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Fiber spaces with solv-geometry: Preliminary report

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1. Introduction

The theory of singular fiber bundles with typical fiber a k-torus T^k has been systematically studied by Conner and Raymond in the 1970's [**6**]. It provides a topological generalization of 3-dimensional Seifert manifolds, and it is called the injective Seifert fiber space construction. This article concerns the structure theory of singular fiber bundles with typical fiber a manifold with a geometry which is locally modelled on a solvable Lie group.

As is the case for Seifert fiber spaces, the structure of a fiber space with solv-geometry, facilitates the construction of diffeomorphisms with prescribed homotopical properties, by starting the construction on the base and subsequent lifting along the fibers. Along these lines we provide rigidity results which reduce the diffeomorphism classification of fiber spaces with solv-geometry to the smooth rigidity properties of their base spaces.

Our main application concerns the smooth rigidity of compact aspherical homogeneous manifolds. We show that these manifolds carry the structure of a singular fiber space with solv-geometry, over a base space which is a non-positively curved locally symmetric orbifold.

CONTENTS

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2. Manifolds with solv-geometry

Let $\mathcal{R}$ be a connected, simply connected solvable Lie group. The semidirect product $\mathrm{Aff}(\mathcal{R}) = \mathcal{R} \rtimes \mathrm{Aut}(\mathcal{R})$, where $\mathrm{Aut}(\mathcal{R})$ is the group of automorphisms of $\mathcal{R}$, is said to be the *affine group* of $\mathcal{R}$. The projection homomorphism $hol : \mathrm{Aff}(\mathcal{R}) \to \mathrm{Aut}(\mathcal{R})$ with respect to the above splitting is called the *holonomy homomorphism*. By letting $\mathcal{R}$ act on itself by left-multiplication, we identify the affine group $\mathrm{Aff}(\mathcal{R})$ with a group of transformations on $\mathcal{R}$.

2.1. Definition. We say that a smooth manifold has a *solv-geometry of type* $\mathcal{R}$ if it can be presented in the form $H\backslash\mathcal{R}$, where H is a torsion-free *virtually solvable* subgroup of $\mathrm{Aff}(\mathcal{R})$ which acts properly on $\mathcal{R}$. Since $\mathcal{R}$ is diffeomorphic to Euclidean space, manifolds with solv-geometry are topologically, smooth aspherical manifolds with universal covering space $\mathbb{R}^n$. Moreover, they are endowed with an affine geometry modelled on $\mathcal{R}$. The particular features of the geometry on $H\backslash\mathcal{R}$ are determined by the restriction of the holonomy homomorphism to H.

2.2. Infra-solvmanifolds. A manifold $H\backslash\mathcal{R}$ is called an *infra-solvmanifold* if the closure of the holonomy $hol(H)$ in $\mathrm{Aut}(\mathcal{R})$ is compact. Infra-solvmanifolds are manifolds with solv-geometry, in particular, H is virtually solvable. Moreover, infra-solvmanifolds carry a natural Riemannian geometry, see [**4, 7, 16, 24, 27**] for further reference.

If $\mathcal{R}$ is isomorphic to the vector space $\mathbb{R}^n$, a simply connected nilpotent Lie group $\mathcal{N}$, respectively, and H is discrete, the infra-solv manifold $H\backslash\mathcal{R}$ is customarily called an Euclidean space form, or an *infra-nilmanifold*, respectively. By classical results of Killing and Hopf, Bieberbach for the Euclidean case (see [**28**]), and results of Gromov [**12**] for the infra-nil case, these *smooth* manifolds may be characterised in terms of curvature properties of their Riemannian connections, as well.

The geometric structure of compact Euclidean space forms, and infra-nil manifolds, has been determined by Bieberbach [**5**], and Auslander [**1**]. In particular, compact Euclidean space forms are finitely and affinely covered by the n-torus, and infra-nilmanifolds $H\backslash\mathcal{N}$ are finitely affinely covered by an n-dimensional nilmanifold $\mathcal{N}/(H \cap \mathcal{N})$. As a matter of fact, the holonomy $hol(H)$ of the presentation $H\backslash\mathcal{N}$ is a finite group, in these cases.

For a general infra-solv manifold, the situation is more complicated than in the infra-nil case. It is known, however, that every infra-solvmanifold is finitely covered by a solvmanifold. That is, it is covered by a homogeneous space of a solvable Lie group. A geometric characterisation of infra-solvmanifolds up to *homeomorphism* is described in [**24**].

Note furthermore, that, as is proved in [**4**], any manifold with solv-geometry $H\backslash\mathcal{R}$ is diffeomorphic to an infra-solvmanifold if $hol(H)$ is contained in a reductive subgroup of $\mathrm{Aut}(\mathcal{R})$, or if $\mathcal{R}$ is nilpotent.

2.3. Smooth Rigidity properties. As implied by a result of Bieberbach [**5**] in 1912, any two homotopic compact Euclidean space forms are affinely diffeomorphic. Now, a corresponding strong rigidity result also holds for infra-nilmanifolds, see [**2, 14**]. Namely, homotopic infra-nil manifolds $H\backslash\mathcal{N}$ and $H'\backslash\mathcal{N}'$, where H and H' are discrete, are affinely diffeomorphic with respect to the canonical bi-invariant affine connections determined by $\mathcal{N}$ and $\mathcal{N}'$. Note, in particular, that in the context of infra-nilmanifolds, the fundamental group H already determines the Lie group $\mathcal{N}$.

One can not expect, in general, to have structure preserving affine diffeomorphisms for homotopic manifolds with solv-geometry. However, weaker analogies of these results survive. In fact, every homotopy equivalence of compact manifolds with solv-geometry $H\backslash\mathcal{R}$ and $H'\backslash\mathcal{R}'$ is induced by a diffeomorphism, provided $hol(H)$ is contained in a reductive subgroup of $\mathrm{Aut}(\mathcal{R})$, or if $\mathcal{R}$ is nilpotent, see [**4**]. In the case of infra-solv manifolds with discrete presentations, the corresponding diffeomorphism may be chosen to be an isometry with respect to suitable left-invariant Riemannian metrics on $\mathcal{R}$ and $\mathcal{R}'$, see [**27**]. Note that the smooth rigidity of infra-solv manifolds, and the (more general) rigidity of manifolds with solv-geometry, provide an extension of Mostow's rigidity result for solvmanifolds [**19**].

