数理解析研究所講究録
1519 巻 2006 年 137-145

REPRESENTATION PROPERTY OF WEIGHTED HARMONIC BERGMAN FUNCTIONS ON THE UPPER HALF-SPACES

KYESOOK NAM

1. INTRODUCTION

Let $\mathbf{H}$ denote the upper half space $\mathbf{R}^{n-1} \times \mathbf{R}_+$ where $\mathbf{R}_+$ denotes the set of all positive real numbers. We will write points $z \in \mathbf{H}$ as $z = (z', z_n)$ where $z' \in \mathbf{R}^{n-1}$ and $z_n > 0$.

For $\alpha > -1$ and $1 \leq p < \infty$, let $b_\alpha^p = b_\alpha^p(\mathbf{H})$ denote the weighted harmonic Bergman space consisting of all real-valued harmonic functions u on $\mathbf{H}$ such that

$$\|u\|_{L_\alpha^p} := \left(\int_{\mathbf{H}} |u(z)|^p \, dV_\alpha(z) \right)^{1/p} < \infty$$

where $dV_\alpha(z) = z_n^\alpha dz$ and dz is the Lebesque measure on $\mathbf{R}^n$. Then we can see easily that the space b_α^p is a Banach space. In particular, b_α^2 is a Hilbert space. Hence, there is a unique Hilbert space orthogonal projection Π_α of L_α^2 onto b_α^2 which is called the weighted harmonic Bergman projection. It is known that this weighted harmonic Bergman projection can be realized as an integral operator against the weighted harmonic Bergman kernel $R_\alpha(z, w)$. See section 2.

The purpose of this paper is to survey [8] concerning the representation property of b_α^p-functions and the interpolation by b_α^p-functions.

In the holomorphic case representation and interpolation properties of Berg man functions have been studied in [5] and [11]. In [5], the representation properties of harmonic Bergman functions, as well as harmonic Bloch functions, were also proved on the unit ball in $\mathbf{R}^n$. See [2] for the interpolation properties of holomorphic (little) Bloch functions. On the setting of the half-space of $\mathbf{R}^n$, Choe and Yi [6] have studied these two properties of harmonic Bergman spaces. In [6], the harmonic (little) Bloch spaces are also considered as limiting spaces of b^p.

2. PRELIMINARIES

First, we introduce the fractional derivative. Let D denote the differentiation with respect to the last component and let $u \in b_\alpha^p$. Then the mean value

This research was supported by KOSEF(R01-2003-000-10243-0) .

property, Jensen's inequality and Cauchy's estimate yield

$$|D^k u(z)| \le c z_n^{-(n+\alpha)/p-k} \tag{2.1}$$

for each $z \in \mathbf{H}$ and for every nonnegative integer k.

Let $\mathcal{F}_\beta$ be the collection of all functions v on $\mathbf{H}$ satisfying $|v(z)| \le c z_n^{-\beta}$ for $\beta > 0$ and let $\mathcal{F} = \cup_{\beta>0}\mathcal{F}_\beta$. If $v \in \mathcal{F}$, then $v \in \mathcal{F}_\beta$ for some $\beta > 0$. In this case, we define the fractional derivative of v of order $-s$ by

$$\mathcal{D}^{-s} v(z) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} v(z', z_n + t)\, dt \tag{2.2}$$

for the range $0 < s < \beta$. (Here, Γ is the Gamma function.)

If $u \in b^p_\alpha$, then for every nonnegative integer k, $D^k u \in \mathcal{F}$ by (2.1). Thus for $s > 0$, we define the fractional derivative of u of order s by

$$\mathcal{D}^s u = \mathcal{D}^{-([s]-s)} D^{[s]} u. \tag{2.3}$$

Here, $[s]$ is the smallest integer greater than or equal to s and $\mathcal{D}^0 = D^0$ is the identity operator. If $s > 0$ is not an integer, then $-1 < [s] - s - 1 < 0$ and $[s] \ge 1$. Thus we know from (2.1) that, for each $z \in \mathbf{H}$ and for every $u \in b^p_\alpha$, the integral

$$\mathcal{D}^s u(z) = \frac{1}{\Gamma([s]-s)} \int_0^\infty t^{[s]-s-1} D^{[s]} u(z', z_n + t)\, dt$$

always makes sense.

