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POSITIVE TOEPLITZ OPERATORS AND HERZ SPACES ON PLURIHARMONIC BERGMAN SPACES

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1. Introduction

For a fixed integer $n \geq 2$, let $\mathbb{C}^n$ denote the Euclidean space of complex dimension n and let $B = B_n$ denote the open unit ball in $\mathbb{C}^n$. For $1 \leq p < \infty$, the pluriharmonic Bergman space $b^p = b^p(B)$ is the set of all complex-valued pluriharmonic functions φ on B such that

$$\|\varphi\|_p = \left\{ \int_B |\varphi|^p \, dV \right\}^{1/p} < \infty$$

where V denotes the Lebesgue volume measure on B. For $1 \leq p \leq \infty$, let $L^p = L^p(V)$ be the Lebesgue spaces on B. Since b^2 is a closed subspace of L^2, it is a Hilbert space.

Since each point evaluation is a bounded linear functional on b^2, for each $z \in B$, there exists a unique function $R_z \in b^2$ which has the reproducing property :

$$\varphi(z) = \int_B \varphi(w) \overline{R_z(w)} \, dV(w) \qquad (z \in B) \tag{1.1}$$

for all $\varphi \in b^2$. More explicitly, the kernel R_z is given by (see [5])

$$R_z(w) = \frac{1}{(1 - w \cdot \overline{z})^{n+1}} + \frac{1}{(1 - z \cdot \overline{w})^{n+1}} - 1 \tag{1.2}$$

for $z, w \in B$. Here, $z \cdot \overline{w} = z_1 \overline{w}_1 + \cdots + z_n \overline{w}_n$ denotes the Hermitian inner product on $\mathbb{C}^n$. From the explicit formula of R_z, one can see that

$$|R_z(w)| \leq \frac{C}{|1 - z \cdot \overline{w}|^{n+1}} \qquad (z, w \in B) \tag{1.3}$$

so that $R_z \in L^\infty$.

Associated with the kernel function R_z is the integral operator

$$R\psi(z) = \int_B \psi(w) R_z(w) \, dV(w) \qquad (z \in B) \tag{1.4}$$

which takes L^p-functions into pluriharmonic functions on B. In fact, R is the Hilbert space orthogonal projection from L^2 onto b^2 and R is a bounded projection from L^p onto b^p for $1 < p < \infty$. The integral transform can be extended to $\mathcal{M}$, the space of all complex Borel measures on B. That is to say, for each $\mu \in \mathcal{M}$, the integral

$$R\mu(z) = \int_B R_z(w) \, d\mu(w) \qquad (z \in B)$$

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defines a function pluriharmonic on B.

For $\mu \in \mathcal{M}$, the Toeplitz operator T_μ with symbol μ is defined by

$$T_\mu f = R(f d\mu)$$

for $f \in b^2 \cap L^\infty$. In case $d\mu = \varphi\, dV$, we write $T_\mu = T_\varphi$. Note that T_μ is defined on a dense subset of b^2, because bounded pluriharmonic functions form a dense subset of b^2. A Toeplitz operator T_μ is called *positive* if $\mu \in \mathcal{M}$ is a positive (finite) Borel measure (we will simply write $\mu \geq 0$).

From now on, we let λ denote the measure on B defined by

$$d\lambda(z) = R_z(z) dV(z).$$

The purpose this lecture is to announce a recent joint work with Choi concerning characterizations of positive Toeplitz operators of Schatten-Herz type (we will define this in the Section 5) in terms of averaging functions and Berezin transforms of symbol functions. Details of proofs will appear elsewhere.

We will often abbreviate inessential constants involved in inequalities by writing $X \lesssim Y$ for positive quantities X and Y if the ratio X/Y has a positive upper bound. Also, we write $X \approx Y$ if $X \lesssim Y$ and $Y \lesssim X$.

2. Averaging functions and Berezin transforms

In this section we define averaging functions and Berezin transforms and briefly review their properties.

