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Notes on the non local connectivity of deformation spaces of Kleinian groups

K. Bromberg*

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In this note we give an outline of the proof that the space of Kleinian punctured torus group is not locally connected. We also present some conjectures on the non local connectivity of other deformation spaces of Kleinian groups and we give some suggestions on how the methods of proof in the punctured torus case may be used to approach these conjectures.

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1 Definitions

We begin by recalling the definitions of the objects we will work with. Fix a group G and let $AH(G)$ be the space of conjugacy classes of discrete, faithfull representations of G in $PSL_2\mathbb{C}$. More explicitly a point in $AH(G)$ is an equivalence class of representations where representations ρ and ρ' are equivalent if there is a $g \in PSL_2\mathbb{C}$ such that $\rho' = g \circ \rho \circ g^{-1}$. A sequence $[\rho_i]$ converges to $[\rho]$ if there exists representatives ρ_i in $[\rho_i]$ and ρ in $[\rho]$ such that for all $g \in G$ the sequence $\rho_i(g)$ convergeres to $\rho(g)$ in $PSL_2\mathbb{C}$. A Kleinian group is a discrete subgroup of $PSL_2\mathbb{C}$ and $AH(G)$ is an example of a deformation space of Kleinian groups mentioned in the title.

As the group $PSL_2\mathbb{C}$ is naturally isomorphic to $\mathrm{Isom}^+(\mathbb{H}^3)$ the space $AH(G)$ can be identified with hyperbolic 3-manifolds with fundamental group G. We now expand on this viewpoint. Let N be a compact 3-manifold with boundary and $\mathcal{P}$ a collection of essential primitive annuli and tori in the boundary of N. We assume that $\mathcal{P}$ contains all the tori in ∂N and that any essential curve in one component of $\mathcal{P}$ is not homotopic in N into another component of $\mathcal{P}$. We the set $AH(N) = AH(\pi_1(N))$ and define the relative deformation

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space $AH(N, \mathcal{P})$ by restricting to representations ρ such that if $g \in \pi_1(N)$ represents a curve that is freely homotopic into $\mathcal{P}$ then $\rho(g)$ is a parabolic element of $PSL_2\mathbb{C}$.

Given a representation ρ in $AH(N, \mathcal{P})$ the image $\rho(\pi_1(N))$ is a Kleinian group which will act properly discontinuously on $\mathbb{H}^3$. The quotient $M_\rho = \mathbb{H}^3/\rho(\pi_1(N))$ is hyperbolic 3-manifold. The representation ρ is an isomorphism from $\pi_1(N)$ to $\pi_1(M_\rho)$ and this determines a map

$$f : N \longrightarrow M_\rho$$

uniquely defined up to homotopy. The pair (M_ρ, ρ) is a marked hyperbolic 3-manifold. Conversely any marked hyperbolic 3-manifold determines a representation in $AH(N)$ or $AH(N, \mathcal{P})$ if the appropriate curves are cusped. There is a bijection between equivalence classes of marked hyperbolic 3-manifolds and $AH(N)$ (or $AH(N, \mathcal{P})$) where the equivalence relation is the standard Teichmüller one. From hereon we will freely refer to points in $AH(N)$ as both representations and marked hyperbolic 3-manifolds.

The interior of $AH(N, \mathcal{P})$ is well understood and has been for some time. Before describing these results we need to clarify what we mean by interior. The deformation space $AH(N, \mathcal{P})$ naturally lies inside the space of all representations of $\pi_1(N)$ with the appropriate elements parabolic. Kapovich ([Kap]) has shown that a neighborhood of $AH(N, \mathcal{P})$ in this representation variety is a manifold. The interior of $AH(N, \mathcal{P})$ is the largest open subset of $AH(N, \mathcal{P})$ in topology of this manifold neighborhood. The dimension of this manifold is exactly the product of the dimensions of the Teichmüller spaces of the components of $\partial N - \mathcal{P}$.