Let $P(z,w)$ be the extended Poisson kernel on $\mathbf{H}$ and put $P_z = P(z,\cdot)$. More explicitly,

$$P_z(w) = P(z,w) = \frac{2}{nV(B)} \frac{z_n + w_n}{|z - \overline{w}|^n}$$

where $z, w \in \mathbf{H}$ and $\overline{w} = (w', -w_n)$ and B is the open unit ball in $\mathbf{R}^n$. It is known that the weighted harmonic Bergman projection Π_α of L^2_α onto b^2_α is given by

$$\Pi_\alpha f(z) = \int_{\mathbf{H}} f(w) R_\alpha(z,w)\, dV_\alpha(w)$$

for all $f \in L^2_\alpha$. Here $R_\alpha(z,w)$ denotes the weighted harmonic Bergman kernel whose explicit formula is given by

$$R_\alpha(z,w) = C_\alpha \mathcal{D}^{\alpha+1} P_z(w) \tag{2.4}$$

where $C_\alpha = (-1)^{[\alpha]+1} 2^{\alpha+1}/\Gamma(\alpha+1)$. Also, it is known that

$$|\mathcal{D}^\beta_{z_n} R_\alpha(z,w)| \le \frac{C}{|z - \overline{w}|^{n+\alpha+\beta}} \tag{2.5}$$

for all $z, w \in \mathbf{H}$. Here, $\beta > -n - \alpha$ and the constant C is dependent only on n, α and β. Using (2.5), we know $R_\alpha(z,\cdot) \in b^q_\alpha$ for all $1 < q \le \infty$. Thus, Π_α

is well defined whenever $f \in L^p_\alpha$ for $1 \le p < \infty$. Also, for $1 \le p < \infty$, $u \in b^p_\alpha$, $z \in \mathbf{H}$, we have the reproducing formula

$$u(z) = \int_{\mathbf{H}} u(w) R_\beta(z,w)\, dV_\beta(w) \tag{2.6}$$

whenever $\beta \ge \alpha$. Furthermore, we have a useful norm equivalence. If $\alpha > -1$, $1 \le p < \infty$ and $(1+\alpha)/p + \gamma > 0$, then

$$\|u\|_{L^p_\alpha} \approx \|w_n^\gamma \mathcal{D}^\gamma u\|_{L^p_\alpha} \tag{2.7}$$

as u ranges over b^p_α.

Set $z_0 = (0,1)$. A harmonic function u on $\mathbf{H}$ is called a Bloch function if

$$\|u\|_{\mathcal{B}} = \sup_{w \in \mathbf{H}} w_n |\nabla u(w)| < \infty,$$

where ∇u denotes the gradient of u. We let $\mathcal{B}$ denote the set of Bloch functions on $\mathbf{H}$ and let $\widetilde{\mathcal{B}}$ denote the subspace of functions in $\mathcal{B}$ that vanish at z_0. Then the space $\widetilde{\mathcal{B}}$ is a Banach space under the Bloch norm $\|\ \|_{\mathcal{B}}$.

A function $u \in \widetilde{\mathcal{B}}$ is called a harmonic little Bloch function if it has the following vanishing condition

$$\lim_{z \to \partial^\infty \mathbf{H}} z_n |\nabla u(z)| = 0$$

where $\partial^\infty \mathbf{H}$ denotes the union of $\partial \mathbf{H}$ and $\{\infty\}$. Let $\widetilde{\mathcal{B}}_0$ denote the set of all harmonic little Bloch functions on $\mathbf{H}$. It is not hard to verify that $\widetilde{\mathcal{B}}_0$ is a closed subspace of $\widetilde{\mathcal{B}}$. Let $\mathcal{C}_0$ denote the set of all continuous functions on $\mathbf{H}$ vanishing at ∞.

