing other than the usual mapping torus of φ extended naturally to the mapping class of Σ_g. If $M \in \mathcal{C}_{g,1}[k]$, we have natural isomorphisms $N_k(\Sigma_g) \cong N_k(M')$ and $N_k(T_M) \cong N_k(\Sigma_g) \times \langle \lambda \rangle$.

Note that these groups are torsion-free nilpotent (hence PTFA). We consider $N_k(\Sigma_g)$ to be a subgroup of $N_k(T_M)$. For simplicity, we denote $N_k(T_M)$ by $N_{k,T}$.

By an argument similar to that in Lemma 3.1, we can show that $H_*(M, i_+(\Sigma_{g,1}); \mathcal{K}_{N_{k,T}}) = 0$. Hence we can define the Reidemeister torsion

$$\tau_{N_{k,T}}(M) := \tau(C_*(M, i_+(\Sigma_{g,1}); \mathcal{K}_{N_{k,T}})) \in K_1(\mathcal{K}_{N_{k,T}})/(\pm N_{k,T}).$$

(See [10], [19] for generalities of Reidemeister torsions) Then we obtain the following factorization formula of $N_{k,T}$-degree for the mapping torus of a homology cylinder.

Theorem 6.4 ([14, Theorem 11.6]). *Let M be a homology cylinder belonging to $C_{g,1}[k]$.*
(1) *For each primitive element $\psi \in H^1(N_{k,T}) = H^1(T_M)$, the $N_{k,T}$-degree $\overline{\delta}^{\psi}_{N_{k,T}}(T_M)$ is finite.*
(2) *We have the equality*

$$\overline{\delta}^{\psi}_{N_{k,T}}(T_M) = \overline{d}^{\psi}_{N_{k,T}}(\tau_{N_{k,T}}(M)) + \overline{d}^{\psi}_{N_{k,T}}\left(\lambda I_{2g} - \overline{r_{k,T}(M)^T}\right) - 2|\psi(\lambda)|,$$

where $r_{k,T} : \mathcal{H}_{g,1} \to GL(2g, \mathcal{K}_{N_{k,T}})$ is defined similarly to the Magnus representation r_k.

Remark 6.5 (The case of $k = 2$). Since $\mathbb{Z}N_{2,T} = \mathbb{Z}N_2(T_M)$ and $\mathcal{K}_{N_{2,T}} = \mathcal{K}_{N_2(T_M)}$ are commutative, we can use the ordinary determinant to calculate the invariants seen above. For $M \in C_{g,1}[2]$, we write Δ_{T_M} for the Alexander polynomial of T_M. By a straightforward computation, we have

$$\Delta_{T_M} \doteq \overline{\tau_{N_{2,T}}(M)} \cdot \det\left(\lambda I_{2g} - \overline{r_{2,T}(M)^T}\right) \cdot (1-\lambda)^{-2},$$

where $\doteq$ means that these equalities hold in $\mathcal{K}_{N_2(T_M)}$ up to $\pm N_2(T_M)$.

7. Acknowledgement

The author would like to express his gratitude to Professor Shigeyuki Morita for his encouragement and helpful suggestions. He also would like to thank Professor Masaaki Suzuki for valuable discussions and advice.

This research was partially supported by the 21-century COE program at Graduate School of Mathematical Sciences, the University of Tokyo, and by JSPS Research Fellowships for Young Scientists.

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Takuya SAKASAI

Graduate School of Mathematical Sciences, the University of Tokyo,

3-8-1 Komaba, Meguro-ku, Tokyo 153-8914, Japan

sakasai@ms.u-tokyo.ac.jp