In this note we will restrict our study to an important special case of the relative deformation space $AH(S \times [0, 1], \partial S \times [0, 1])$ where S is a compact surface with (possibly empty) boundary. In this case one usually simplifies notation by denoting this relative deformation space by $AH(S)$. By work of Marden ([Mar]) and Sullivan ([Sul]) the interior of $AH(S)$ is the space of quasifuchian groups which we denote by $QF(S)$. It is a classical result of Bers ([Bers]) that $QF(S)$ is naturally parameterized by the product of two copies of the Teichmüller space of S.

While we will only be interested in the topology of $AH(S)$, two other deformation spaces will arise naturally in our study which is why we need the general formulation described above. The first of these is a subspace of $AH(S)$ where we add one extra parabolic. To set our notation in this cases we let $N = S \times [0, 1]$ and let $\mathcal{P} = \partial S \times [0, 1]$. Then $AH(S) = AH(N, \mathcal{P})$ as defined above. Now fix an an essential, non-peripheral simple closed curve γ on S and let A be an annulus with core curve γ. Then we define $\mathcal{P}'$ to be the union of $A \times \{1\}$ and $\mathcal{P}$. The deformation space $AH(N, \mathcal{P}')$ is the first of the auxiliary subspaces we will use.

The deformation space $AH(N, \mathcal{P}')$ will have two components which we can distinguish geometrically. To do so we choose an orientation for N and for hyperbolic space. Then for each marked hyperbolic 3-manifold (M, f) in $AH(N, \mathcal{P}')$ the marking map f can be chosen to be an orientation preserving embedding of N in M. Using this embedding we distinguish between the two components by whether the cusp associated to γ is above or below the image of N in M. We call the component where the cusp is above $AH^+(N,\mathcal{P}')$ and the component where the cusp is below $AH^-(N,\mathcal{P}')$. We also note that the interior of $AH(N,\mathcal{P}')$ will have two components each of which is naturally identified with the product of the Teichmüller space of S and the Teichmüller space of $S\backslash\gamma$.

For the second auxiliary deformation space we need to change the topology of N. The product $A \times (1/4, 3/4)$ is a solid torus in N. We let $\hat{N}$ be N with this solid torus removed. The paring locus $\hat{P}$ for $\hat{N}$ is then the union of $\mathcal{P}$ and the torus boundary of $A \times [1/4, 3/4]$. In some ways $AH(\hat{N},\hat{\mathcal{P}})$ is very different from the two other deformation spaces we have mentioned. One striking difference is that $AH(\hat{N},\hat{\mathcal{P}})$ has an infinite number components. However, as we will see below, representations in $AH(\hat{N},\hat{\mathcal{P}})$ are built out of representations in $AH(N,\mathcal{P}')$. Furthermore, the interior of each component of $AH(\hat{N},\hat{\mathcal{P}})$ can be identified with the product of two copies of the Teichmüller space of S, just as for $QF(S)$, the interior of $AH(S)$. The relationship between the three deformation spaces will be spelled out in more detail below and it will be a key part of our approach to non-local connectivity of deformation spaces.

2 Punctured torus groups

Let $\hat{T}$ be the a torus with one boundary component. Here is our main result.

Theorem 2.1 *The space of Kleinian punctured torus groups, $AH(\hat{T})$, is not locally connected.*

We now begin our outline of the proof of this theorem.

The fundamental group of the punctured torus, $\hat{T}$, is the free group on two generators. We fix a presentation

$$\pi_1(\hat{T}) = \langle a, b \rangle.$$

We use the same notation as above with $S = \hat{T}$ and the element b of $\pi_1(\hat{T})$ freely homotopic to γ. Then a representation in $AH(\hat{T})$ is a discrete, faithful representation of the free group with the extra condition that the image of the commutator, $[a, b]$, is parabolic. For representations that also lie $AH(N,\mathcal{P}')$ the image of b will also be parabolic.

One of the advantages of working with the punctured torus is that the space $AH(N,\mathcal{P}')$ can be described fairly explicitly.