For $\mu \in \mathcal{M}$, its Berezin transform $\widetilde{\mu}$ is a function on B defined by

$$\widetilde{\mu}(z) = \int_B |r_z(w)|^2\, d\mu(w) \qquad (z \in B)$$

where

$$r_z = \frac{R_z}{\|R_z\|_2}$$

is the normalized reproducing kernel. For $\varphi \in L^1$, we define $\widetilde{\varphi} = \widetilde{\mu}$ where $d\mu = \varphi\, dV$. The notion of Berezin transform can be extended to non-integrable functions which belong to some weighted Lebesgue spaces. See Proposition 4.5 below.

Fix $z \in B$. Let P_z be the orthogonal projection of $\mathbb{C}^n$ onto the subspace $\langle z \rangle$ generated by z, and let $Q_z = I - P_z$ be the projection on the orthogonal complement of $\langle z \rangle$. To be quite explicit, $P_0 = 0$ and

$$P_z(w) = \frac{w \cdot \bar{z}}{|z|^2}\, z \qquad if\ z \neq 0.$$

For $z, w \in B$, $z \neq 0$, define

$$\varphi_z(w) = \frac{z - P_z(w) - (1 - |z|^2)^{\frac{1}{2}} Q_z(w)}{1 - w \cdot \bar{z}}.$$

Each φ_z is a biholomorphic self-map of B and $\varphi_z \circ \varphi_z$ is the identity on B. Note that

$$1 - |\varphi_z(w)|^2 = \frac{(1 - |z|^2)(1 - |w|^2)}{|1 - z \cdot \overline{w}|^2} \qquad (z,\ w \in B).$$

See Section 2 of [5] for details.

The pseudo-hyperbolic ball $E_r(z)$ with center $z \in B$ and radius $r \in (0,1)$ is defined by

$$E_r(z) = \varphi_z(rB).$$

Since φ_z is an involution, $w \in E_r(z)$ if and only if $|\varphi_z(w)| < r$. It is known that

$$|E_r(z)| \approx (1-|z|^2)^{n+1}.$$

Here, and in what follows, we write $|E| = V(E)$ for all Borel sets for $E \subset B$. For $r \in (0,1)$ fixed, the averaging function $\widehat{\mu}_r$ is defined by

$$\widehat{\mu}_r(z) = \frac{\mu[E_r(z)]}{|E_r(z)|} \qquad (z \in B).$$

Also, we let $\widehat{\mu}_r = \widehat{\varphi}_r$ for $d\mu = \varphi\, dV$.

The estimate of the kernel function along the diagonal is easily seen. Namely, it is straightforward from (1.2) to see that

$$R_z(z) = \|R_z\|_2^2 \approx (1-|z|)^{-n-1} \tag{2.1}$$

for $z \in B$. This estimate continues to hold even for points staying sufficiently close to the diagonal in the sense that there exists some $r_0 \in (0,1)$ for which we have

$$R_z(w) \approx (1-|z|)^{-n-1} \tag{2.2}$$

whenever $z \in B$ and $w \in E_{r_0}(z)$ (see Lemma 2.1 of [2]). The following proposition says that averaging functions over balls of small radii are dominated by Berezin transforms. This consequence will be useful for our purpose.

Proposition 2.1. *Given $r \in (0, r_0)$, there exits a constant C_r such that $\widehat{\mu}_r \le C_r \widetilde{\mu}$ for $\mu \ge 0$.*

The following proposition comes from Lemma 3.1 of [2].

Proposition 2.2. *Given $r, s \in (0,1)$, there exists a constant $C_{r,s}$ such that $\widehat{\mu}_{r,s} \le C_r \widehat{(\widehat{\mu}_s)}_r$ for $\mu \ge 0$.*

Using subharmonicity and Fubini's theorem, we get the following: given $r \in (0,1)$, there is a constant C_r such that

$$\int_B f\, d\mu \le C_r \int_B f \widehat{\mu}_r dV$$

for all $f \ge 0$ subharmonic on B and $\mu \ge 0$.

The next proposition is an immediate consequence of the above estimate.