Because $R_\alpha(z,\cdot)$ is not in L^1_α, $\Pi_\alpha f$ is not well defined for $f \in L^\infty$. So we need the following modified Bergman kernel. For $z, w \in \mathbf{H}$, define

$$\widetilde{R}_\alpha(z,w) = R_\alpha(z,w) - R_\alpha(z_0,w).$$

Then, there is a constant $C = C(n,\alpha)$ such that

$$|\widetilde{R}_\alpha(z,w)| \le C \left(\frac{|z - z_0|}{|z - \overline{w}|^{n+\alpha} |z_0 - \overline{w}|} + \frac{|z - z_0|}{|z - \overline{w}| |z_0 - \overline{w}|^{n+\alpha}} \right) \tag{2.8}$$

for all $z, w \in \mathbf{H}$. Thus, (2.8) implies that $\widetilde{R}_\alpha(z,\cdot) \in L^1_\alpha$ for each fixed $z \in \mathbf{H}$ and thus we can define $\widetilde{\Pi}_\alpha$ on L^∞ by

$$\widetilde{\Pi}_\alpha f(z) = \int_{\mathbf{H}} f(w) \widetilde{R}_\alpha(z,w)\, dV_\alpha(w)$$

for $f \in L^\infty$. It turns out that $\widetilde{\Pi}_\alpha$ is a bounded linear map from L^∞ onto $\widetilde{\mathcal{B}}$. Also, $\widetilde{\Pi}_\alpha$ has the following property: If $\gamma > 0$ and $v \in \widetilde{\mathcal{B}}$ then

$$\widetilde{\Pi}_\alpha(w_n^\gamma \mathcal{D}^\gamma v)(z) = Cv(z) \tag{2.9}$$

where $C = C(\alpha, \gamma)$. The Bloch norm is also equivalent to the normal derivative norm : If $\gamma > 0$, then

$$\|u\|_{\mathcal{B}} \approx \|w_n^{\gamma} \mathcal{D}^{\gamma} u\|_{\infty} \tag{2.10}$$

as u ranges over $\widetilde{\mathcal{B}}$. (See [7] for details.)

3. Technical Lemmas

We first introduce a distance function on $\mathbf{H}$ which is useful for our purposes. The pseudohyperbolic distance between $z, w \in \mathbf{H}$ is defined by

$$\rho(z, w) = \frac{|z - w|}{|z - \overline{w}|}.$$

This ρ is an actual distance. (See [6].) Note that ρ is horizontal translation invariant and dilation invariant. In particular,

$$\rho(z, w) = \rho(\phi_a(z), \phi_a(w)) \tag{3.1}$$

for $z, w \in \mathbf{H}$ where $\phi_a (a \in \mathbf{H})$ denotes the function defined by

$$\phi_a(z) = \left(\frac{z' - a'}{a_n}, \frac{z_n}{a_n} \right)$$

for $z \in \mathbf{H}$. Note that the Jacobian of ϕ_a^{-1} is a_n^n. For $z \in \mathbf{H}$ and $0 < \delta < 1$, let $E_\delta(z)$ denote the pseudohyperbolic ball centered at z with radius δ. Note that $\phi_z(E_\delta(z)) = E_\delta(z_0)$ by the invariance property (3.1). Also, simple calculation shows that

$$E_\delta(z) = B\left(\left(z', \frac{1 + \delta^2}{1 - \delta^2} z_n \right), \frac{2\delta}{1 - \delta^2} z_n \right) \tag{3.2}$$

so that $B(z, \delta z_n) \subset E_\delta(z) \subset B(z, 2\delta(1 - \delta)^{-1} z_n)$ where $B(z, r)$ denotes the Euclidean ball centered at z with radius r. From (3.2), we have two lemmas. For proofs of the following lemmas, see [6].