Proposition 2.2 *Every representation of $\pi_1(\hat{T})$, with the image of b and $[a, b]$ parabolic, is conjugate to a unique representation of the form ρ_μ where*

$$\rho_\mu(a) = \begin{pmatrix} i\mu & i \\ i & 0 \end{pmatrix} \text{ and } \rho_\mu(b) = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}.$$

We can use these representations to identify $AH(N, \mathcal{P}')$ with a subspace of $\mathbb{C}$. Namely, define the *Maskit slice*, $\mathcal{M}$, to be the set of μ in $\mathbb{C}$ that $\rho_\mu \in AH(N, \mathcal{P}')$. We will need some simple facts about the Maskit slice.

Lemma 2.3 *1. $\mathcal{M}$ is has two connected components each of which is homeomorphic to the closed upper half plane in $\mathbb{C}$.*

2. If μ is in $\mathcal{M}$ then $\mu \pm 2$ are in $\mathcal{M}$.

3. The components of $\mathcal{M}$ are not bounded by a horizontal line.

The first of these facts, while simple to state, is not so simple to prove. It comes from Minsky's solution of the ending lamination conjecture ([Min]). For the second fact, one can take the marked hyperbolic manifold associated to ρ_μ and choose a marking map in a different homotopy class to get the representations $\rho_{\mu+2}$ and $\rho_{\mu-2}$ which implies that these two representations are in $AH(N, \mathcal{P}')$ and that $\mu \pm 2$ are in $\mathcal{M}$. To prove the third fact one uses the fact that representations with an extra element parabolic will lie on the boundary of $\mathcal{M}$. One can then find two representations, both of which have an extra parabolic, that do not have the same imaginary part.

We let $\mathcal{M}^+$ be those μ such that ρ_μ is in $AH^+(N, \mathcal{P}')$ and we similarly define $\mathcal{M}^-$. Note that $\mathcal{M}$ has exactly two components one of which will lie in the upper half plane while the other will lie in the lower half plane. We can assume that orientations have been chosen such that $\mathcal{M}^+$ lies in the upper half plane.

We now use $\mathcal{M}$ to build $AH(\hat{N}, \hat{\mathcal{P}})$. First we observe that $\pi_1(\hat{N})$ is an HNN-extension and has presentation

$$\pi_1(\hat{N}) = \langle a, b, c | [b, c] \rangle.$$

Using this presentation we can extend our representations of $\pi_1(N)$ to representations of $\pi_1(\hat{N})$. For each $\beta \in \mathbb{C}$ define a representation $\rho_{\mu,\beta}$ of $\pi_1(\hat{N})$ by setting $\rho_{\mu,\beta}(a) = \rho_\mu(a)$, $\rho_{\mu,\beta}(b) = \rho_\mu(b)$ and

$$\rho_{\mu,\beta}(c) = \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}.$$

We then have:

Proposition 2.4 *Every representation in $AH(\hat{N}, \hat{\mathcal{P}})$ is conjugate to a unique representation $\rho_{\mu,\beta}$.*

For each μ in the interior of $\mathcal{M}^+$ define a subset $\mathcal{A}_\mu$ of $\widehat{\mathbb{C}}$ by

$$\mathcal{A}_\mu = \{\beta \in \mathbb{C} | \rho_{\mu,\beta} \in AH(\hat{N}, \hat{\mathcal{P}}) \text{ and } \operatorname{Im} \beta > 0\} \cup \{\infty\}.$$

We then define the subset $\mathcal{A}$ of $\mathbb{C} \times \widehat{\mathbb{C}}$ by

$$\mathcal{A} = \{(\mu, \beta) \in \mathbb{C} \times \widehat{\mathbb{C}} | \mu \in \operatorname{int} \mathcal{M}^+ \text{ and } \beta \in \mathcal{A}_\mu\}.$$