Proposition 2.3. *Given $r \in (0,1)$, there exists a constant C_r such that $\widetilde{\mu} \le C_r \widetilde{(\widehat{\mu}_r)}$ for $\mu \ge 0$.*

3. Schatten classes and Herz spaces

In this section we introduce Schatten classes and Herz spaces. We first recall Schatten class operators. For a compact operator T on b^2, let $\{s_m(T)\}$ be the nonzero eigenvalues with multiplicity of $|T| = (T^*T)^{1/2}$ arranged so that the sequence is non-increasing, where T^* denotes the Hilbert space adjoint of T. This sequence is called the singular value sequence of T. For $1 \le p < \infty$, we say that T is a Schatten p-class operator if the singular value sequence $\{s_m(T)\}$ belongs to ℓ^p. Let S_p be the space of all Schatten p-class operators on b^2. The space S_p is then a Banach space equipped with the norm

$$\|T\|_{S_p} = \left\{ \sum_m |s_m(T)|^p \right\}^{1/p}.$$

See [6], for example, for more information and related facts. Also, we denote by S_∞ the class of all bounded linear operators on b^2 and let $\|T\|_{S_\infty}$ denote the operator norm $\|T\|$ of $T \in S_\infty$.

The following proposition, taken from Theorem 3.13 of [2], expresses the characterization for a positive Toeplitz operator to be a member of the Schatten class S_p. Note that the case $p = \infty$ gives characterizations for boundedness, which is also included in Proposition 3.2 below. We did so for easier reference later.

Proposition 3.1. *Let $1 \le p \le \infty$, $r \in (0,1)$ and $\mu \ge 0$. Then the following conditions are equivalent:*

(a) $T_\mu \in S_p$.
(b) $\widetilde{\mu} \in L^p(\lambda)$.
(c) $\widehat{\mu}_r \in L^p(\lambda)$.

Moreover, the equivalences $\|T_\mu\|_{S_p} \approx \|\widetilde{\mu}\|_{L^p(\lambda)} \approx \|\widehat{\mu}_r\|_{L^p(\lambda)}$ hold.

We also mention that corresponding characterizations for boundedness and compactness come from Theorem 3.9 and Theorem 3.12 of [2]. Here, L_0 denote the space of all bounded functions f on B such that $f(z) \to 0$ as $|z| \to 1$.

Proposition 3.2. *Let $r \in (0,1)$ and $\mu \ge 0$. Then the following conditions are equivalent:*

(a) *T_μ is bounded (compact) on b^2.*
(b) $\widetilde{\mu} \in L^\infty$ (L_0).
(c) $\widehat{\mu}_r \in L^\infty$ (L_0).
(d) *The inclusion $J_\mu : b^2 \subset L^2(\mu)$ is bounded (compact).*

Moreover, the equivalences $\|T_\mu\| \approx \|\widetilde{\mu}\|_{L^\infty} \approx \|\widehat{\mu}_r\|_{L^\infty} \approx \|J_\mu\|^2$ hold.

The above proposition yields the following result.

Proposition 3.3. *Let $\mu \in \mathcal{M}$ and assume that $T_{|\mu|}$ is bounded on b^2. Then T_μ is bounded on b^2 and*

$$\|T_\mu\| \le C\|T_{|\mu|}\|$$

for some constant C independent of μ. If, in addition, $T_{|\mu|}$ is compact on b^2, then T_μ is also compact on b^2.

Finally, we recall the Herz spaces on the ball. We let

$$A_m = \{z \in B \ : \ r_m \le |z| < r_{m+1}\}$$

where $r_m = 1 - 2^{-m}$ for each integer $m \ge 0$. We will write χ_m for the characteristic function of A_m for each m. Also, given $\mu \in \mathcal{M}$, we let $\mu\chi_m$ stand for the restriction of μ to A_m for each m. Let α be real and $1 \le p, q \le \infty$. Then the Herz space $\mathcal{K}_q^{p,\alpha}$ is the space consisting of all functions $f \in L^p_{\text{loc}}(V)$ such that

$$\|f\|_{\mathcal{K}_q^{p,\alpha}} = \left\| \left\{ 2^{-m\alpha} \|f\chi_m\|_{L^p} \right\} \right\|_{\ell^q} < \infty.$$

Equipped with the norm above, the space $\mathcal{K}_q^{p,\alpha}$ is a Banach space.