3. Fiber structures

3.1. Fiber spaces with $\mathcal{R}$-geometry. Let $\mathcal{R}$ be a simply connected solvable Lie group. Let X be a manifold on which $\mathcal{R}$ acts freely and properly with quotient manifold

$$W = \mathcal{R}\backslash X .$$

We let $\mathsf{p} : X \to W$ denote the projection map of the corresponding principal $\mathcal{R}$-bundle. Moreover, we let $\mathrm{Diff}(X,\mathcal{R})$ denote the normaliser of $\mathcal{R}$ in $\mathrm{Diff}(X)$ and $\mathrm{Diff}^1(X,\mathcal{R})$ the kernel of the $\mathrm{Diff}(X,\mathcal{R})$-action on the quotient W. Given a compatible trivialisation $X = \mathcal{R}\times W$, $\mathrm{Aff}(\mathcal{R})$ acts on X by extending from the first factor, and, in this way, embeds as a subgroup of $\mathrm{Diff}^1(X,\mathcal{R})$. We put $\mathrm{Aff}(\mathcal{R}\times W)$ for this subgroup of $\mathrm{Diff}^1(X,\mathcal{R})$, and call it the affine group of $\mathcal{R}\times W$.

We introduce now our main concept.

3.1.1. *Definition.* Let $\mathsf{p} : X \to W$ be a principal $\mathcal{R}$-bundle. Let $H \leq \mathrm{Diff}(X,\mathcal{R})$ be a Lie group normalising $\mathcal{R}$. We put $\Delta = H \cap \mathrm{Diff}^1(X,\mathcal{R})$ and $\Theta = H/\Delta$. Since $H \leq \mathrm{Diff}(X,\mathcal{R})$, Θ acts on W. We assume that the following conditions are satisfied:

(1) H acts properly on X.
(2) Θ acts properly discontinuously on W.
(3) There exist compatible coordinates $X = \mathcal{R}\times W$ for p such that $\Delta \leq \mathrm{Aff}(\mathcal{R}\times W)$.

DEFINITION 3.1. We call data $(X,\mathcal{R},H)$ as above which satisfy (1), (2), (3) a *fiber space with $\mathcal{R}$-geometry.*

To every fiber space with $\mathcal{R}$-geometry, there is associated a singular fibration of the form

$$\Delta\backslash\mathcal{R} \longrightarrow X/H \xrightarrow{\mathsf{q}} W/\Theta \tag{3.1}$$

and an associated group extension

$$1 \to \Delta \to H \to \Theta \to 1 . \tag{3.2}$$

Accordingly, we will also call the map $\mathsf{q} : X/H \to W/\Theta$ a fiber space with $\mathcal{R}$-geometry. If $\Delta\backslash\mathcal{R}$ is compact we say that $(X,\mathcal{R},H)$ has compact fibers.

3.1.2. *Fiber types.* Consider the finite group $\Theta_w \leq \Theta$ which is the stabiliser of $w \in W$. Accordingly, we can distinguish two principal fiber types for the fiber space $\mathsf{q} : X/H \to W/\Theta$:

Non-singular fibers: These are the fibers $\bar{F}_{\bar{w}}$ over points $\bar{w} \in \Theta\backslash W$ with $\Theta_w = \{1\}$. In this case, $H_w = \Delta_w$ and $\bar{F}_{\bar{w}}$ identifies with $\Delta_w\backslash\mathcal{R}_w$.

Singular fibers: These are the fibers, where $\Theta_w \neq \{1\}$. Then $H_w \leq H$ is a finite extension group of Δ which projects onto Θ_w. The singular fiber $\bar{F}_{\bar{w}}$ identifies with $H_w\backslash\mathcal{R}_w$.

Various situations may occur. If Θ is torsion-free then W/Θ is a manifold, all fibers are non-singular, and (3.1) is a differentiable locally trivial fibration with fiber $\Delta\backslash\mathcal{R}$.

3.1.3. *Affine geometry on the fibers.* Since $H \leq \mathrm{Diff}(X, \mathcal{R})$, the action of H preserves the affine structure on the fibres of $\mathsf{p} : X \to W$. Hence, the fibres of

$$\mathsf{q} : X/H \to W/\Theta$$

inherit an affine geometry modelled on $\mathcal{R}$. In fact, H_w acts affinely on F_w, and restricting H_w to F_w defines a homomorphism $H_w \to \mathrm{Aff}(F_w)$. In particular, the fibers $\bar{F}_{\bar{w}} = F_w/H_w$ of q are spaces with $\mathcal{R}$-geometry. The geometry on $\bar{F}_{\bar{w}}$ is determined by the induced *local holonomy homomorphism*

$$hol_w : H_w \longrightarrow \mathrm{Aff}(F_w)/\mathcal{R}_w \cong \mathrm{Aut}(\mathcal{R}) .$$

We remark further that, by condition (3) of Definition 3.1, *the fiber-stabilising group Δ naturally identifies with a subgroup of* $\mathrm{Aff}(\mathcal{R})$, and this embedding determines the geometry of the generic fibers of q completely.

DEFINITION 3.2. Assume that $\mathsf{q} : X/H \to W/\Theta$ is a fiber space of type $\mathcal{R}$. If in addition to (1) - (3) the condition

(4) The closure of the local holonomy groups $hol_w(H_w) \leq \mathrm{Aff}(F_w)/\mathcal{R}_w$ is compact, for all $w \in W$,

is satisfied, then q is called an *infra-solv fiber space* with fiber modelled on $\mathcal{R}$.

Note that the condition (4) is satisfied if and only if $\Delta\backslash\mathcal{R}$ is an infra-solv manifold. Another important special case arises if the holonomy of Δ is trivial:

DEFINITION 3.3. Assume that Δ is contained in $\mathcal{R}$ and that H is a discrete group. Then $\mathsf{q} : M = X/H \to W/\Theta$ is called a *Seifert bundle* with $\mathcal{R}$-fiber.