For each μ in the interior of $\mathcal{M}^+$ we can define a map

$$\Phi : U \longrightarrow AH(N, \mathcal{P})$$

where U is a neighborhood of (μ, ∞) in $\mathcal{A}$ and $\Phi(\mu, \infty) = \rho_\mu$. Theorem 2.1 then follows from from the following two results:

Theorem 2.5 *The map Φ is a local homeomorphism.*

Theorem 2.6 *There exists μ in the interior of $\mathcal{M}$ such that $\mathcal{A}$ is not locally connected at (μ, ∞).*

We will give an outline of the construction of the map Φ but let the reader refer to [Br] for the proof of Theorem 2.5. Given Lemma 2.3 the proof of Theorem 2.6 is not so difficult and we will provide some detail here.

Here is the construction of Φ. For points of the form (μ, ∞), Φ is easy to define for we just set $\Phi(\mu, \infty) = \rho_\mu$. The construction is more involved when for points (μ, β) with $\beta \neq \infty$ and in fact we can only define the map when $|\beta|$ is large.

Let $\hat{M}_{\mu,\beta}$ be the quotient hyperbolic 3-manifold for the representation $\rho_{\mu,\beta}$. Then $\hat{M}_{\mu,\beta}$ is a complete hyperbolic structure on the interior of $\hat{N}$. When $|\beta|$ is large we can find a hyperbolic structure $M_{\mu,\beta}$ on the interior of N with the following properties.

1. There is an embedding
$$\phi_{\mu,\beta} : \hat{M}_{\mu,\beta} \longrightarrow M_{\mu,\beta}$$
that is bi-Lipschitz in the complement of the unique rank two cusp of $\hat{M}_{\mu,\beta}$ and extends to a conformal map between the conformal boundaries of $\hat{M}_{\mu,\beta}$ and $M_{\mu,\beta}$.

2. The image of c in $M_{\mu,\beta}$ is trivial.

Now choose a π_1-injective map
$$f_{\mu,\beta} : \hat{T} \longrightarrow \hat{M}_{\mu,\beta}$$
that induces the representation ρ_μ on $\pi_1(\hat{T})$. The composition $\phi_{\mu,\beta} \circ f_{\mu,\beta}$ will then be a homotopy equivalence and the pair $(M_{\mu,\beta}, \phi_{\mu,\beta} \circ f_{\mu,\beta})$ will be a marked hyperbolic 3-manifold in $AH(N, \mathcal{P})$. We then define $\Phi(\mu, \beta) = (M_{\mu,\beta}, \phi_{\mu,\beta} \circ f_{\mu,\beta})$.

After constructing Φ the proof of Theorem 2.5 use the Drilling Theorem of [BB] along with Minsky's ending lamination theorem for punctured torus groups ([Min]).

We now move on to the proof of Theorem 2.6. The first thing we need is the following description of $\mathcal{A}$.

Proposition 2.7 *The representation $\rho_{\mu,\beta}$ is in $AH(\hat{N}, \hat{\mathcal{P}})$ if and only if there is an integer n such that $\mu - sn\beta$ is in $\mathcal{M}^+$ and $\mu - s(n+1)\beta$ is in $\mathcal{M}^-$ where s is the sign of β.*

Sketch of Proof. To see that the conditions are necessary we need to see that if they fail then we can find a subgroup where the restriction of the representation is not discrete and faithful. In particular it is not hard to see that if the conditions fail there is an integer k such that $\mu - k\beta$ is not in $\mathcal{M}$. Then the representation restricted to the subgroup generated $c^{-k}a$ and b will not be discrete and faithful.