Given γ real, let V_γ denote the weighted measure on B defined by

$$dV_\gamma(z) = (1 - |z|^2)^\gamma \, dV(z).$$

Let $1 \le p < \infty$ and α real. Then, given $m \ge 0$, we have $1 - |z|^2 \approx 2^{-m}$ for $z \in A_m$ and thus we obtain

$$\begin{aligned} 2^{-m\alpha}\|f\chi_m\|_{L^p} &= \left\{ \int_{A_m} \left(2^{-m\alpha}|f(z)|\right)^p \, dV(z) \right\}^{1/p} \\ &\approx \left\{ \int_{A_m} (1 - |z|^2)^{\alpha p} |f(z)|^p \, dV(z) \right\}^{1/p} \\ &\approx \|f\chi_m\|_{L^p(V_{\alpha p})} \end{aligned}$$

and this estimate is uniform in m. It follows that

$$\|f\|_{\mathcal{K}_q^{p,\alpha}} \approx \left\| \left\{ \|f\chi_m\|_{L^p(V_{\alpha p})} \right\} \right\|_{\ell^q} \tag{3.1}$$

for $1 \le q \le \infty$. In particular, since $\lambda \approx V_{-n-1}$, we have

$$\|f\|_{\mathcal{K}_q^{p,-(n+1)/p}} \approx \left\| \left\{ \|f\chi_m\|_{L^p(\lambda)} \right\} \right\|_{\ell^q}$$

and this estimate is easily seen to be valid even for $p = \infty$. So, equipped with the norm of $\mathcal{K}_q^{p,-(n+1)/p}$, the space $\mathcal{K}_q^p(\lambda)$ is precisely the same as $\mathcal{K}_q^{p,-(n+1)/p}$ for the full range $1 \le p, q \le \infty$. Also, note that

$$\mathcal{K}_p^p(\lambda) \approx L^p(\lambda) \tag{3.2}$$

for $1 \le p \le \infty$. That is, these two spaces are the same as sets and have equivalent norms as Banach spaces.

Note that Hölder's inequality holds in the Herz space as follows :

$$\left| \int_B f\bar{g} \, dV \right| \le \|f\|_{\mathcal{K}_q^{p,\alpha}} \|g\|_{\mathcal{K}_{q'}^{p',-\alpha}} \tag{3.3}$$

for functions $f \in \mathcal{K}_q^{p,\alpha}$ and $g \in \mathcal{K}_{q'}^{p',-\alpha}$ for the full range $1 \le p, q \le \infty$ and arbitrary α real (see [1] for details). Here, and in what follows, p' is the conjugate exponent of p.

4. Various mapping properties

For our main result (Theorem 5.2) in the next section, we need to establish various mapping properties of the Berezin transform. We begin with the Herz norm estimates of the kernel function. For that purpose, we need the following fact (see Proposition 1.4.10 of [5]). Here dS is the surface area measure on ∂B, the boundary of B.

Lemma 4.1. *For $-1 < \alpha < \infty$ and c real, let*

$$J_c(z) = \int_{\partial B} \frac{dS(\zeta)}{|1 - z \cdot \overline{\zeta}|^{n+c}}$$

and

$$I_{\alpha,c}(z) = \int_B \frac{(1-|w|)^\alpha}{|1 - z \cdot \overline{w}|^{n+1+\alpha+c}}\, dV(w)$$

for $z \in B$. Then the following estimates hold:

$$J_c(z) \approx I_{\alpha,c}(z) \approx \begin{cases} 1 & \text{if } c < 0 \\ \log \frac{1}{1-|z|^2} & \text{if } c = 0 \\ \frac{1}{(1-|z|^2)^c} & \text{if } c > 0 \end{cases}$$

as $|z| \to 1$.

