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Ford domains of a certain hyperbolic 3-manifold whose boundary consists of a pair of once-punctured tori

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1 Introduction

The following is our initial problem.

Problem 1.1. Characterize the Ford domains of hyperbolic structures on a 3-manifold which has a pair of punctured tori as boundary.

In this section we see some background materials to Problem 1.1.

Both knots depicted in Figure 1 are hyperbolic. Moreover, each is of genus 1. In fact, they have (once-)punctured tori depicted in the figure as Seifert surfaces. One can see that K_2 is obtained from K_1 by performing a Dehn surgery on α.

One major difference between the two knots is that K_1 is a fibered knot while K_2 is not. This causes a difference in the proof of the Thurston's Hyperbolization Theorem for Haken manifolds. Following it, one can construct the complete hyperbolic structure of finite volume on the complement of each

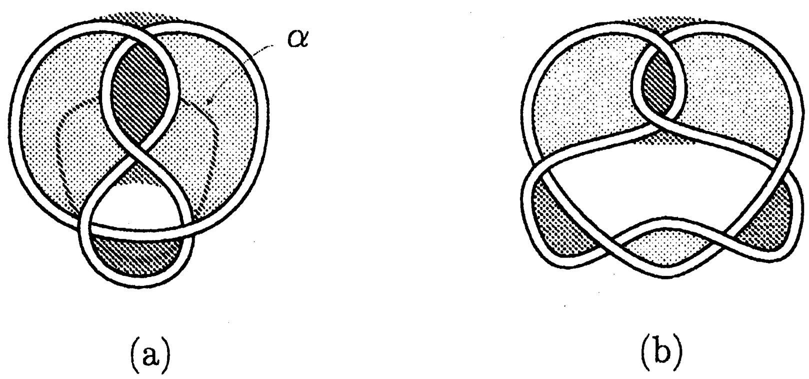

Figure 1: (a) the figure-8 knot $K_1 = 4_1$, (b) $K_2 = 6_1$

K_1 and K_2 by cutting along essential surfaces in the manifold several times to obtain a finite number of balls, construct hyperbolic structures on each component, then deforming and gluing back the structures until one obtains a hyperbolic structure on the original manifold. The argument which guarantees that the final gluing is possible is as follows. If one cuts the original manifold along a fiber surface as in the case of K_1, then, by the "double limit theorem", one can find a hyperbolic structure which is invariant under the gluing map. On the other hand, if one cuts the manifold along a non-fiber surface as in the case of K_2, then one needs to define a map on the space of geometrically finite hyperbolic structures whose fixed point gives a structure which is invariant under the gluing map, for the map show the "fixed point theorem".

As is explained in Section 5, the Jorgensen theory tells in detail the combinatorial structures of the Ford domains of hyperbolic structures on punctured torus bundles. So, we expect to understand in detail the hyperbolic structures on manifolds with non-fiber surfaces from the combinatorial structures of Ford domains. Problem 1.1 is the first step to the attempt to fill in the box with "???" in the following table.

	analytic	combinatorial (genus= 1)
fiber surface	double limit theorem	Jorgensen theory
non-fiber surface	fixed point theorem	???

In this paper, we study a manifold which is obtained from the product of the punctured torus and the interval by performing Dehn surgery along an essential simple closed curve in a level-surface. In Section 3, we will see some topological property of such a manifold. In Section 6, we give a parametrization of the space of geometrically finite minimally parabolic hyperbolic structures on the manifold. Finally, in Section 7, we obtain some numerical results.

2 Ford domain

Throughout the paper, we use the upper half space model for the hyperbolic 3-space $\mathbb{H}^3$. We shall identify the boundary plane of the upper half space with the complex plane $\mathbb{C}$.

Definition 2.1. For an element of $PSL(2,\mathbb{C})$, $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, which does not stabilize ∞, we define the following.