Lemma 3.1. *Let $z, w \in \mathbf{H}$. Then*

$$\frac{1 - \rho(z, w)}{1 + \rho(z, w)} \leq \frac{z_n}{w_n} \leq \frac{1 + \rho(z, w)}{1 - \rho(z, w)}.$$

This lemma implies the following lemma.

Lemma 3.2. *Let $z, w \in \mathbf{H}$. Then*

$$\frac{1 - \rho(z, w)}{1 + \rho(z, w)} \leq \frac{|z - \overline{s}|}{|w - \overline{s}|} \leq \frac{1 + \rho(z, w)}{1 - \rho(z, w)}$$

for all $s \in \mathbf{H}$.

The following lemma is used to prove the representation theorem. If α is a nonnegative integer, then it is proved in [6].

Lemma 3.3. *Let $\alpha > -1$ and β be real. Then*

$$\left|z_n^\beta R_\alpha(s,z) - w_n^\beta R_\alpha(s,w)\right| \le C\rho(z,w)\frac{z_n^\beta}{|z-\overline{s}|^{n+\alpha}}$$

whenever $\rho(z,w) < 1/2$ and $s \in \mathbf{H}$.

Let $\alpha > -1$ and let $1 \le p < \infty$. Define Π_β on the weighted Lebesque space L^p_α by

$$\Pi_\beta f(z) = \int_{\mathbf{H}} f(w)R_\beta(z,w)\,dV_\beta(w)$$

for $f \in L^p_\alpha$ and $z \in \mathbf{H}$. Then we have the following two lemmas from [7].

Lemma 3.4. *Suppose $\alpha > -1$, $1 \le p < \infty$ and $\alpha + 1 < (\beta+1)p$. Then Π_β is bounded projection of L^p_α onto b^p_α.*

Lemma 3.5. *For $b < 0, -1 < a + b$, there exists a constant $C = C(a,b)$ such that*

$$\int_{\mathbf{H}} \frac{w_n^{a+b}}{|z-\overline{w}|^{n+a}}\,dw \le Cz_n^b$$

for every $z, w \in \mathbf{H}$.

Lemma 3.6. *Let $\alpha > -1$, $1 \le p < \infty$ and let $(1+\alpha)/p + \gamma > 0$. Suppose $0 < \delta < 1$. Then*

$$z_n^{n+p\gamma}|\mathcal{D}^\gamma u(z)|^p \le \frac{C}{\delta^{n+pk}}\int_{E_\delta(z)} |u(w)|^p\,dw$$

for all $z \in \mathbf{H}$ and for every u harmonic on $\mathbf{H}$ where $k = [\gamma]$ if $\gamma > -1$ and $k = 0$ if $\gamma \le -1$. The constant $C = C(n,p,\gamma)$ is independent of δ.

If γ satisfies the condition of Lemma 3.6, we can show $\mathcal{D}^\gamma u$ is harmonic on $\mathbf{H}$. If γ is a nonnegative integer, then $\mathcal{D}^\gamma u$ is harmonic on $\mathbf{H}$, because it is a partial derivative of a harmonic function. If γ is not a nonnegative integer, we see also $\mathcal{D}^\gamma u$ is harmonic on $\mathbf{H}$ by passing the Laplacian through the integral.

The notation $|E|$ denotes the Lebesque measure of a Borel subset E of $\mathbf{H}$. Let $|E|_\alpha$ denote $V_\alpha(E)$. The following lemma is proved by using the mean value property and Cauchy's estimates. The notation $d(E,F)$ denotes the euclidean distance between two sets E and F.