3.2. Standard actions and standard fiber spaces. Let U denote a simply connected nilpotent Lie group. Let $\Delta \leq \mathrm{Aff}(U)$ be a subgroup which acts properly on U with compact quotient. Then Δ is called *standard* if $\Delta \leq UT$, where $T \leq \mathrm{Aut}(U)$ is a (split) d-subgroup of the linear algebraic group $\mathrm{Aut}(U)$. An action $\rho : \Gamma \to \mathrm{Aff}(U)$ is said to be *standard* if it is an effective properly discontinuous action such that $\rho(\Gamma)$ is standard.

We can associate to every solvable affine action on a simply connected nilpotent Lie group U a unique standard action. In fact, as proved in [**4**, Theorem 1.2], standard Γ-actions on U are unique up to conjugacy in $\mathrm{Aff}(U)$. Moreover, U is uniquely determined by Γ, see [**4**, §3.2].

DEFINITION 3.4. Let (X, U, H) be a fiber space, where U is a simply connected nilpotent Lie group. We call (X, U, H) a *standard fiber space* if the affine action of $\Delta = H \cap \mathrm{Diff}^1(X, U)$ on U is standard. Let $(X, \tilde{H}, \rho)$ be a proper action, and let $H = \rho(\tilde{H})$. Then $(X, \tilde{H}, \rho)$ is called a *standard action* if (X, U, H) is a standard fiber space. In addition, if $\Delta = \Gamma$ is a discrete group then the standard fiber space (X, U, H) is called a *standard* Γ*- fiber space.*

3.2.1. *Coordinate expression of group actions and* T*-compatible maps.* Let $\tau \in \mathrm{Diff}(X)$ be a diffeomorphism which preserves the fibers of the bundle projection $\mathsf{p} : X \to W$. We let $\bar{\tau} : W \to W$ denote the induced diffeomorphism of W. Compatible coordinates $X = \mathcal{R} \times W$ determine a family of diffeomorphisms $\psi_{\tau,w} \in \mathrm{Diff}(\mathcal{R})$ such that (with respect to the coordinates) the action of τ on X is expressed as

$$\tau\,(r, w) = (\psi_{\tau,w}\, r, \bar{\tau} w) \ . \tag{3.3}$$

Let T be a maximal torus in the Zariski-closure of the adjoint image of $\mathcal{R}$ in $\mathrm{Aut}(\mathcal{R})$. We let $\mathrm{Aff}(\mathcal{R}, T)$ denote the subgroup of elements in $\mathrm{Aff}(\mathcal{R})$ whose linear parts stabilise T. A diffeomorphism $\tau \in \mathrm{Diff}(X, \mathcal{R})$ is called a T*-compatible map* if $\psi_{\tau,w} \in \mathrm{Aff}(\mathcal{R}, T)$, for all $w \in W$. It follows that the T-compatible maps of $\mathrm{Diff}(\mathcal{R}, X)$ form a subgroup $\mathrm{Diff}(X, \mathcal{R}, T)$.

We show in [**3**] that the equivalence classification of certain fiber spaces with solvable geometry reduces to the classification of standard fiber spaces.

THEOREM 3.5. *Let* $(X, \mathcal{R}, H)$ *be a fiber space with compact fibers such that* $H \leq \mathrm{Diff}(X, \mathcal{R}, T)$ *and the fiber stabilising group* $\Delta = H \cap \mathrm{Diff}^1(X, \mathcal{R})$ *is virtually solvable. Then* $(X, \mathcal{R}, H)$ *is equivalent to the standard fiber space* $(Y, \mathbf{U}, H')$*, where* $\mathbf{U}$ *is the unipotent shadow of* Δ *and* $H' \leq \mathrm{Diff}(Y, \mathbf{U}, T)$ *is isomorphic to* H*.*

See [**4**], for the definition of the unipotent shadow.

The next result shows that the diffeomorphism classification of standard fiber spaces reduces to the corresponding classification of standard Γ- spaces: Let (X, U, H) be a standard fiber space, where U is a simply connected nilpotent Lie group. As usual we let $\Delta = H \cap \mathrm{Diff}^1(X, U)$ denote the fiber preserving subgroup of H. Let Δ^0 denote the connected component of Δ and U_0 the unipotent radical of the Zariski-closure $\overline{\Delta^0}$.

PROPOSITION 3.6. *Let* (X, U, H) *be a standard fiber space, and let* U_0 *be the unipotent radical of the Zariski-closure* $\overline{\Delta^0}$*. Then the following hold:*

1. *The action of* H *on* X *decends to an action of* H/Δ^0 *on* $(X/U_0, U/U_0)$ *which has fiber stabilising group* $\Gamma = \Delta/\Delta^0$*.*
2. *The fiber space* $(X/U_0, U/U_0, H/\Delta_0)$ *is standard.*
3. *The natural map* $X \to X/U_0$ *defines a diffeomorphism of fiber spaces*
$$(X, U, H) \to (X/U_0, U/U_0, H/\Delta_0) \ .$$
4. *If* H *acts freely on* X *then* H/Δ^0 *acts effectively on* X*, and the fiber space* $(X/U_0, U/U_0, H/\Delta^0)$ *is a standard* Γ*- fiber space.*

4. Iterated Seifert fibering

Let (X, U, π) be a standard fiber space, with discrete fiber stabilising group Γ. We let $\rho : \pi \to \mathrm{Diff}(X)$ denote the action of π on X. Recall that the action of Γ on

U is standard, and, hence, the Fitting hull $F = F_{\rho(\Gamma)}$, ie., the hull of the maximal normal nilpotent subgroup $\mathrm{Fitt}(\rho(\Gamma))$ of $\rho(\Gamma)$, is a connected normal subgroup of U. Note that the vector space $V = U/F$ acts on X/F. Let us furthermore put $W = X/F$ and $\Theta = \pi/\Gamma$.