Lemma 4.2. *Let $1 \le p, q \le \infty$ and assume $-1/p < \alpha < 2(n+1) - (n+1)/p$. Then there exists a constant $C = C_{\alpha,p,q}$ such that*

$$\|R_z^2\|_{\mathcal{K}_q^{p,\alpha}} \le \frac{C}{(1-|z|)^{2(n+1)-(n+1)/p-\alpha}}$$

for $z \in B$.

The following proposition ensures that the Berezin transform continuously takes the Herz spaces $\mathcal{K}_q^p(\lambda)$ into L^∞.

Proposition 4.3. *Let $1 \le p, q \le \infty$. There exists a constant $C = C_{p,q}$ such that*

$$\|\widetilde{\varphi}\|_{L^\infty} \le C \|\varphi\|_{\mathcal{K}_q^p(\lambda)}$$

for functions $\varphi \in \mathcal{K}_q^p(\lambda)$.

As a corollary we see that Toeplitz operators with $\mathcal{K}_q^p(\lambda)$-symbols are compact when q is finite.

Corollary 4.4. *Let $1 \le p \le \infty$, $1 \le q < \infty$ and $\varphi \in \mathcal{K}_q^p(\lambda)$. Then T_φ is compact on b^2.*

We now turn to the boundedness of the Berezin transform on the spaces $\mathcal{K}_q^{p,\alpha}$ for certain range of parameters. As a preliminary step we establish the boundedness of the Berezin transform on the weighted Lebesgue spaces $L^p(V_\gamma)$. We actually prove a more general version in the same way as Theorem 1.9 of [3].

Given α and β real, we let

$$\Psi_{\alpha,\beta} f(z) = (1-|z|^2)^\alpha \int_B \frac{f(w)}{(1 - z \cdot \overline{w})^{n+1+\alpha+\beta}}\, dV_\beta(w)$$

and

$$\Phi_{\alpha,\beta}f(z) = (1-|z|^2)^\alpha \int_B \frac{f(w)}{|1-z\cdot\overline{w}|^{n+1+\alpha+\beta}}\, dV_\beta(w)$$

for $z \in B$.

Proposition 4.5. *Let $1 \le p < \infty$ and α, β, γ be real. Then the following conditions are equivalent:*

(a) *$\Psi_{\alpha,\beta}$ is bounded on $L^p(V_\gamma)$.*
(b) *$\Phi_{\alpha,\beta}$ is bounded on $L^p(V_\gamma)$.*
(c) $-p\alpha < \gamma + 1 < p(\beta+1)$.

The following shows that $\mathcal{K}_q^{p,\alpha}$-boundedness of the Berezin transform.

Proposition 4.6. *Let $1 \le p < \infty$, $1 \le q \le \infty$ and α be real. If $-(n+1) - 1/p < \alpha < 1/p'$, then the Berezin transform is bounded on $\mathcal{K}_q^{p,\alpha}$. In particular, the Berezin transform is bounded on $\mathcal{K}_q^p(\lambda)$.*

Let $\mathfrak{M}_+$ be the class of all positive (possibly infinite) measurable functions on B. Also, note that the Berezin transform takes $\mathfrak{M}_+$ into itself.

The following lemma is needed for the proof of the implication (b) $\Longleftrightarrow$ (c) in our main result(Theorem5.2).

Lemma 4.7. *Let $f \in \mathfrak{M}_+$. Then*

$$\|\widetilde{f\chi_k}\|_{L^\infty} \lesssim \sum_{m=0}^{\infty} \frac{\|f\chi_m\|_{L^\infty}}{2^{|m-k|}}, \qquad k = 0, 1, \ldots$$

5. Main results

In this section we state our main result Theorem 5.2 and observe some consequences. We start with a simple covering lemma. A straightforward computation shows that the pseudohyperbolic ball $E_r(z)$ consists of all w that satisfy

$$\frac{|P_z(w) - C_z|^2}{r^2\rho_z^2} + \frac{|Q_z(w)|^2}{r^2\rho_z} < 1,$$

where $C_z = (1-r^2)z/(1-r^2|z|^2)$ and $\rho_z = (1-|z|^2)/(1-r^2|z|^2)$. Thus $E_r(z)$ is an ellipsoid with center at C_z.