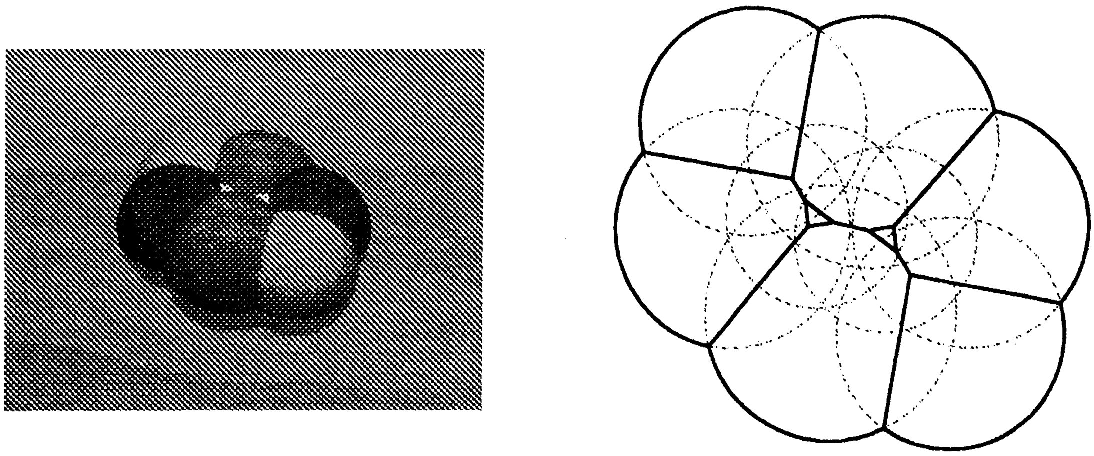

Figure 2: Ford domain of a cyclic Kleinian group

- The *isometric circle* of γ, denoted by $I(\gamma)$, is the circle in the complex plane with center $-d/c$ and radius $1/|c|$. The exterior of $I(\gamma)$ is denoted by $E(\gamma)$.
- The *isometric hemisphere* of γ, denoted by $Ih(\gamma)$, is the hemisphere in the upper half space centered at a point in $\mathbb{C}$ with equator $I(\gamma)$. The exterior of $Ih(\gamma)$ is denoted by $Eh(\gamma)$.

Definition 2.2. For a Kleinian group Γ, let Γ_∞ be the stabilizer of ∞ in Γ. The *Ford domain* of Γ in $\mathbb{C}$ (resp. $\mathbb{H}^3$), denoted by $P(\Gamma)$ (resp. $Ph(\Gamma)$), is defined by

$$P(\Gamma) = \bigcap_{\gamma \in \Gamma - \Gamma_\infty} E(\gamma), \qquad Ph(\Gamma) = \bigcap_{\gamma \in \Gamma - \Gamma_\infty} Eh(\gamma).$$

Remark 2.3. The Ford domain $P(\Gamma)$ (resp. $Ph(\Gamma)$) is not a fundamental domain for the action of Γ on $\Omega(\Gamma)$ (resp. $\mathbb{H}^3$), whenever Γ_∞ is nontrivial, where $\Omega(\Gamma)$ is the "domain of discontinuity" of Γ on which Γ acts discontinuously. Even in the case, the intersection of the Ford domain and a fundamental domain for Γ_∞ is actually a fundamental domain for Γ.

Example 2.4. The Ford domain $Ph(\langle\gamma\rangle)$ of the cyclic Kleinian group $\langle\gamma\rangle$, generated by a loxodromic element $\gamma \in PSL(2,\mathbb{C})$ which does not stabilize ∞, is as depicted in Figure 2. Every "face" of $Ph(\Gamma)$ is supported by an isometric hemisphere; there are 8 faces in this example. The characterization of combinatorial structures of the Ford domains of cyclic Kleinian groups is given by Jorgensen [7] (cf. [5]).

Example 2.5. The image Γ of the holonomy representation of a complete hyperbolic structure on the punctured torus of finite area is a fuchsian group.

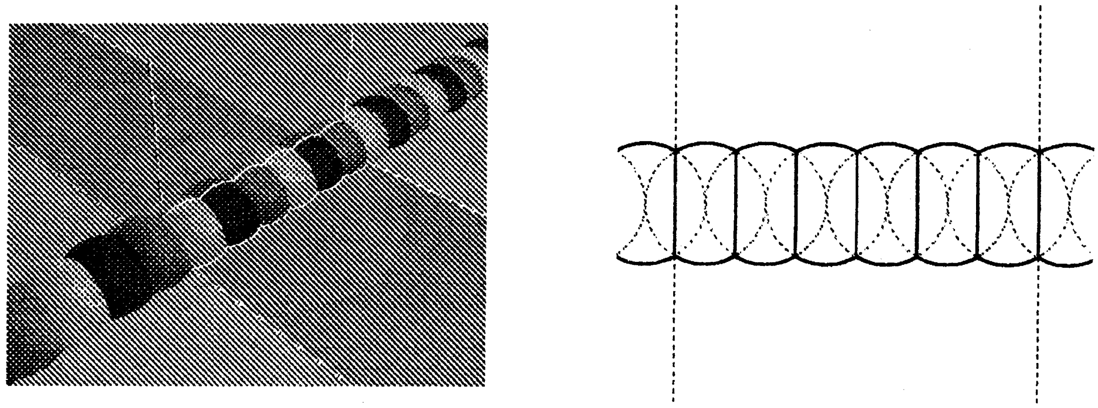

Figure 3: Ford domain of a fuchsian punctured torus group

The fuchsian group is also regarded as a Kleinian group in a natural way, and the Ford domain $Ph(\Gamma)$ is as depicted in Figure 3. In this situation, the vertical plane lying on the real axis of $\mathbb{C}$ is naturally identified with the hyperbolic plane on which the fuchsian group Γ acts. Then the intersection of Γ and the plane is equal to the "Ford domain" of the fuchsian group.

3 Manifolds with a pair of punctured tori as boundary

We denote the one-holed torus by T_0. Let γ be an essential simple closed curve on the level surface $T_0 \times \{0\}$ of the product manifold $T_0 \times [-1, 1]$, and denote by M_0 the exterior of γ, i.e., $M_0 = T_0 \times [-1, 1] - \operatorname{Int} N(\gamma)$, where $N(\gamma)$ is a regular neighborhood of γ. For each sign $\epsilon = \pm$, we denote the one-holed torus $T_0 \times \{\epsilon 1\} \subset \partial M_0$ by T_0^ϵ. We define the *slopes* (= free homotopy classes) μ and λ in $\partial N(\gamma)$ as follows. μ is the meridian slope of γ, i.e., μ is represented by an essential simple closed curve which bounds a disk in $N(\gamma)$, and λ is the slope represented by the intersection of $\partial N(\gamma)$ and the annulus $\gamma \times [0, 1]$. Then $\{\mu, \lambda\}$ generates $H_1(\partial N(\gamma))$.