4.1. Induced Seifert fiberings.

LEMMA 4.1. *If (X, F) is the principal bundle defined by the subgroup $F \subset U$, then there is an induced quotient principal bundle $(X/F, V)$ such that the following hold:*

(i) *The action of π normalises F.*
(ii) *The quotient action of π on X/F normalises V.*
(iii) *There is an induced properly discontinuous action $(X/F, \pi/\mathrm{Fitt}(\Gamma), \hat{\rho})$.*

Moreover, we show in [**3**]:

PROPOSITION 4.2.

i) *The actions $(X/F, V, \pi/\mathrm{Fitt}(\Gamma))$ define a standard fiber space which is Seifert.*
ii) *The actions (X, F, π) define a Seifert fiber space.*

We obtain the following equivariant commutative (and exact) diagram of Seifert actions:

$$\begin{tikzcd}
(F, \mathrm{Fitt}(\Gamma)) \arrow[r, equal] \arrow[d] & (F, \mathrm{Fitt}(\Gamma)) \arrow[d] & \\
(U, \Gamma) \arrow[r] \arrow[d] & (X, \pi, \rho) \arrow[r] \arrow[d] & (W, \Theta) \arrow[d, equal] \\
(V, \Gamma/\mathrm{Fitt}(\Gamma)) \arrow[r] & (X/F, \pi/\mathrm{Fitt}(\Gamma), \hat{\rho}) \arrow[r] & (W, \Theta).
\end{tikzcd} \tag{4.1}$$

4.1.1. *Seifert fiberings.* We briefly recall the definition of Seifert fiber spaces. Let (X, N) be a principal N-bundle, where N is a connected simply connected Lie group. Let π be a subgroup of $\mathrm{Diff}(X, N)$ which acts properly discontinuously on X.

DEFINITION 4.3. Actions (X, N, π) as above are said to define a Seifert fiber space if they satisfy the following conditions

1. $\pi_N = N \cap \pi$ is a discrete uniform subgroup of N.
2. $\Theta_N = \pi/\pi_N$ acts properly discontinuously on $W = X/N$.
3. (X, N) admits a trivialisation $X = N \times W$.

Let (X, π, ρ) be a properly discontinuous action on X. Then the actions (X, N, π) are called a *Seifert action* if $(X, N, \rho(\pi))$ defines a Seifert fiber space

Remark, if $\pi_N = \Gamma$ then (X, N, π) is also a fiber space with N-geometry in the sense of Definition 3.1. Given two Seifert actions (X, N, π) and (X, N', π'), where N and N' are simply connected nilpotent Lie groups, an isomorphism $\phi : \pi \longrightarrow \pi'$ is called a *compatible isomorphism* with respect to the Seifert actions, if

i) ϕ is a compatible map of actions, that is, $\phi(\ker \rho) = \ker \rho'$ and $\phi(\Gamma) = \Gamma'$.
ii) ϕ respects the Seifert structure, that is, $\phi(\pi_N) = \pi'_N$.

4.2. Application of Seifert rigidity. We now arrive at the smooth rigidity of standard fiber spaces (X, U, π):

THEOREM 4.4. *Let $\phi : \pi \to \pi'$ be a compatible isomorphism. Then every equivariant diffeomorphism $(\bar{f}, \bar{\phi}) : (X/U, \Theta) \to (X/U', \Theta')$ lifts to an equivalence of fiber spaces*

$$(f, \phi) : (X, U, \pi) \to (X', U', \pi') .$$

Therefore, if (W, Θ) is smoothly rigid then (X, π) is smoothly rigid.

PROOF. **First step.** There are induced isomorphisms of groups $\hat{\phi} : \pi/\mathrm{Fitt}(\Gamma) \to \pi/\mathrm{Fitt}(\Gamma')$ and $\bar{\phi} : \Theta \to \Theta'$, and a commutative diagram as follows:

(4.2)
$$\begin{tikzcd}
\pi \arrow[r, "\phi"] \arrow[d] & \pi' \arrow[d] \\
\pi/\mathrm{Fitt}(\Gamma) \arrow[r, "\hat{\phi}"] \arrow[d] & \pi'/\mathrm{Fitt}(\Gamma') \arrow[d] \\
\Theta \arrow[r, "\bar{\phi}"] & \Theta'
\end{tikzcd}$$

The compatibility of ϕ means that

$$\phi(\Gamma) = \Gamma' \text{ and } \phi(\ker \rho) = \ker \rho' ,$$

where $\rho : \pi \to \mathrm{Diff}(X)$ and $\rho' : \pi' :\to \mathrm{Diff}(X')$ denote the actions of π and π'. Now the induced isomorphisms are compatible with the iterated Seifert fiber space structure. We need:

LEMMA 4.5.

(i) *The isomorphism ϕ is compatible with respect to the Seifert actions (X, F, π) and (X', F', π').*
(ii) *The isomorphism $\hat{\phi}$ is compatible with the Seifert actions $(X/F, V, \pi/\mathrm{Fitt}(\Gamma))$ and $(X'/F', V', \pi'/\mathrm{Fitt}(\Gamma'))$.*

Second step. Using the Seifert lifting for nilpotent fiber (cf. [**14**]), we can construct the lift (f, ϕ) of $(\bar{f}, \bar{\phi})$ subsequently along the vertical Seifert fiberings, as in the following diagram:

$$\begin{tikzcd}
(X, F, \pi) \arrow[r, "(f,\phi)"] \arrow[d] & (X, F, \pi') \arrow[d] \\
(X/F, V, \pi/\mathrm{Fitt}(\Gamma)) \arrow[r, "(\hat{f},\hat{\phi})"] \arrow[d] & (X'/F', V', \pi'/\mathrm{Fitt}(\Gamma') \arrow[d] \\
(W, \Theta) \arrow[r, "(\bar{f},\bar{\phi})"] & (W, \Theta') .
\end{tikzcd}$$

$\square$

5. Aspherical homogeneous manifolds

A manifold is called homogeneous if it has a transitive action of a Lie group, and it is called aspherical if its universal covering space is contractible. We show, cf. Theorem 5.8, that every aspherical homogeneous manifold carries the structure of a singular fiber space with solv-geometry on the fibers, over a base space which is a non-positively curved locally symmetric orbifold. As a consequence, see Theorem 5.13, we establish that every isomorphism between fundamental groups of aspherical homogeneous manifolds is induced by a diffeomorphism. That is, aspherical homogeneous manifolds are smoothly rigid. This extends Mostow's well known rigidity result for solvmanifolds [**19**] .