Lemma 3.7. *Suppose u is harmonic on some proper open subset Ω of $\mathbf{R}^n$. Let $\alpha > -1$ and let $1 \le p < \infty$. Then, for a given open ball $E \subset \Omega$,*

$$\int_E |u(z) - u(a)|^p\,dV_\alpha(z) \le C\frac{|E|^{p/n}|E|_\alpha}{d(E,\partial\Omega)^{n+p}}\int_\Omega |u(w)|^p\,dw$$

for all $a \in E$. The constant C depends only on n, α and p.

4. Representation theory

Let $\{z_m\}$ be a sequence in $\mathbf{H}$ and let $0 < \delta < 1$. We say that $\{z_m\}$ is δ-separated if the balls $E_\delta(z_m)$ are pairwise disjoint or simply say that $\{z_m\}$ is separated if it is δ-separated for some δ. Also, we say that $\{z_m\}$ is a δ-lattice if it is $\delta/2$-separated and $\mathbf{H} = \bigcup E_\delta(z_m)$. Note that any "maximal" $\delta/2$-separated sequence is a δ-lattice.

From [4] and [6], we have the following three lemmas.

Lemma 4.1. *Fix a $1/2$-lattice $\{a_m\}$ and let $0 < \delta < 1/8$. If $\{z_m\}$ is a δ-lattice, then we can find a rearrangement $\{z_{ij} : i = 1, 2, \ldots, j = 1, 2, \ldots, N_i\}$ of $\{z_m\}$ and a pairwise disjoint covering $\{D_{ij}\}$ of $\mathbf{H}$ with the following properties:*
(a) $E_{\delta/2}(z_{ij}) \subset D_{ij} \subset E_\delta(z_{ij})$
(b) $E_{1/4}(a_i) \subset \cup_{j=1}^{N_i} D_{ij} \subset E_{5/8}(a_i)$
(c) $z_{ij} \in E_{1/2}(a_i)$
for all $i = 1, 2, \ldots,$ and $j = 1, 2, \ldots, N_i$.

Lemma 4.2. *Let $r > 0$ and let $0 < r\eta < 1$. If $\{z_m\}$ is an η-separated sequence, then there is a constant $M = M(n, r, \eta)$ such that more than M of the balls $E_{r\eta}(z_m)$ contain no point in common.*

Lemma 4.3. *Let N_i be the sequence defined in Lemma 4.1. Then*

$$\sup_i N_i \leq C\delta^{-n}$$

for some constant C depending only on n.

Analysis similar to that for the proof of Lemma 3.4 shows the following lemma which will be used in the proof of Proposition 4.5.

Lemma 4.4. *Let $\alpha > -1$, $1 \leq p < \infty$ and $\alpha + 1 < (\beta + 1)p$. For $f \in L^p_\alpha$, define*

$$\Phi_\beta f(z) = \int_{\mathbf{H}} f(w) \frac{w_n^\beta}{|z - \overline{w}|^{n+\beta}}\, dw$$

for $z \in \mathbf{H}$. Then, $\Phi_\beta : L^p_\alpha \to L^p_\alpha$ is bounded.

Let $\{z_m\}$ be a sequence in $\mathbf{H}$. Let $\alpha > -1$, $1 \leq p < \infty$ and $\alpha + 1 < (\beta + 1)p$. For $(\lambda_m) \in l^p$, let $Q_\beta(\lambda_m)$ denote the series defined by

$$Q_\beta(\lambda_m)(z) = \sum \lambda_m z_{mn}^{(n+\beta)(1-1/p)+(\beta-\alpha)/p} R_\beta(z, z_m) \tag{4.1}$$

for $z \in \mathbf{H}$. For a sequence $\{z_m\}$ good enough, $Q_\beta(\lambda_m)$ will be harmonic on $\mathbf{H}$. We say that $\{z_m\}$ is a b^p_α-representing sequence of order β if $Q_\beta(l^p) = b^p_\alpha$. Lemma 4.4 implies the following proposition which shows $Q_\beta(l^p) \subset b^p_\alpha$ if the underlying sequence is separated.