For the sufficiency of the conditions we directly build the hyperbolic 3-manifold. By our assumption the restriction of the representation to the subgroup generated by $c^{-n}a$ and b and the subgroup generated by $c^{-(n+1)}a$ and b are discrete and faithful. In fact these two subgroups correspond to the representations $\rho_{\mu - sn\beta}$ and $\rho_{\mu - s(n+1)\beta}$ of $\pi_1(\hat{T})$. Let $M_{\mu - sn\beta}$ and $M_{\mu - s(n+1)\beta}$ be the corresponding hyperbolic 3-manifolds. The "top" of the convex core of $M_{\mu - sn\beta}$ and the "bottom" of the convex core $M_{\mu - s(n+1)\beta}$ will be bounded by totally geodesic triply punctured spheres. From each manifold we remove the component of the complement of the convex core that is bounded by this triply punctured sphere and then glue what is left of the two manifolds together along their boundary surfaces. We then get a new complete hyperbolic manifold homeomorphic to the interior of $\hat{N}$. By carefully marking this hyperbolic manifold with $\hat{N}$ (note that this marking map may not be homotopic to a homeomorphism) we get a marked hyperbolic manifold in $AH(\hat{N}, \hat{\mathcal{P}})$ that induces the representation $\rho_{\mu,\beta}$ on $\pi_1(\hat{N})$. For more details of this construction see [Br]. $\boxed{2.7}$

By combining Lemma 2.3 and Proposition 2.7 we can show that $\mathcal{A}$ is not locally connected and prove Theorem 2.6.

Skectch of Proof of Theorem 2.6. We begin with three observations.

1. From Proposition 2.7 we see that the sets $\mathcal{A}_\mu$ are unions of sets of the form
$$\mathcal{A}_{\mu,n} = \{\beta \in \mathbb{C} | \operatorname{Im} \beta > 0, \mu - n\beta \in \mathcal{M}^+, \mu - (n+1)\beta \in \mathcal{M}^-\}$$
and the point $\infty \in \widehat{\mathbb{C}}$.

2. By (2) of Lemma 2.3 the sets $\mathcal{A}_{\mu,n}$ and therefore $\mathcal{A}_\mu$ are translation invariant: If $\beta \in \mathcal{A}_\mu$ then $\beta \pm 2 \in \mathcal{A}_\mu$.

3. Using (3) of Lemma 2.3 we can find a $\mu \in \mathcal{M}^+$ and a positive integer n such that $\mathcal{A}_{\mu,n}$ has a bounded component in $\mathbb{C}$. This is because each $\mathcal{A}_{\mu,n}$ is a scaled and translated copy of $\mathcal{M}^+$ intersected with a scaled and translated copy of $\mathcal{M}^-$. The amount of scaling is completely determined by n and the choice of μ gives us complete control over how the two sets are translated with respect to each other. Since these two sets are not bounded by horizontal lines if they are translated so that they just intersect the intersection will have bounded components.

Together these three observations allows us to find a $\mu \in \mathcal{M}^+$ such that $\mathcal{A}_\mu$ has a component that is bounded in $\mathbb{C}$. Since $\mathcal{A}_\mu$ is translation invariant this means that there are an infinite number of bounded components which accumulate on $\infty \in \widehat{\mathbb{C}}$ and therefore $\mathcal{A}_\mu$ is not locally connected at ∞.

This is not quite the same as showing that $\mathcal{A}$ is not locally connected (μ, ∞) but it is not too difficult to see that this is, in fact, the case. We refer to [Br] for the details of this argument. $\boxed{2.6}$

Theorems 2.5 and 2.6 together imply our main result, Theorem 2.1, completing our sketch of its proof.

3 Higher dimensional deformation spaces

We now address the following conjecture.

Conjecture 3.1 *$AH(S)$ is not locally connected for any hyperbolic surface S.*

In this section we assume that S is a compact surface with boundary whose Teichmüller space is not one complex dimensional. In particular, S is a hyperbolic surface but not a punctured torus or 4-times punctured sphere. We then define N, $\hat{N}$, $\mathcal{P}$, $\mathcal{P}'$ and $\hat{\mathcal{P}}$ as in section 1.