5.1. Presentations of aspherical homogeneous spaces. A manifold M together with a presentation $M = G/H$ is called a homogeneous space. The homogeneous space $M = G/H$ is said to be *irreducible* if G does not contain a proper subgroup which acts transitively on M. It is called *locally effective* if H^0 does not contain any non-trivial connected normal subgroup of G.

Let us put $X = G/H^0$. Now G acts transitively on the homogeneous space X by left-multiplication. Note also that the subgroup $\mathrm{N}_G(H^0)$ of G acts on X by multiplication from the right. We call the group of transformations of X which is generated by those two actions the automorphism group $\mathrm{Aut}(X)$ of the homogeneous space X. In this way, we obtain a natural projection homomorphism

$$G \times \mathrm{N}_G(H^0) \longrightarrow \mathrm{Aut}(X)$$

onto an effective transformation group of X. Moreover, by the inclusion $H \hookrightarrow \mathrm{N}_G(H_0)$, the group $\pi = H/H^0$ embeds as a discrete subgroup of $\mathrm{Aut}(X)$ which acts properly on X, and we have a natural diffeomorphism

$$G/H = X/\pi .$$

The following has been observed by Gorbatsevich, cf. [**8**]:

PROPOSITION 5.1. *Let G be a simply connected Lie group and $H \leq G$ a closed subgroup such that $M = G/H$ is compact and aspherical. If G acts locally effectively on M then:*

1. *H^0 is solvable.*
2. *The solvable radical R of G is simply connected.*
3. *G is isomorphic to a semi-direct product of Lie groups $R \rtimes (\widetilde{\mathrm{SL}_2\mathbb{R}})^n$.*

PROOF. For (1), see [**8**, Theorem 3.1]. We prove (2). Let $G = R\tilde{S}$ be a Levi decomposition of G. Thus R is the solvable radical of G, $\tilde{S}$ is a maximal semi-simple subgroup, and $R \cap \tilde{S}$ is discrete. Recall that $\pi_2(G) = 1$ for any Lie group G (cf. [**13**]). Thus the homotopy exact sequence of the Lie-group fibration $R \to G \to G/R$ gives rise to the short exact sequence

$$1 \to \pi_1(R) \to \pi_1(G) \to \pi_1(G/R) \to 1 .$$

Since G is simply connected it follows that R and G/R are simply connected. As $\tilde{S} \cap R \to \tilde{S} \to G/R$ is a covering, the discrete subgroup $\tilde{S} \cap R$ must be trivial, and $\tilde{S}$ is simply connected. Therefore, in particular, G is a semidirect product $R \rtimes \tilde{S}$. By our assumption, G/H is compact and aspherical. A compact Lie group which acts locally effectively on a closed aspherical manifold is a torus, cf. [**6**]. Hence, the maximal compact subgroup of $\tilde{S}$ is a torus. It follows that each of the simple factors of $\tilde{S}$ is locally isomorphic to $\mathrm{SL}(2,\mathbb{R})$. Therefore, $\tilde{S}$ is isomorphic to $(\widetilde{\mathrm{SL}_2\mathbb{R}})^n$. □

5.2. Structure decomposition of S. Let S be a semi simple factor of G. Choose an enumeration S_i, $i = 1, \dots, n$, for the connected normal, simple subgroups of S. This provides an identification

$$S = S_1 \times \cdots \times S_n = (\widetilde{\mathrm{SL}_2\mathbb{R}})^n$$

which is uniquely defined up to a permutation of the factors. We let

$$\mathsf{p} : S \longrightarrow S^* = S_1^* \times \cdots \times S_n^* = (\mathrm{PSL}_2\,\mathbb{R})^n$$

denote the quotient mapping onto the adjoint form of S. It has kernel $\mathcal{Z}^n$, where $\mathcal{Z}$ denotes the center of $\widetilde{\mathrm{SL}_2\mathbb{R}}$. Furthermore, let $p_i : S \to S_i$ denote the projection map onto the factor S_i, $\mathsf{p}_i : S \to S_i^*$, $p_i^* : S^* \to S_i^*$ the projection maps from S onto the simple factors S_i^* of S^*.

We study direct product decompositions of the form $S = P \times Q$, where P and Q are normal connected subgroups of S. We have a corresponding decomposition $S^* = P^* \times Q^*$ of the adjoint form as well. We let $p_P : S \to P$ and $p_Q : S \to Q$ denote the projection maps corresponding to the decomposition of S. Furthermore, we let $\mathsf{p}_Q : S \to Q^*$ denote the projection map onto the adjoint of Q.

Definition 5.2. Let $H \leq S$ be a uniform subgroup of S. Let $S = P \times Q$ be the unique decomposition of S which satisfies $p_Q(H^0) = \{1\}$, and $p_i(H^0) \neq \{1\}$, for all $i \leq k$. Then P and Q are called the *canonical factors of S* relative to the uniform subgroup H.

Let $G = R \rtimes S$ be as above, $H \leq G$ a uniform subgroup, R the solvable radical of G, and $S \cong (\widetilde{\mathrm{SL}_2\mathbb{R}})^n$ a maximal semi-simple connected subgroup of G. Let $\mathsf{p}_Q : G \to Q^*$ be the natural projection onto Q^*. Then $\ker \mathsf{p}_Q = R \rtimes (P \times \mathcal{Z}^\ell)$. Let furthermore K denote a maximal connected compact subgroup of $\mathrm{SL}(2,\mathbb{R})$ and AN the maximal upper triangular subgroup. Next recall from **[1]** that every subgroup H of a Lie group has a unique maximal normal solvable subgroup $\mathrm{rad}(H)$ which is called the *discrete solvable radical* of H. With this notation we have:

Theorem 5.3 (Structure decomposition). *Let $M = G/H$ be a compact homogeneous space which is locally effective. Then there exists a unique decomposition $S = P \times Q$ of the semi-simple part S of G such that the following hold:*