For a pair of coprime integers (p, q), let $M(p, q)$ be the result of Dehn filling on M_0 with slope $p\mu + q\lambda$, i.e., the manifold obtained from M_0 by gluing the solid torus V by an orientation-reversing homeomorphism $\partial V \to \partial N(\gamma) \subset \partial M_0$ so that the meridian of V is identified with a simple closed curve on $\partial N(\gamma)$ of slope $p\mu + q\lambda$. We regard M_0 as a submanifold of $M(p, q)$ by using the canonical embedding.

Proposition 3.1. *For any pair of coprime integers (p, q), $M(p, q)$ is homeomorphic to the handlebody of genus 2.*

Set $P = \partial T \times [-1, 1]$. In contrast to Proposition 3.1, we should be careful about the manifold pair $(M(p,q), P)$.

Proposition 3.2. *The surfaces $T_0^{\pm}$ is incompressible in $M(p,q)$ if and only if $(p,q) \neq (0, \pm 1)$. In this case, it follows that $(M(p,q), P)$ is an atoroidal Haken pared manifold (in the sense of [9]).*

By the Thurston's Hyperbolization Theorem for Haken pared manifolds (cf. [9, Theorem 1.43]), we obtain the following corollary.

Corollary 3.3. *For any pair of coprime integers $(p,q) \neq (0, \pm 1)$, $M(p,q)$ admits a complete geometrically finite hyperbolic structure with the parabolic locus P.*

We may choose simple closed curves α and β in the level surface $T_0 \times \{0\}$ which intersect at a single point such that α is parallel to γ. Let $\alpha^{\pm}$ and $\beta^{\pm}$ be the simple closed curves in $T_0^{\pm}$ which are parallel to α and β respectively. Then α^{ϵ} and β^{ϵ} freely generate $\pi_1(T_0^{\epsilon})$ for each $\epsilon = \pm$. Let $\kappa^{\epsilon} = [\alpha^{\epsilon}, \beta^{\epsilon}]$ $(\epsilon = \pm)$ be the commutator of α^{ϵ} and β^{ϵ}, which is represented by a peripheral loop of T_0^{ϵ}. Then, by the van Kampen's theorem, $\pi_1(M_0)$ is canonically isomorphic to the free product of the two free groups $\langle \alpha^-, \beta^- \rangle$ and $\langle \alpha^+, \beta^+ \rangle$ with the subgroups $\langle \alpha^-, \kappa^- \rangle$ and $\langle \alpha^+, \kappa^+ \rangle$ amalgamated under the mapping $(\alpha^-, \kappa^-) \mapsto (\alpha^+, \kappa^+)$. Then the following lemma is immediate.

Lemma 3.4. *The fundamental group of M_0 (resp. $M(p,q)$) is isomorphic to the group with presentation $\langle A, B^-, B^+ \mid [A, B^-] = [A, B^+] \rangle$ (resp. $\langle A, B^-, B^+ \mid [A, B^-] = [A, B^+], \{(B^+)^{-1} B^-\}^p A^q = 1 \rangle$) by the map which sends the quadruple $(\alpha^-, \alpha^+, \beta^-, \beta^+)$ to (A, A, B^-, B^+). In particular, there is a canonical surjection $\Phi : \pi_1(T_0^-) * \pi_1(T_0^+) \to \pi_1(M(p,q))$ whose restriction to each $\pi_1(T_0^{\pm})$ is injective whenever $(p,q) \neq (0, \pm 1)$.*

By Lemma 3.4, the image of the holonomy representation of a hyperbolic structure on $M(p,q)$ is an amalgamated free product of two "punctured torus groups". In the following section, we give a brief review on such groups.

4 Punctured torus groups

Definition 4.1. Let $\rho_0 : \pi_1(T) \to PSL(2, \mathbb{R}) \subset PSL(2, \mathbb{C})$ be the holonomy representation of a complete hyperbolic structure on the punctured torus of finite area.

- A representation $\rho : \pi_1(T) \to PSL(2, \mathbb{C})$ is a *quasiconformal deformation* of ρ_0 if there is a quasiconformal homeomorphism $w : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that $\rho = w \circ \rho_0 \circ w^{-1}$, i.e., $\rho(g) = w \circ \rho_0(g) \circ w^{-1}$ for any $g \in \pi_1(T)$.

- The *quasifuchsian space* $\mathcal{QF}$ of the punctured torus is the space of conjugacy classes of quasiconformal deformations of ρ_0. We regard $\mathcal{QF}$ as a subspace of the space $\mathcal{R}$ of $PSL(2,\mathbb{C})$-representations of $\pi_1(T)$.

- We denote the closure of $\mathcal{QF}$ in $\mathcal{R}$ by $\overline{\mathcal{QF}}$.