We now repeat the construction from the previous section in our more general setting. First assume b is an element of $\pi_1(S)$ that is freely homotopic to the curve γ. Then we can give $\pi_1(\hat{N})$ the presentation

$$\pi_1(\hat{N}) = \langle \pi_1(S), c | [b, c] \rangle.$$

We then can extend a representation ρ in $AH(N, \mathcal{P}')$ to a representation of $\pi_1(\hat{N})$. To do so we assume that ρ has been conjugated so that

$$\rho(b) = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}.$$

Then for each $\beta \in \mathbb{C}$ define the extension of ρ to a representation ρ_β of $\pi_1(\hat{N})$ by setting

$$\rho_\beta(c) = \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}.$$

We then define subsets $\mathcal{A}_\rho$ of $\widehat{\mathbb{C}}$ by

$$\mathcal{A}_\rho = \{\beta \in \mathbb{C} | \operatorname{Im} \beta > 0 \text{ and } \rho_\beta \in \mathrm{AH}(\hat{N}, \hat{\mathcal{P}})\} \cup \{\infty\}.$$

Note by fixing $\rho(b)$ we have not uniquely determined ρ within its conjugacy class. However, it is easy to check that for any representation ρ' conjugate to ρ with $\rho'(b) = \rho(b)$ we have $\mathcal{A}_{\rho'} = \mathcal{A}_\rho$. As before we define $\mathcal{A}$ by

$$\mathcal{A} = \{(\rho, \beta) | \rho \in \operatorname{int} AH^+(N, \mathcal{P}') \text{ and } \beta \in \mathcal{A}_\rho\}.$$

Finally for any ρ in $\operatorname{int} AH^+(N, \mathcal{P}')$ on a neighborhood U of (ρ, ∞) we can define a map

$$\Phi : U \longrightarrow AH(S)$$

with $\Phi(\rho, \infty) = \rho$. We then make the following conjectures.

Conjecture 3.2 *There exists ρ in $\operatorname{int} AH^+(N, \mathcal{P}')$ such that $\mathcal{A}$ is not locally connected at (ρ, β).*

Conjecture 3.3 *If $\mathcal{A}$ is not locally connected at (ρ, ∞) then $AH(S)$ is not locally connected at ρ.*

Obviously, Conjectures 3.2 and 3.3 imply Conjecture 3.1.

One might hope to prove Conjecture 3.3 by showing that Φ is a local homeomorphism at (ρ, ∞). Unfortunately, although Φ is a bijection onto a neighborhood of ρ, it is most likely not continuous. The phenomena one needs to worry about arises in the work of Kerkchoff and Thurston ([KT]). They show that for Teichmüller spaces of dimension > 1 there are examples where the canonical bijection between two Bers' slices is not continuous and therefore not a homeomorphism. A variant of their construction should lead to a similar lack of continuity for Φ.

However, in [Br], we prove the following result which is strong evidence for Conjecture 3.3.

Theorem 3.4 *The map Φ restricted to the pre-image of $QF(S) \cup \{\rho\}$ is a local homeomorphism onto its image for any $\rho \in \operatorname{int} AH^+(N, \mathcal{P}')$. In particular, if $\mathcal{A}$ is not locally connected at (ρ, ∞) then $QF(S) \cup \{\rho\}$ is not locally connected at ρ.*

For the rest of this section we concentrate on how one might approach Conjecture 3.2. We need to find a replacement for the Maskit slice and then prove a version of Lemma 2.3 for this new object.

For convenience we will assume that γ is non-separating. Let R be the compact surface whose genus is one less than the genus of S and has two more boundary components then does S. By gluing two of the boundary components of R together we obtain S and we can assume that γ is the image of the two glued curves.

Choose a basepoint for the fundamental group of $\pi_1(R)$ and pick two loops that only intersect at the basepoint with each loop freely homotopic to a distinct component of ∂R. These loops represent elements a and b of $\pi_1(R)$ and we orient them such that the element ab^{-1} is freely homotopic to a simple curve. We then write $\pi_1(S)$ as an HNN-extension of $\pi_1(R)$ by setting

$$\pi_1(S) = \langle \pi_1(R), c | cac^{-1} = b \rangle.$$

This identifies $\pi_1(R)$ as a subgroup of $\pi_1(S)$ and it will allow us to both extend representations of $\pi_1(R)$ to representations of $\pi_1(S)$ and to restrict representations of $\pi_1(S)$ to $\pi_1(R)$.