1. *$\mathsf{p}_Q(H)$ is a discrete uniform subgroup of Q^*.*
2. *H is conjugate to a subgroup of $R \rtimes \big((\mathcal{Z} \times AN)^k \times Q\big)$.*
3. *$H \cap \ker \mathsf{p}_Q$ is the discrete solvable radical $\mathrm{rad}(H)$ of H.*

Corollary 5.4 (Structure fibration). *Let $M = G/H$ be a compact aspherical homogeneous space which is locally effective. Then there exists a closed subgroup $L \leq G$ and a singular fibration of the form*

$$L/\mathrm{rad}(H) \longrightarrow G/H \longrightarrow L\backslash G/H = (\mathbb{H}^2)^\ell/\mathsf{p}_Q(H)\,. \tag{5.1}$$

Here the base space $(\mathbb{H}^2)^\ell/\mathsf{p}_Q(H)$ is an orbifold modelled on a product of copies of the hyperbolic plane, and the non-singular fibers of (5.1) are diffeomorphic to a compact aspherical homogeneous space $L/\mathrm{rad}(H)$ with solvable fundamental group.

PROOF. We put $L = R \rtimes (P \times \tilde{K}^\ell) \leq R \rtimes (P \times Q) = G$. Consider the left-multiplication action of L on G. There is a corresponding L-principal bundle

$$L \longrightarrow G \xrightarrow{\mathsf{q}} L\backslash G = \left(\mathbb{H}^2\right)^\ell . \tag{5.2}$$

Here, right-multiplication turns $L\backslash G$ into a homogeneous G-space, and the map q is a G-homomorphism. The map

$$L\backslash G \to \left(\mathbb{H}^2\right)^\ell = \tilde{K}^\ell\backslash Q = K^\ell\backslash Q^* , \; Lg \mapsto K^\ell \mathsf{p}_{Q(g)} \tag{5.3}$$

is a G-equivariant diffeomorphism, where G acts on $\left(\mathbb{H}^2\right)^\ell$ by isometries via the homomorphism $\mathsf{p}_Q : G \to Q^*$.

Taking the quotient by H gives rise to the fibration (5.1). Since $\mathsf{p}_Q(H)$ is discrete, the base is an orbifold. In particular, for all points $q \in (\mathbb{H}^2)^\ell$, the stabilizer $\mathsf{p}_Q(H)_q$ is finite.

By definition, the image of $\bar{q} \in (\mathbb{H}^2)^\ell/\mathsf{p}(H)$ of $q \in (\mathbb{H}^2)^\ell$ is non-singular if $\mathsf{p}_Q(H)_q = \{1\}$. Thus, for such point the stabilizer of $\mathsf{q}^{-1}(q)$ under H is $\ker \mathsf{p}_Q \cap H = \mathrm{rad}(H)$. Hence, the fiber over $\bar{q}$ in (5.1) is obtained by taking the quotient of L by an action of $\mathrm{rad}(H)$ which is twisted depending on q. Note that, by Theorem 5.3, $\mathrm{rad}(H) = \ker \mathsf{p}_Q \cap H$ is a closed subgroup of L. Over the base point $q = L$, the fiber quotient is diffeomorphic to the coset space $L/\mathrm{rad}(H)$. It is easy to see that the twisted actions of $\mathrm{rad}(H)$ on L are conjugate to the standard action of $\mathrm{rad}(H)$ by right-multiplication in the group $\mathrm{Diff}(L)$. Hence all such fiber quotients are actually diffeomorphic to $L/\mathrm{rad}(H)$. Since L is diffeomorphic to Euclidean space, $L/\mathrm{rad}(H)$ is an aspherical homogeneous space. □

5.3. Solv-geometry on the fibers. We will show now that the non-singular fibers of the *structure fibration* (5.1) of a compact aspherical homogeneous space $M = G/H$ carry a natural (infra-) solv geometry. Actually, as we will show, see Corollary 5.10, this also implies that the non-singular fiber of the structure fibration is diffeomorphic to a homogeneous space of a solvable Lie group.

5.3.1. *Solvable group actions on G.* We keep our notation $G = R \rtimes S$, where R is the solvable radical. Henceforth, $S = P \times Q$ will always denote the structure decomposition, where $P = (\widetilde{\mathrm{SL}_2\mathbb{R}})^k$, $Q = (\widetilde{\mathrm{SL}_2\mathbb{R}})^\ell$, are as defined in Theorem 5.3.

Let us define now an action of the solvable group

$$\mathcal{R}_1 = \tilde{K}^n \times \left(R \rtimes (AN)^k\right)$$

on the space G. For this, we let $\tilde{K}^n \leq S$ act on G by left-multiplication and the right-side factor $R \rtimes (AN)^k$ of $\mathcal{R}_1$ by multiplication from the right. Note that $\mathcal{R}_1$ acts freely. We study here how this action of $\mathcal{R}_1$ interacts with the right-multiplication action of H on G. In the following we shall thus identify $\mathcal{R}_1$ with its corresponding subgroup of $\mathrm{Diff}(G)$, and H with the group $R_H \leq \mathrm{Diff}(G)$. Here we let $R_g : G \to G$ denote the right-multiplication map $g_1 \mapsto g_1 g$.

LEMMA 5.5. *After replacing H by a conjugate with an element of S, the following hold:*

1. *R_H normalizes $\mathcal{R}_1$ in* $\mathrm{Diff}(G)$.
2. *$R_H \cap \mathcal{R}_1$ is uniform in $\mathcal{R}_1$.*
3. *$R_H \cap \mathcal{R}_1$ is of finite index in $R_{\mathrm{rad}(H)}$.*

PROOF. As implied by Theorem 5.3, after replacing H with a conjugate if required, we may assume that $\mathrm{rad}(H)$ is contained as a uniform subgroup in

$$G_{\mathrm{rad}(H)} = R \rtimes \left((\mathcal{Z} \times AN)^k \times \mathcal{Z}^\ell\right) ,$$

and, moreover, $\mathrm{rad}(H) = G_{\mathrm{rad}(H)} \cap H$, since $G_{\mathrm{rad}(H)} \cap H = \ker \mathsf{p}_Q \cap H$.