Proposition 4.2. *Let ρ be an arbitrary element of $\overline{\mathcal{QF}}$. Then the quotient manifold $\mathbb{H}^3/\operatorname{Im}\rho$ is homeomorphic to the product space $T \times (-1,1)$. The domain of discontinuity of the Kleinian group $\operatorname{Im}\rho$ is the disjoint union of two $(\operatorname{Im}\rho)$-invariant subsets $\Omega^{\pm}$ which correspond to the "ends" $e^- = T \times (-1,-1+\delta)$ and $e^+ = T \times (1-\delta,1)$ of $T \times (-1,1)$ respectively, and each Ω^{ϵ} $(\epsilon = \pm)$ satisfies one of the following conditions.*

(i) *Ω^{ϵ} is homeomorphic to the open disk, and $\Omega^{\epsilon}/\operatorname{Im}\rho$ is homeomorphic to T.*

(ii) *Ω^{ϵ} is the countable union of open disks, and $\Omega^{\epsilon}/\operatorname{Im}\rho$ is homeomorphic to the thrice-punctured sphere.*

(iii) *Ω^{ϵ} is empty.*

Definition 4.3. The end satisfying one of the conditions (i) and (ii) of Proposition 4.2 is said to be *geometrically finite*, and one satisfying the condition (iii) is said to be *geometrically infinite.*

Definition 4.4. For every $\rho \in \overline{\mathcal{QF}}$, the *end invariant* $\lambda^{\epsilon}(\rho)$ of each end e^{ϵ} of $\mathbb{H}^3/\operatorname{Im}\rho$ is defined to be a point of the Thurston compactification, canonically identified with $\overline{\mathbb{H}^2}$, of the Teichmüller space of T as follows. Let Ω^{ϵ} be the subset of the domain of discontinuity of $\operatorname{Im}\rho$ corresponding to the end e^{ϵ}.

(i) If Ω^{ϵ} is homeomorphic to the open disk, then $\lambda^{\epsilon}(\rho) \in \mathbb{H}^2$ is the marked conformal structure on T defined by $\Omega^{\epsilon}/\operatorname{Im}\rho$.

(ii) If Ω^{ϵ} is the countable union of open disks, then $\lambda^{\epsilon}(\rho) \in \partial\mathbb{H}^2$ is the marked conformal structure on T with nodes defined by $\Omega^{\epsilon}/\operatorname{Im}\rho$.

(iii) If Ω^{ϵ} is empty, then there is a sequence of closed geodesics in $\mathbb{H}^3/\operatorname{Im}\rho$ which exits the end e^{ϵ}. $\lambda^{\epsilon}(\rho) \in \partial\mathbb{H}^2$ is defined to be the limit of the sequence.

Theorem 4.5 (Minsky [11]). *The end invariant map $\boldsymbol{\lambda} = (\lambda^-,\lambda^+) : \overline{\mathcal{QF}} \to \overline{\mathbb{H}^2} \times \overline{\mathbb{H}^2} - \operatorname{diag}(\partial\mathbb{H}^2)$ is a bijection and its inverse is a continuous map.*

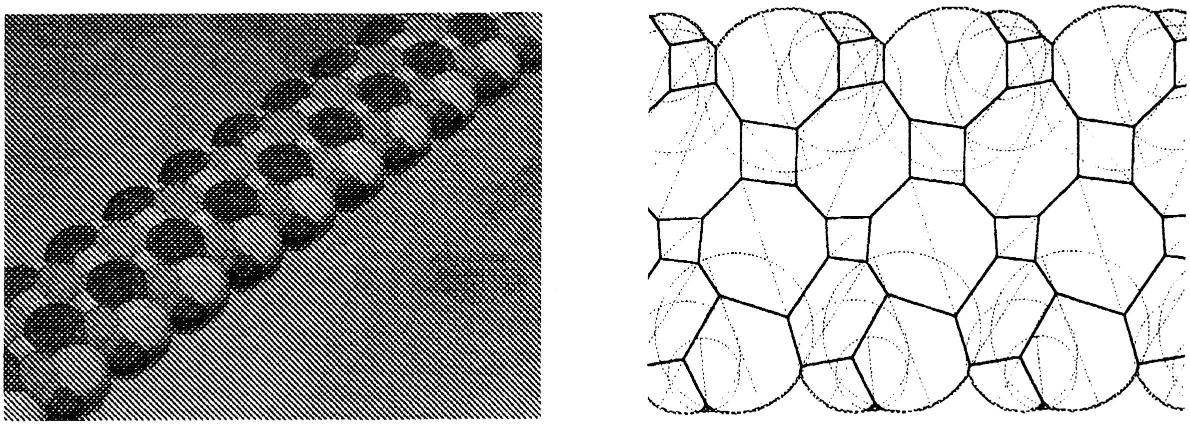

Figure 4: Ford domain of a generic quasifuchsian punctured torus group

Remark 4.6. A *punctured torus group* is a Kleinian group which is freely generated by two elements with parabolic commutator. In fact, Theorem 4.5 is true for the set of all punctured torus groups (Solution of the Ending Lamination Conjecture for punctured torus). In particular, it is proved that the set of all (marked) punctured torus groups is equal to $\overline{\mathcal{QF}}$.

Example 4.7. The Ford domain of a generic quasifuchsian punctured torus group looks like Figure 4. Its combinatorial structure is described by using the "side parameter" defined in Definition 5.2. The upper and lower boundary components in the right figure define two spines of T. By following $\partial Ph(\rho)$ from the lower component to the upper, one finds the sequence of Whitehead moves connecting the two spines (cf. [3]).