Proposition 4.5. *Let $\alpha > -1$, $1 \leq p < \infty$ and $\alpha + 1 < (\beta + 1)p$. Suppose $\{z_m\}$ is a δ-separated sequence. Then $Q_\beta : l^p \to b^p_\alpha$ is bounded.*

The following theorem is the b^p_α-representation result under the lattice density condition.

Theorem 4.6. *Let $\alpha > -1$, $1 \le p < \infty$ and $\alpha + 1 < (\beta+1)p$. Then there exists $\delta_0 > 0$ with the following property: Let $\{z_m\}$ be a δ-lattice with $\delta < \delta_0$ and let $Q_\beta : l^p \to b^p_\alpha$ be the associated linear operator as in (4.1). Then there is a bounded linear operator $\mathcal{P}_\beta : b^p_\alpha \to l^p$ such that $Q_\beta \mathcal{P}_\beta$ is the identity on b^p_α. In particular, $\{z_m\}$ is a b^p_α-representing sequence of order β.*

Since $\mathcal{D}^\gamma u$ is harmonic and we have (2.7), we can have similar result with Proposition 4.8 of [6].

Proposition 4.7. *Let $\alpha > -1$, $1 \le p < \infty$ and let $(1+\alpha)/p + \gamma > 0$. If $\{z_m\}$ is a δ-lattice with δ sufficiently small, then*

$$\|u\|^p_{L^p_\alpha} \approx \sum z_{mn}^{n+\alpha+p\gamma} |\mathcal{D}^\gamma u(z_m)|^p$$

as u ranges over b^p_α.

Let $\{z_m\}$ be a sequence in $\mathbf{H}$ and let $\beta > -1$. For $(\lambda_m) \in l^\infty$, let

$$\widetilde{Q}_\beta(\lambda_m)(z) = \sum \lambda_m z_{mn}^{n+\beta} \widetilde{R}_\beta(z, z_m) \tag{4.2}$$

for $z \in \mathbf{H}$. We say that $\{z_m\}$ is a $\widetilde{\mathcal{B}}$-representing sequence of order β if $\widetilde{Q}_\beta(l^\infty) = \widetilde{\mathcal{B}}$. We also say that $\{z_m\}$ is a $\widetilde{\mathcal{B}}_0$-representing sequence of order β if $\widetilde{Q}_\beta(\mathcal{C}_0) = \widetilde{\mathcal{B}}_0$. Then we have the result which shows that a separated sequence represents a part of the whole space.

Proposition 4.8. *Let $\beta > -1$ and suppose $\{z_m\}$ is a δ-separated sequence. Then, $\widetilde{Q}_\beta : l^\infty \to \widetilde{\mathcal{B}}$ is bounded. In addition, $\widetilde{Q}_\beta$ maps $\mathcal{C}_0$ into $\widetilde{\mathcal{B}}_0$.*

If γ is a positive integer, then the following lemma is proved in [6].

Lemma 4.9. *Let $\gamma > 0$. Then*

$$|z_n^\gamma \mathcal{D}^\gamma u(z) - w_n^\gamma \mathcal{D}^\gamma u(w)| \le C\rho(z,w)\|u\|_{\mathcal{B}}$$

for all $z, w \in \mathbf{H}$ and $u \in \widetilde{\mathcal{B}}$.

The following theorem is the limiting version of the b^p_α-representation theorem.

Theorem 4.10. *Let $\beta > -1$. Then there exists a positive number δ_0 with the following property: Let $\{z_m\}$ be a δ-lattice with $\delta < \delta_0$ and let $\widetilde{Q}_\beta : l^\infty \to \widetilde{\mathcal{B}}$ be the associated linear operator as in (4.2). Then there exists a bounded linear operator $\widetilde{\mathcal{P}}_\beta : \widetilde{\mathcal{B}} \to l^\infty$ such that $\widetilde{Q}_\beta \widetilde{\mathcal{P}}_\beta$ is the identity on $\widetilde{\mathcal{B}}$. Moreover, $\widetilde{\mathcal{P}}_\beta$ maps $\widetilde{\mathcal{B}}_0$ into $\mathcal{C}_0$. In particular, $\{z_m\}$ is a both $\widetilde{\mathcal{B}}$-representing and $\widetilde{\mathcal{B}}_0$-representing sequence of order β.*

Lemma 4.9 yields the following result for $\widetilde{\mathcal{B}}$ analogous to Proposition 4.7.