Note that H centralizes the left-multiplication action of $\tilde{K}^n \leq \mathcal{R}_1$. But then H also normalizes the second factor $R \rtimes (AN)^k$ of $\mathcal{R}_1$, considered as a subgroup of G. Since both groups act by right-multiplication, H normalizes the action of $\mathcal{R}_1$. Hence, (1) holds.

Recall now that every representation of $\widetilde{\mathrm{SL}_2\mathbb{R}}$ factors through $\mathrm{SL}(2,\mathbb{R})$. Therefore, a finite index subgroup of the center $\mathcal{Z}^n$ of S centralizes R. This subgroup is thus contained in the center of G. Letting $G_{\mathrm{rad}(H)}$ act by right-multiplication on G, we deduce that the intersection $R_{G_{\mathrm{rad}(H)}} \cap \mathcal{R}_1$ in $\mathrm{Diff}(G)$ is a finite index normal subgroup of $R_{G_{\mathrm{rad}(H)}}$ and it is also a uniform subgroup of $\mathcal{R}_1$. Since $\mathrm{rad}(H)$ is uniform in $G_{\mathrm{rad}(H)}$, all this implies that $R_H \cap \mathcal{R}_1$ is uniform in $\mathcal{R}_1$. Hence, (2) holds.

Since $\mathcal{R}_1$ is normalized by H, $R_H \cap \mathcal{R}_1$ is a solvable normal subgroup of R_H, and therefore it is contained in the radical $R_{\mathrm{rad}(H)}$. Since $R_{G_{\mathrm{rad}(H)}} \cap \mathcal{R}_1$ is of finite index in $R_{G_{\mathrm{rad}(H)}}$, (3) follows from $\mathrm{rad}(H) \leq G_{\mathrm{rad}(H)}$. □

Recall next the construction of the structure fibration for G/H which is obtained from the L-principal bundle (5.2)

$$L \longrightarrow G \xrightarrow{\mathsf{q}} \left(\mathbb{H}^2\right)^\ell$$

by dividing the right-multiplication of H. We show now that $\mathcal{R}_1$ acts simply transitively on the fibers of q. This gives rise to another principal bundle

$$\mathcal{R}_1 \longrightarrow G \xrightarrow{\mathsf{q}} (\mathbb{H}^2)^\ell \tag{5.4}$$

with the same projection map q. Our assertion is implied by the following:

LEMMA 5.6. *The identity of G induces an equivalence of orbit spaces*

$$\mathcal{R}_1 \backslash G \longrightarrow L \backslash G .$$

PROOF. Clearly, the decomposition

$$G = L \cdot (AN)^\ell \tag{5.5}$$

gives a trivialization of G as an L-bundle. Hence, every L-orbit in G has a unique representative $v \in (AN)^\ell$. Let $r_1 = (k, ru) \in \mathcal{R}_1$, where $k \in \tilde{K}^n$, $r \in R$, $u \in (AN)^k \leq P$. We compute the action of r_1 as

$$r_1 \cdot v = r^{kv} k u v .$$

Thus, we see that $\mathcal{R}_1 \cdot v = Lv$. The Lemma follows. □

5.3.2. *Affine solv-geometry on the fibers.* By Lemma 5.6, the decomposition (5.5) corresponds to a trivialization

$$G = \mathcal{R}_1 \times (AN)^\ell = \mathcal{R}_1 \times (\mathbb{H}^2)^\ell \tag{5.6}$$

of the $\mathcal{R}_1$-space $\mathsf{q} : G \to (\mathbb{H}^2)^\ell$. With respect to this product decomposition we may let the group $\mathrm{Aff}(\mathcal{R}_1)$ act on the first factor, and on G via trivial extension to the second factor. In particular, $\mathrm{Aff}(\mathcal{R}_1)$ acts on the fibers of the $\mathcal{R}_1$-principal bundle (5.4). Henceforth, we identify $\mathrm{Aff}(\mathcal{R}_1)$ with the corresponding subgroup

$\mathrm{Aff}(\mathcal{R}_1 \times (AN)^\ell)$ of $\mathrm{Diff}(G)$. With our conventions, the action of $\mathcal{R}_1$ on G identifies $\mathcal{R}_1$ as the natural normal subgroup of $\mathrm{Aff}(\mathcal{R}_1)$.

LEMMA 5.7. *The following holds:* $\mathrm{Aff}(\mathcal{R}_1) \cap R_H = R_{\mathrm{rad}(H)}$.

PROOF. Using (5.3) and Lemma 5.6 it follows that $R_h : G \to G$ stabilizes all fibers of (5.4) if and only if $h \in \ker \mathsf{p}_Q \cap H = \mathrm{rad}(H)$. Therefore, $R_H \cap \mathrm{Aff}(\mathcal{R}_1) \leq R_{\mathrm{rad}(H)}$.

To finish the proof of the Lemma, it remains to show that $R_h : G \mapsto G$ defines an element of $\mathrm{Aff}(\mathcal{R}_1)$, for all $h \in \mathrm{rad}(H)$: By Theorem 5.3, $\mathrm{rad}(H)$ is contained in the subgroup

$$G_{\mathrm{rad}(H)} = R \rtimes (AN)^k \times \mathcal{Z}^n \leq G .$$

By definition, right-multiplication with $R \rtimes (AN)^k$ is, a factor of $\mathcal{R}_1 = \tilde{K}^n \times (R \rtimes (AN)^k) \leq \mathrm{Diff}(G)$, and thus, in particular, $R_{R\rtimes(AN)^k}$ is contained in $\mathrm{Aff}(\mathcal{R}_1)$. It remains to understand the right-multiplication action of $\mathcal{Z}^n$. For this, we first compute the evaluation map $o_w : \mathcal{R}_1 \to G$, at $w \in (AN)^\ell$. Let $(k, ru) \in \mathcal{R}_1$. Then:

$$\begin{aligned} o_w(k, ru) = (k, ru) \cdot w &= kwu^{-1}r^{-1} \\ &= (r^{-1})^{ku^{-1}w} ku^{-1}w . \end{aligned}$$