Fix a framing $\{\alpha, \beta\} \subset H_1(T)$ and a peripheral element K of $\pi_1(T)$.

Definition 4.8. We call a pair of elements, (A, B), of $\pi_1(T)$ a *generator pair* if A and B generates $\pi_1(T)$ and satisfies $ABA^{-1}B^{-1} = K$. For such a pair, A (resp. B) is called a *left* (resp. *right*) *generator*, or simply a *generator*.

Remark 4.9. The situation may be more clear if we introduce the notion of *elliptic generator triple*, for which we need to extend the group $\pi_1(T)$ to the fundamental group of the orbifold obtained as the quotient space of T by the hyperelliptic involution (cf. [3]).

One can see that every generator in the above sense has a simple closed curve in T as a representative.

Definition 4.10. For each generator X which represents an element $p\alpha + q\beta \in H_1(T)$, the *slope*, $s(X)$, of X is defined by $p/q \in \widehat{\mathbb{Q}} := \mathbb{Q} \cup \{\infty\}$.

Definition 4.11. The Farey triangulation of $\mathbb{H}^2$ is an ideal triangulation consisting of the ideal triangles $\{\gamma\sigma_0 \mid \gamma \in PSL(2,\mathbb{Z})\}$, where σ_0 is the ideal triangle with vertices $\infty, 0, 1 \in \partial\mathbb{H}^2$.

Lemma 4.12. *The following holds.*

1. *For any generator pair (A, B), the slopes of A, AB and B span an ideal triangle in the Farey triangulation.*

2. *For any ideal edge (resp. ideal triangle) σ in the Farey triangulation, there is a generator pair (A, B) such that the slopes of A and B (resp. A, AB and B) span σ.*

The space of type-preserving representations is parametrized as follows (cf. [3]).

Definition 4.13. Let $\mathcal{R}_0$ be the space of type-preserving irreducible representations of $\pi_1(T)$ to $SL(2,\mathbb{C})$ up to conjugation, and set $\mathcal{M} = \{(x,y,z) \in \mathbb{C}^3 \mid x^2+y^2+z^2 = xyz\} - \{(0,0,0)\}$. Fix a generator pair (A_0, B_0) and let $\Psi : \mathcal{R}_0 \to \mathbb{C}^3$ be the map which sends $[\rho]$ to $(\mathrm{Tr}\,\rho(A_0), \mathrm{Tr}(A_0B_0), \mathrm{Tr}(B_0))$.

Proposition 4.14. *The image of Ψ is equal to $\mathcal{M}$, and it is a homeomorphism onto the image.*

Definition 4.15. An element of $\mathcal{M}$ is called a *Markoff triple*.

5 Jorgensen theory

In this section, we briefly review the work of Jorgensen [8] on the characterization of combinatorial structures of punctured torus groups. (See [3] for a complete proof of Jorgensen's results for quasifuchsian punctured torus groups.) In what follows, for any $\rho \in \overline{\mathcal{QF}}$, we denote $P(\mathrm{Im}\,\rho)$ (resp. $Ph(\mathrm{Im}\,\rho)$) by $P(\rho)$ (resp. $Ph(\rho)$) for simplicity.

Theorem 5.1. *For any $\rho \in \mathcal{QF}$, $P(\rho) \subset \mathbb{C}$ consists of precisely two connected components $P^{\pm}(\rho)$, where $P^-(\rho)$ (resp. $P^+(\rho)$) is the component which is lower (resp. higher) than the other in $\mathbb{C}$. For each $\epsilon \in \{-,+\}$, there is a sequence $\{A_j^\epsilon\}$ of generators of $\pi_1(T)$ such that $\partial P^\epsilon(\rho)$ is the union of circular edges e_j^ϵ $(j \in \mathbb{Z})$ with the following property.*

(i) *For any $j, k \in \mathbb{Z}$, it follows that $s(A_{j+3k}^\epsilon) = s(A_j^\epsilon)$, and the three slopes $s(A_0^\epsilon), s(A_1^\epsilon), s(A_2^\epsilon)$ span a triangle σ^ϵ of $\mathcal{D}$.*

(ii) *For any $j \in \mathbb{Z}$, e_j^ϵ is contained in $I(\rho(A_j^\epsilon))$.*

(iii) *If we denote by* θ_j^ϵ *the half of the angle of* e_j^ϵ *in* $I(\rho(A_j^\epsilon))$, *then*

$$\theta_0^\epsilon + \theta_1^\epsilon + \theta_2^\epsilon = \pi/2.$$

Definition 5.2 (side parameter). For any $\rho \in \mathcal{QF}$, we define the two points $\nu^\pm(\rho)$ in $\mathbb{H}^2$ as follows. For each $\epsilon \in \{-,+\}$, let σ^ϵ be the triangle in $\mathcal{D}$ determined by Theorem 5.1. Then $\nu^\epsilon(\rho)$ is the point in the triangle σ^ϵ with barycentric coordinate $(\theta_0^\epsilon, \theta_1^\epsilon, \theta_2^\epsilon)$. The point $\boldsymbol{\nu}(\rho) = (\nu^-(\rho), \nu^+(\rho)) \in \mathbb{H}^2 \times \mathbb{H}^2$ is called the *side parameter* of ρ.