Proposition 4.11. *Let $\gamma > 0$. Let $\{z_m\}$ be a δ-lattice with δ sufficiently small. Then*

$$\|u\|_{\mathcal{B}} \approx \sup_m z_{mn}^{\gamma} |\mathcal{D}^{\gamma} u(z_m)|$$

as u ranges over $\widetilde{\mathcal{B}}$.

5. Interpolation theory

Let $\{z_m\}$ be a sequence on $\mathbf{H}$. Let $\alpha > -1$, $1 \le p < \infty$ and $(1+\alpha)/p+\gamma > 0$. For $u \in b^p_\alpha$, let $T_\gamma u$ denote the sequence of complex numbers defined by

$$T_\gamma u = \left(z_{mn}^{(n+\alpha)/p+\gamma} \mathcal{D}^{\gamma} u(z_m) \right). \tag{5.1}$$

If $T_\gamma(b^p_\alpha) = l^p$, we say that $\{z_m\}$ is a b^p_α-interpolating sequence of order γ.

The following two lemmas are used to prove that separation is necessary for b^p_α-interpolation.

Lemma 5.1. *Let $\alpha > -1$, $1 \le p < \infty$ and $(1+\alpha)/p+\gamma > 0$. Let $\{z_m\}$ be a b^p_α-interpolating sequence of order γ. Then $T_\gamma : b^p_\alpha \to l^p$ is bounded.*

The following lemma is a b^p_α-version of Lemma 4.9 concerning $\widetilde{\mathcal{B}}$-functions. If γ is a nonnegative integer, then the following lemma is proved in [6].

Lemma 5.2. *Let $\alpha > -1$, $1 \le p < \infty$ and $(1+\alpha)/p+\gamma > 0$. Then,*

$$\left| z_n^{(n+\alpha)/p+\gamma} \mathcal{D}^{\gamma} u(z) - w_n^{(n+\alpha)/p+\gamma} \mathcal{D}^{\gamma} u(w) \right| \le C\rho(z,w) \|u\|_{L^p_\alpha}$$

for all $z, w \in \mathbf{H}$ and $u \in b^p_\alpha$.

Proposition 5.3. *Let $\alpha > -1$, $1 \le p < \infty$ and $(1+\alpha)/p+\gamma > 0$. Every b^p_α-interpolating sequence of order γ is separated.*

For interpolation, we need the sufficient separation condition.

Theorem 5.4. *Let $\alpha > -1$, $1 \le p < \infty$ and $(1+\alpha)/p+\gamma > 0$. Then there exists a positive number δ_0 with the following property: Let $\{z_m\}$ be a δ-separated sequence with $\delta > \delta_0$ and let $T_\gamma : b^p_\alpha \to l^p$ be the associated linear operator as in (5.1). Then there is a bounded linear operator $S_\gamma : l^p \to b^p_\alpha$ such that $T_\gamma S_\gamma$ is the identity on l^p. In particular, $\{z_m\}$ is a b^p_α-interpolating sequence of order γ.*

Let $\gamma > 0$ and let $\{z_m\}$ be a sequence in $\mathbf{H}$. For $u \in \widetilde{\mathcal{B}}$, define

$$\widetilde{T}_\gamma u = \left(z_{mn}^{\gamma} \mathcal{D}^{\gamma} u(z_m) \right). \tag{5.2}$$

Then (2.10) implies the operator

$$\widetilde{T}_\gamma : \widetilde{\mathcal{B}} \to l^\infty$$

is bounded. If $\widetilde{T}_\gamma(\widetilde{\mathcal{B}}) = l^\infty$, $\{z_m\}$ is called a $\widetilde{\mathcal{B}}$-interpolating sequence of order γ. Also, if $\widetilde{T}_\gamma(\widetilde{\mathcal{B}}_0) = \mathcal{C}_0$, $\{z_m\}$ is called a $\widetilde{\mathcal{B}}_0$-interpolating sequence of order γ.