Now for $z \in \mathcal{Z}^n$, let $\mu_z : R \rtimes (AN)^k) \to R \rtimes (AN)^k)$ denote the automorphism which is induced by conjugation with z inside the group G. We compute:

$$\begin{aligned} o_w(k, ru)z &= (r^{-1})^{ku^{-1}w} ku^{-1}wz \\ &= (r^{-1})^{ku^{-1}w} (zk)u^{-1}w \\ &= o_w(zk, r^{z^{-1}} u) \\ &= o_w(zk, \mu_{z^{-1}}(ru)). \end{aligned}$$

Extending μ_z to an element of $\mathrm{Aut}(\mathcal{R}_1)$, we can thus write

$$o_w(k, ru) = o_w(z\mu_{z^{-1}}(k, ru)) . \tag{5.7}$$

This shows that $R_z : G \to G$ acts as an element of $\mathrm{Aff}(\mathcal{R}_1)$ on the fibers of (5.4). Hence, $R_{G_{\mathrm{rad}(H)}} \leq \mathrm{Aff}(\mathcal{R}_1)$. In particular, this implies $R_{\mathrm{rad}(H)} \leq \mathrm{Aff}(\mathcal{R}_1)$. □

THEOREM 5.8. *Let $M = G/H$ be a compact aspherical homogeneous space. Then M has an infra-solv fiber space structure*

$$S \longrightarrow M \longrightarrow \left(\mathbb{H}^2\right)^\ell / \mathsf{p}_Q(H)$$

over the locally symmetric orbifold $\left(\mathbb{H}^2\right)^\ell / \mathsf{p}_Q(H)$ with the infra-solv manifold $S = \mathrm{rad}(H) \backslash \mathcal{R}_1$ as non-singular fiber.

PROOF. Theorem 5.3 and Lemma 5.5 imply that the data $X = G$, $\Delta = \mathrm{rad}(H)$, $W = (\mathbb{H}^2)^\ell$ with the action of R_H on G satisify the conditions (1) - (3) of Definition 3.1. That is, the structure fibration $M \to \left(\mathbb{H}^2\right)^\ell / \mathsf{p}_Q(H)$ turns M into a fiber-space with $\mathcal{R}_1$-geometry. By Lemma 5.7, it also satisfies condition (4) with respect to the decomposition (5.6). Therefore, the structure fibration actually inherits a $\mathcal{R}_1$-geometry. Finally, by equation (5.7), $hol(\mathrm{rad}(H)) \leq \mu(\mathcal{Z}^n) \leq \mathrm{Aut}(\mathcal{R}_1)$ is a finite group. Thus, the geometry on the fibers is infra-solv of type $\mathcal{R}_1$. □

We also get:

COROLLARY 5.9. *Let π be the fundamental group of a compact aspherical homogeneous space. Let $\Delta = \mathrm{rad}(\Pi)$ denote the discrete solvable radical of* Π. *Then Δ is a Wang group, and $\Theta = \Pi/\Delta$ is isomorphic to a lattice in* $(\mathrm{PSL}_2\,\mathbb{R})^{\ell}$.

COROLLARY 5.10. *The non-singular fiber of the structure fibration of an aspherical homogeneous space is diffeomorphic to a solv-manifold.*

Both corollaries are a consequence of:

PROPOSITION 5.11. *The solvable radical of $\pi = H/H_0$ is a Wang group.*

5.4. Rigidity of compact aspherical homogeneous manifolds. Let G/H be a compact aspherical homogeneous manifold. By Theorem 5.8, G/H has a infra-solv fiber space structure

$$\mathrm{rad}(H)\backslash\mathcal{R}_1 \longrightarrow G/H \longrightarrow \left(\mathbb{H}^2\right)^{\ell}/\mathsf{p}_Q(H)$$

over the locally symmetric orbifold $\left(\mathbb{H}^2\right)^{\ell}/\mathsf{p}_Q(H)$, with the infra-solv manifold $\mathrm{rad}(H)\backslash\mathcal{R}_1$ as non-singular fiber.

We show in [**3**]:

LEMMA 5.12. *$H \leq \mathrm{Diff}(G, \mathcal{R}_1, T)$.*

Here $T \leq \mathrm{Aut}(\mathcal{R}_1)$ is defined as in section 3.2.1. Note that the fiber stabilising group Δ is isomorphic to $\mathrm{rad}(H) = H \cap \mathrm{Diff}^1(G, \mathcal{R}_1)$ which is virtually solvable.

We now arrive at:

THEOREM 5.13. *Let $h : M \to M'$ be a homotopy-equivalence between compact aspherical homogeneous manifolds M and M'. Then there exists a diffeomorphism $\Phi : M \to M'$ which is homotopic to h.*

PROOF. Let $M = G/H$, $M' = G'/H'$ be presentations, and let $\phi : H/H^0 \to H'/H'^0$ correspond to the isomorphism of fundamental groups induced by h. By Theorem 3.5 and the subsequent Proposition 3.6, there are two standard Γ-fiber space structures $(G/H^0, \mathrm{U}_L, H/H^0)$, $(G'/H'^0, \mathrm{U}'_{L'}, H'/H'^0)$, which are associated to M and M', accordingly. Note that the associated group extension $1 \to \Gamma \to \pi \longrightarrow \mathsf{p}_Q(H) \to 1$ is characteristic, since Γ is the the radical of π. Hence, the isomorphism ϕ is compatible with the fiber space structures on M and M', and it induces an isomorphism $\bar{\phi} : \mathsf{p}_Q(H) \to \mathsf{p}_Q(H')$. Now we need the following well known fact, cf. [**3, 21**]:

PROPOSITION 5.14. *The smooth rigidity holds for the actions* $(\mathsf{p}_Q(H), (\mathbb{H}^2_{\mathbb{R}})^{\ell})$, $(\mathsf{p}_Q(H'), (\mathbb{H}^2_{\mathbb{R}})^{\ell})$.

Applying Theorem 4.4, we can construct an equivariant diffeomorphism (f, ϕ) : $(G/H^0, \mathrm{U}_L, H/H^0) \to (G'/H'^0, \mathrm{U}'_{L'}, H'/H'^0)$. In particular, G/H is diffeomorphic to G'/H'. $\square$

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