Theorem 5.3. (1) *For any* $\rho \in \mathcal{QF}$, *the combinatorial structure of* $Ph(\rho)$ *is described by using* $\boldsymbol{\nu}(\rho)$.

(2) *The map* $\boldsymbol{\nu} : \mathcal{QF} \to \mathbb{H}^2 \times \mathbb{H}^2$ *is a homeomorphism.*

The following theorem gives an extension of the side parameter to $\overline{\mathcal{QF}}$. (See [1] for an outline.)

Theorem 5.4. *The map* $\boldsymbol{\nu} : \mathcal{QF} \to \mathbb{H}^2 \times \mathbb{H}^2$ *is extended to a map* $\boldsymbol{\nu} = (\nu^-, \nu^+) : \overline{\mathcal{QF}} \to \overline{\mathbb{H}^2} \times \overline{\mathbb{H}^2} - \mathrm{diag}(\partial\mathbb{H}^2)$ *with the following property.*

(1) *For any* $\rho \in \overline{\mathcal{QF}}$, *the combinatorial structure of* $Ph(\rho)$ *is described by using* $\boldsymbol{\nu}(\rho)$.

(2) *The map* $\boldsymbol{\nu}$ *is surjective, and it is continuous in the strong topology on* $\overline{\mathcal{QF}}$.

(3) *For each* $\epsilon = \pm$, $\nu^\epsilon(\rho) \in \partial\mathbb{H}^2$ *if and only if* $\lambda^\epsilon(\rho) \in \partial\mathbb{H}^2$. *Moreover, under the mutually equivalent conditions, it follows that* $\nu^\epsilon(\rho) = \lambda^\epsilon(\rho)$.

Since the fundamental group of a punctured torus bundle contains the fundamental group of the fiber surface as a normal subgroup, we obtain the following corollary, which is first proved by Lackenby [10] with a topological argument.

Corollary 5.5. *For any hyperbolic punctured torus bundle over the circle, the Ford domain of the image of the holonomy representation of the complete hyperbolic structure is dual to the "Jorgensen's triangulation" (cf. [6]).*

6 Deformation space for $M(p,q)$

Fix a pair of coprime integers $(p,q) \neq (0, \pm 1)$, and set $M = M(p,q)$. We shall denote by $\mathcal{MP}$ the space of geometrically finite hyperbolic structures on the pared manifold (M, P) with the parabolic locus P. Then, by Corollary 3.3, $\mathcal{MP}$ is not empty, and hence is isomorphic to the square of the Teichmüller space $\mathrm{Teich}(T) \times \mathrm{Teich}(T)$ by the Marden's isomorphism theorem.

By using a presentation of $\pi_1(M)$, we can embed $\mathcal{MP}$ into an affine space.

Definition 6.1. Let $\mathcal{E} : \mathcal{MP} \to \mathcal{R}_0 \times \mathcal{R}_0$ be the map defined as follows. For any element of $\mathcal{MP}$, let $\rho : \pi_1(M) \to SL(2, \mathbb{C})$ be (a lift of) the holonomy representation. Then its image by $\mathcal{E}$ is defined to be $(\rho|_{\pi_1(T_0^-)}, \rho|_{\pi_1(T_0^+)})$. (Since $\mathcal{MP}$ is connected, it is well-defined by fixing a base-point and a lift at the point.)

Let $\widehat{\Psi} = \Psi^- \times \Psi^+ : \mathcal{R}_0 \times \mathcal{R}_0 \to \mathcal{M} \times \mathcal{M}$ be the product map, where each Ψ^ϵ $(\epsilon = \pm)$ is defined from the generator pair $(\alpha^\epsilon, \beta^\epsilon)$. By Lemma 3.4 and the Covering Theorem (cf. [4]), we obtain the following proposition.

Proposition 6.2. *The image of $\mathcal{E}$ is contained in*

$$(\mathcal{QF} \times \mathcal{QF}) \cap \widehat{\Psi}^{-1}\left(\left\{\left((x^-, y^-, z^-), (x^+, y^+, z^+)\right) \mid x^- = x^+\right\}\right).$$

Remark 6.3. One obtains another polynomial equation in $\mathcal{M} \times \mathcal{M}$ for $\mathcal{E}(\mathcal{MP})$ from the relation coming from the Dehn filling.

7 Ford domains for structures in $\mathcal{MP}$

To answer Problem 1.1 for the pared manifold (M, P) with a coprime integers $(p, q) \neq (0, \pm 1)$, we will follow the following program.

(1) Construct a geometrically finite hyperbolic structure on the pared manifold $(M_0, P \cup \partial N(\gamma))$ with the parabolic locus $P \cup \partial N(\gamma)$.

(2) Construct a geometrically finite hyperbolic structure in $\partial \mathcal{MP}$ by hyperbolic Dehn surgery on the structure obtained in (1).

(3) By using the "geometric continuity" argument, which is used in the Jorgensen theory, characterize the combinatorial structures of Ford domains of the structures in $\mathcal{MP}$.

Step (1) in the program is already done (see Figure 5), which is obtained from Jorgensen's characterization; just take the "double" of the Ford domain of a double cusp group.