Let ρ be a representation in $\pi_1(R)$ and assume that

$$\rho(b) = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}.$$

Let C be an element of $PSL_2\mathbb{C}$ such that $C\rho(a)C^{-1} = \rho(b)$. We then define an extension ρ_0 of ρ to $\pi_1(S)$ by setting $\rho_0(c) = C$. After fixing this initial representation we can define a whole family of extensions by multiplying C on the left by a parabolic element that commutes with $\rho(b)$. In particular for each $\mu \in \mathbb{C}$ define the extension ρ_μ by setting

$$\rho_\mu(c) = \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix} C.$$

We are interested in the subset where these extensions are in $AH(N, \mathcal{P}')$. We define

$$\mathcal{M}_\rho = \{\mu \in \mathbb{C} | \rho_\mu \in AH(N, \mathcal{P}')\}.$$

We also break $\mathcal{M}_\rho$ into two sets by setting $\mathcal{M}_\rho^+$ to be those parameters where ρ_μ is in $AH^+(N, \mathcal{P}')$ and $\mathcal{M}_\rho^-$ is the subset of parameters where ρ_μ is in $AH^-(N, \mathcal{P}')$. Note that if we pick a different representation ρ' conjugate to ρ (with $\rho'(b) = \rho(b)$) and a different C then $\mathcal{M}_{\rho'}$ may not be the same as $\mathcal{M}_\rho$. However, the only difference will be that $\mathcal{M}_{\rho'}$ may be a translated copy of $\mathcal{M}_\rho$.

Note that every representation in $AH(N, \mathcal{P}')$ is of the form ρ_μ where ρ is a representation in $AH(R)$ and μ is in $\mathcal{M}_\rho$. The following result is a generalization of Proposition 2.7.

Proposition 3.5 *Let ρ be a representation in $QF(R)$ and assume that μ is in $\mathcal{M}_\rho^+$. Then $\beta \neq \infty$ is in $\mathcal{A}_{\rho_\mu}$ if and only if there exists an integer n such that $\mu - n\beta$ is in $\mathcal{M}_\rho^+$ and $\mu - (n+1)\beta$ is in $\mathcal{M}_\rho^-$.*

Using this proposition one would like to follow the proof of Theorem 2.6 to prove Conjecture 3.2. Along with Proposition 2.7 the key facts we need are that $\mathcal{M}_\rho$ is translation invariant and that the components of $\mathcal{M}_\rho$ are not bounded by horizontal lines. The first statement is easy to prove as the proof in the punctured torus case works in general. Therefore the key to proving Conjecture 3.2 is:

Conjecture 3.6 *The components of $\mathcal{M}_\rho$ are not bounded by horizontal lines.*

One expects that the boundary of $\mathcal{M}_\rho$ is nowhere differentiable which is a much stronger statement. Note that the proof for the punctured torus will not work in this case as for a generic ρ, the subset $\mathcal{M}_\rho$ will not have any representations with extra parabolics much less two distinct ones. We also note that for the purposes of proving Conjecture 3.2 one probably only needs to find one $\mathcal{M}_\rho$ whose components are not bounded by horizontal lines. This should be easier.

4 The Bers' slice

The approach that we outlined in the previous section should be useful in studying a fairly wide class of deformation spaces. The key property that we use is that N contains a properly embedded incompressible and boundary incompressible annulus whose core curve represents a primitive element of $\pi_1(N)$. In the special case we have examined, where N is an I-bundle over a surface, there are many such annuli. However, we really only needed one. The opposite situation is when the pared manifold $(N, \mathcal{P})$ is acylindrical. Rather than delve into a discussion of acylindrical manifolds we look at a Bers' slice as a Bers' slice is a subspace of $AH(S)$ that behaves much like the deformation space of an acylindrical manifold.