Lemma 5.1. *Given $r \in (0,1)$, let N be an integer with $N > \log_2\{(1+r)/(1-r)\}$. Then*

$$E_r(z) \subset \bigcup_{k=m-N}^{m+N} A_k$$

for $z \in A_m$ and $m \ge N$. Here, $A_t = \emptyset$ if $t < 0$.

Given $1 \le p, q \le \infty$, the space $S_{p,q}$ consists of all Toeplitz operators T_μ of Schatten-Herz (p,q)-type, meaning that $T_{\mu\chi_m} \in S_p$ for each m and the sequence $\{\|T_{\mu\chi_m}\|_{S_p}\}$ belongs to ℓ^q. The norm of $T_\mu \in S_{p,q}$ is given by

$$\|T_\mu\|_{S_{p,q}} = \left\| \{\|T_{\mu\chi_m}\|_{S_p}\} \right\|_{\ell^q}.$$

Now our main result is stated and we also show the equivalences of the associated norms.

Theorem 5.2. *Let $1 \le p, q \le \infty$, $r \in (0,1)$ and $\mu \ge 0$. Then the following conditions are equivalent:*

(a) $T_\mu \in S_{p,q}$.

(b) $\widetilde{\mu} \in \mathcal{K}_q^p(\lambda)$.

(c) $\widehat{\mu}_r \in \mathcal{K}_q^p(\lambda)$.

Moreover, the equivalences $\|T_\mu\|_{S_{p,q}} \approx \|\widetilde{\mu}\|_{\mathcal{K}_q^p(\lambda)} \approx \|\widehat{\mu}_r\|_{\mathcal{K}_q^p(\lambda)}$ hold.

In the remainder of this section we observe some consequences. Given $r \in (0,1)$, it is not hard to see that the averaging operator $\varphi \mapsto \widehat{\varphi}_r$ is $L^p(\lambda)$-bounded for $p = 1$ or $p = \infty$ and thus for all $1 \le p \le \infty$ by the Riesz-Thorin interpolation theorem. It turns out that the averaging operator is bounded on each of the Herz spaces $\mathcal{K}_q^p(\lambda)$. Combining this fact with Theorem 5.2, we have the boundedness of the Berezin transform on $K_q^\infty(\lambda)$ which is missing in Proposition 4.6.

Corollary 5.3. *Let $1 \le p, q \le \infty$ and $r \in (0,1)$. Then the averaging operator $\varphi \mapsto \widehat{\varphi}_r$ is bounded on $\mathcal{K}_q^p(\lambda)$. Also, the Berezin transform is bounded on $\mathcal{K}_q^p(\lambda)$.*

Next, the following is an immediate consequence of (3.2), Proposition 3.1 and Theorem 5.2.

Corollary 5.4. *Let $1 \le p \le \infty$ and $\mu \ge 0$. Then $T_\mu \in S_{p,p}$ if and only if $T_\mu \in S_p$.*

Also, we observe that the operator norm of positive Toeplitz operators are dominated by their $S_{p,q}$-norms. This consequence in turn implies that operators in $S_{p,q}$ are all compact for finite q.

Corollary 5.5. *Let $1 \le p, q \le \infty$. Assume $\mu \in \mathcal{M}$ and $T_{|\mu|} \in S_{p,q}$. Then*

$$\|T_\mu\| \le C \|T_{|\mu|}\|_{S_{p,q}}$$

for some constant $C = C_{p,q}$ independent of μ. If $q < \infty$ and $d\mu = \varphi dV$ in addition, then T_μ is compact on b^2.

6. Remarks

M. Loaiza, M. López-García and S. Pérez-Esteva ([4]) first introduce mixed norm spaces associated with Schatten classes and decomposed a given positive Toeplitz operator into a family of local operators and then characterized membership in those spaces in terms of the so-called Herz spaces in the holomorphic case on the unit disk. In the case of harmonic Bergman space, B. Choe, H. Koo and K. Na [1] obtained the analogous results on the ball and removed some restriction of the main result of [4].

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