Step (2) is done by studying the Ford domains after hyperbolic Dehn surgery (cf. [2]). Figure 6 is the Ford domain for a structure in $\partial \mathcal{MP}$ for

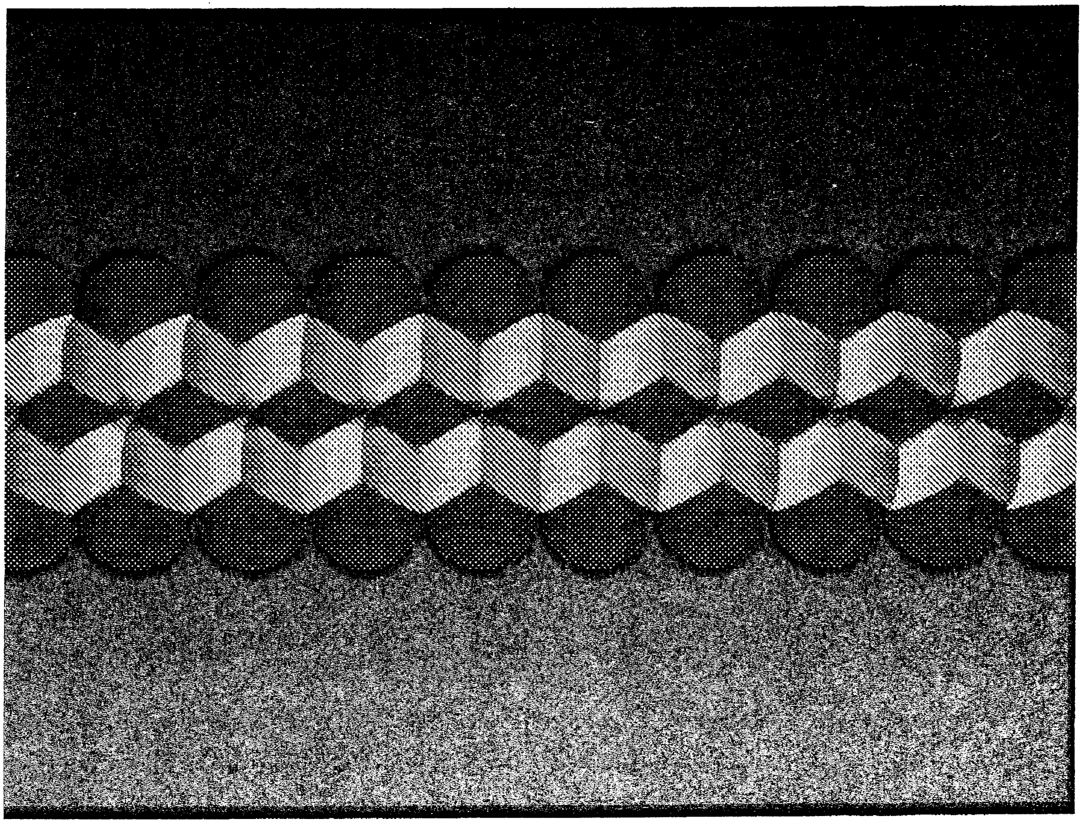

Figure 5: Ford domain of a structure on $(M_0, P \cup \partial N(\gamma))$

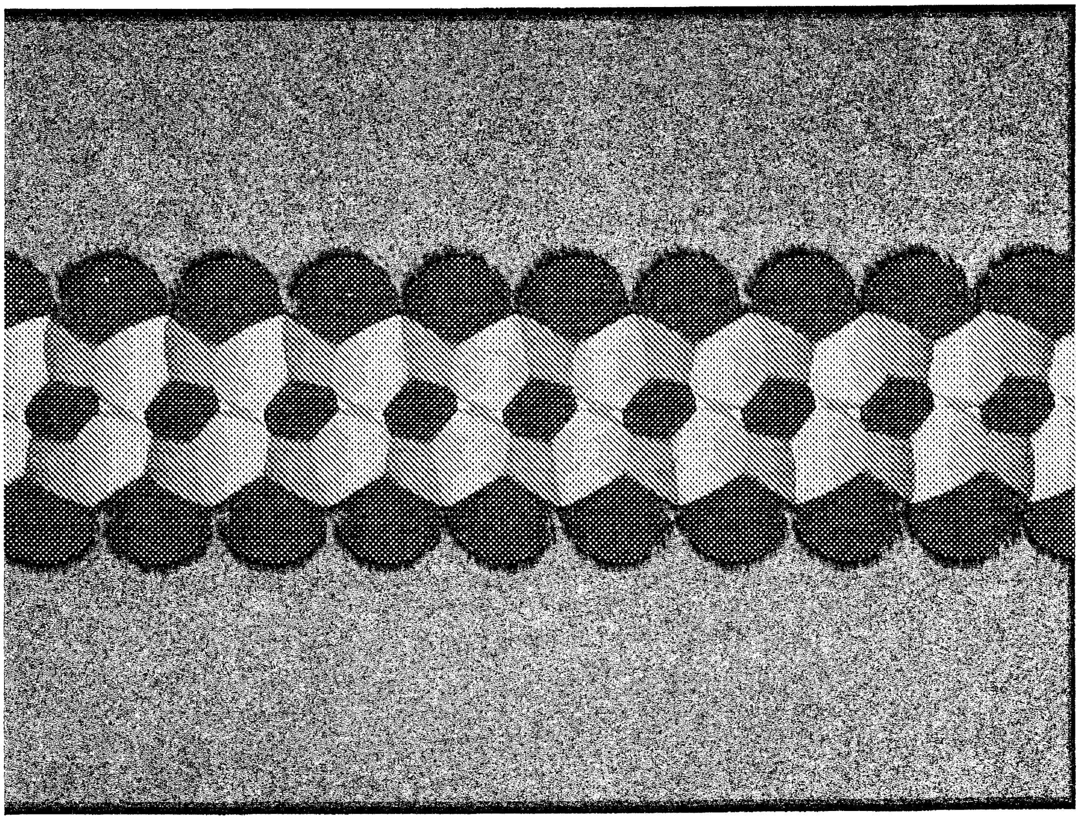

Figure 6: Ford domain of a structure in $\partial\mathcal{MP}$ for $(p, q) = (3, 5)$

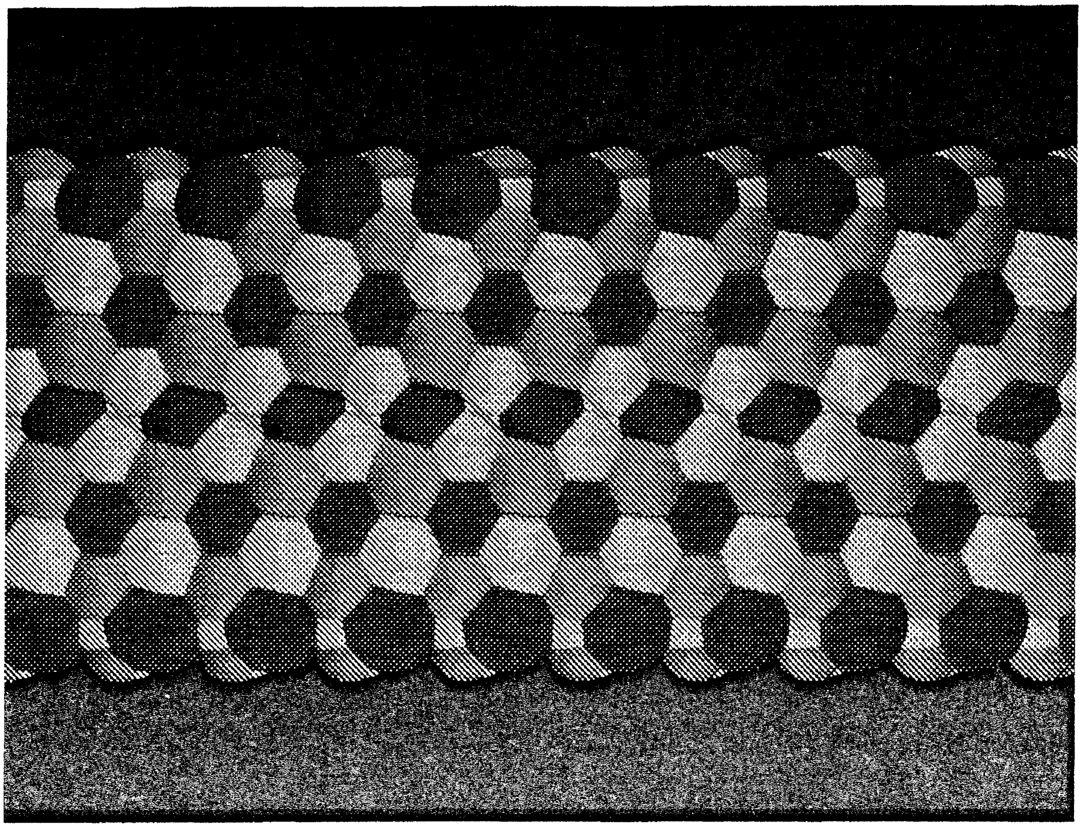

Figure 7: Ford domain corresponding to the "fixed point" in $\mathcal{MP}$

$(p, q) = (3, 5)$. It is roughly the combination of the Ford domain obtained by Step (1) and the Ford domain of some cyclic Kleinian group (see Figure 2).

Let $\mathcal{MP}_{\mathrm{sym}}$ be the subspace of $\mathcal{MP}$ consisting of the structures whose image by $\widehat{\Psi} \circ \mathcal{E}$ is of the form $((x, y, z), (x, z, y))$. For a gluing map $T_0^- \to T_0^+$ with certain "symmetry", the invariant hyperbolic structure is contained in $\mathcal{MP}_{\mathrm{sym}}$. The parameters of some of such structures are explicitly determined. Figure 7 is the Ford domain of such a structure in $\mathcal{MP}_{\mathrm{sym}}$ for $(p, q) = (3, 5)$.

Conjecture 7.1. *(1) An analogue of Jorgensen's theory is valid for* $\mathcal{MP}_{\mathrm{sym}}$.

(2) For any $(p, q) \neq (0, \pm 1)$ *and any "symmetric" pseudo-Anosov homeomorphism* $\varphi : T_0^- \to T_0^+$, *the Ford domain of the complete hyperbolic structure on* M/φ *has a "good" combinatorial structure. (This should be a corollary to the assertion (1).)*

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