Here is how we define it. The quasifuchsian groups, $QF(S)$, are parameterized by the product of two copies of the Teichmüller space of S. If we fix a conformal structure in one of the coordinates then we get a slice of $QF(S)$ that is parameterized by a single copy of the Teichmüller space. The closure of this slice in $AH(S)$ is called a Bers' slice.

An alternative definition will be more useful for us. Let $(N, \mathcal{P})$ be a pared manifold and let X be a conformal structure defined on some of the components of $\partial N - \mathcal{P}$. Then we define $AH(N, \mathcal{P}, X)$ to be those marked hyperbolic 3-manifolds in $AH(N, \mathcal{P})$ where the marking map can be chosen to be a conformal map from X to the conformal boundary of the hyperbolic manifold. We now revert to our standard notation for the surface S and let X be a conformal structure on $S \times \{0\}$ which is a component of $\partial N - \mathcal{P}$. Then the Bers' slice is the deformation space $AH(N, \mathcal{P}, X)$. The deformation space $AH(N, \mathcal{P}', X)$ lies on the boundary of $AH(N, \mathcal{P}, X)$.

To study the topology of $AH(N, \mathcal{P}, X)$ we define for each ρ in $AH(N, \mathcal{P}', X)$ the spaces

$$\mathcal{B}_\rho = \{\beta \in \mathbb{C} | \operatorname{Im} \beta > 0 \text{ and } \rho_\beta \in AH(\hat{N}, \hat{\mathcal{P}}, X)\} \cup \{\infty\}$$

and

$$\mathcal{B} = \{(\rho, \beta) | \rho \in \operatorname{int} AH^+(N, \mathcal{P}', X) \text{ and } \beta \in \mathcal{B}_\rho\}.$$

As before for any $\rho \in \operatorname{int} AH^+(N, \mathcal{P}', X)$ we have a map

$$\Phi : U \longrightarrow AH(N, \mathcal{P}, X)$$

defined on a neighborhood U of (ρ, ∞) in $\mathcal{B}$. One when then like to prove the appropriately modified versions of Conjectures 3.2 and 3.3 which would imply that the Bers' slice is not locally connected. We note the methods of [Br] also imply an appropriately modified version of Theorem 3.4 for the Bers' slice.

The key difference between the the Bers' slice and all of $AH(S)$ is seen in the following proposition which is the Bers' slice version of Proposition 3.5.

Proposition 4.1 *Let ρ be a representation in $QF(R)$ and assume that μ is in $\mathcal{M}_\rho^+$. Then $\beta \neq \infty$ is in $\mathcal{B}_{\rho_\mu}$ if and only if $\mu - \beta$ is in $\mathcal{M}_\rho^-$.*

Again we would like to follow the proof of Theorem 2.6 to show that $\mathcal{B}$ is not locally connected. However, here we run into more serious problems. The subsets $\mathcal{B}_{\rho_\mu}$ are essentially just $\mathcal{M}_\rho^-$. If we want $\mathcal{B}_{\rho_\mu}$ to not be locally connected at infinity then we need the following conjecture:

Conjecture 4.2 *If S is not a punctured torus or 4-times punctured spheres then there exists ρ such $\mathcal{M}_\rho$ has bounded components.*

This conjecture seems to be difficult. It would be interesting to test it by trying to draw computer pictures. We also note that it is necessary to assume that S is not a punctured torus or 4-times punctured sphere because in both cases $\mathcal{M}$ is homeomorphic to $AH(N, \mathcal{P}')$ which has exactly two components. If there where any bounded components in $\mathcal{M}$ there would be an infinite number of components by translation invariance. For other surfaces the sets $\mathcal{M}_\rho$ are only a slice of $AH(N, \mathcal{P}')$. Since one expects that $AH(N, \mathcal{P}')$ has a fractal boundary taking such a slice should lead to bounded components.

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