The following proposition shows that separation is also necessary for $\widetilde{\mathcal{B}}_0$ interpolation. Since we have Lemma 4.9, the proof of the following proposition is the same as that of Proposition 5.6 in [6].

Proposition 5.5. *Let $\gamma > 0$. Every $\widetilde{\mathcal{B}}$-interpolating sequence of order γ is separated. Also, every $\widetilde{\mathcal{B}}_0$-interpolating sequence of order γ is separated.*

Theorem 5.6. *Let $\gamma > 0$. Then there exists a positive number δ_0 with the following property: Let $\{z_m\}$ be a δ-separated sequence with $\delta > \delta_0$ and let $\widetilde{T}_\gamma : \widetilde{\mathcal{B}} \to l^\infty$ be the associated linear operator as in (5.2). Then there exists a bounded linear operator $\widetilde{S}_\gamma : l^\infty \to \widetilde{\mathcal{B}}$ such that $\widetilde{T}_\gamma \widetilde{S}_\gamma$ is the identity on l^∞. Moreover, $\widetilde{S}_\gamma$ maps C_0 into $\widetilde{\mathcal{B}}_0$. In particular, $\{z_m\}$ is a both $\widetilde{\mathcal{B}}$-interpolating and $\widetilde{\mathcal{B}}_0$-interpolating sequence of order γ.*

References

[1] E. Amar, *Suites d'interpolation pour les classes de Bergman de la boule du polydisque de $\mathbb{C}^n$*, Canadian J. Math. **30** (1978), 711–737.

[2] K. R. M. Attle, *Interpolating sequences for the derivatives of Bloch functions*, Glasgow Math. J. **34** (1992), 35-41.

[3] S. Axler, P. Bourdon and W. Ramey, *Harmonic function theory*, Springer-Verlag, New York 1992.

[4] B. R. Choe, H. Koo and H. Yi, *Positive Toeplitz operators between the harmonic Bergman spaces*, Potential Analysis **17** (2002), 307–335.

[5] R. R. Coifman and R. Rochberg, *Representation theorems for holomorphic and harmonic functions in L^p*, Astérisque **77** (1980), 11–66.

[6] B. R. Choe and H. Yi, *Representations and interpolations of harmonic Bergman functions on half-spaces*, Nagoya Math.J. **151** (1998), 51–89.

[7] H. Koo, K. Nam, and H. Yi, *Weighted harmonic Bergman functions on half-spaces* J. Korean Math. Soc. **42** (2005), no. 5, 975–1002.

[8] K. Nam, *Representations and interpolations of weighted harmonic Bergman functions,* Rocky Mountain J. Math. **36** (2006), no. 1, 237–263.

[9] F. Ricci and M. Taibleson, *Boundary values of harmonic functions in mixed norm spaces and their atomic structure*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **10** (1983), no. 1, 1–54.

[10] F. Ricci and M. Taibleson, *Representation theorems for harmonic functions in mixed norm spaces on the half plane*, Rend. Circ. Mat. Palermo (2) (1981), suppl. 1, 121–127.

[11] R. Rochberg, *Interpolation by functions in Bergman spaces*, Michigan Math. J. **29** (1982), 229–236.

Department of Mathematics, Hanshin University, Yangsan-dong, Osan-si, Gyeonggi-do, 447-791, Korea

E-mail address: ksnam@hs.ac.kr