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NOTE ON THE BEREZIN TRANSFORM ON HERZ SPACES

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ABSTRACT. In a previous work of the author with Koo and Na, the Berezin transform is shown to be bounded on Herz spaces $\mathcal{K}_q^{p,\alpha}$ on the unit ball of $\mathbf{R}^n$, when the parameters p, q and α belong to a ceratin range. In this note that parameter range is shown to be also necessary. In addition, the parameter q is extended to the full range $0 \leq q \leq \infty$. Also, the pointwise growth estimate of the Berezin transform on Herz spaces is obtained in certain cases.

1. INTRODUCTION

For a fixed integer $n \geq 2$, let $B = B_n$ denote the open unit ball in $\mathbf{R}^n$. Let R be the kernel function on $B \times B$ defined by

$$R(x,y) = \frac{1}{\omega_n} \cdot \frac{1}{[x,y]^n} \left\{ \left(\frac{1 - |x|^2|y|^2}{[x,y]} \right)^2 - \frac{4|x|^2|y|^2}{n} \right\} \tag{1.1}$$

for $x, y \in B$ where ω_n is the volume of B and

$$[x,y] = \sqrt{1 - 2x \cdot y + |x|^2|y|^2}.$$

Here, $x \cdot y$ denotes the dot product of $x, y \in \mathbf{R}^n$. This kernel R is the well-known harmonic Bergman kernel for B which has the the reproducing property for harmonic Bergman functions on B. More explicitly, if f is a square-integrable harmonic function on B, then

$$f(x) = \int_B f(y) R(x,y)\, dy \tag{1.2}$$

for $x \in B$. See [1] for more information and related facts.

Given a positive Borel measure μ on B (we write $\mu \geq 0$, for brevity), the *(harmonic) Berezin transform* $\widetilde{\mu}$ is a function on B defined by

$$\widetilde{\mu}(x) = R(x,x)^{-1} \int_B |R(x,y)|^2 \, d\mu(y)$$

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for $x \in B$. Also, for a complex Borel measure μ on B, we define $\widetilde{\mu}$ similarly. Let V be the volume measure on B. In case $d\mu = f\,dV$, we let $\widetilde{f} = \widetilde{\mu}$ for brevity.

Note that we have

$$R(x,x) = \frac{1}{\omega_n} \cdot \frac{1}{(1-|x|^2)^n} \left\{ (1+|x|^2)^2 - \frac{4|x|^4}{n} \right\}$$

and thus

$$\widetilde{\mu}(x) \approx (1-|x|)^n \int_B |R(x,y)|^2 \, d\mu(y)$$

for $\mu \geq 0$.

Given α real, let V_α denote the weighted measure on B defined by

$$dV_\alpha(x) = (1-|x|)^\alpha \, dx.$$

and let $L^p_\alpha = L^p(B, dV_\alpha)$. In case $\alpha = 0$, we let $L^p = L^p_0$. Clearly, the Berezin transform is a bounded linear operator on L^∞. More generally, when $1 \leq p \leq \infty$, the precise range of parameters for L^p_α on which the Berezin transform is bounded is proved in [2]:

$$\textit{The Berezin transform is bounded on } L^p_\alpha \iff -n < \frac{\alpha+1}{p} < 1.$$

The purpose of this note is to find the precise range of parameters for certain mixed norm spaces, called Herz spaces, which is briefly recalled below.

We decompose B into a family of annuli A_m given by

$$A_m = \{x \in B \ : \ r_m \leq |x| < r_{m+1}\}$$

where

$$r_m = 1 - 2^{-m}$$

for each integer $m \geq 0$. Now, given α real and $0 < p, q \leq \infty$, the *Herz* space $\mathcal{K}^{p,\alpha}_q$ is defined to be the space consisting of all functions $f \in L^p_{\text{loc}}(V)$ such that

$$\|f\|_{\mathcal{K}^{p,\alpha}_q} := \left\| \left\{ 2^{-m\alpha} \, \|f\chi_m\|_{L^p} \right\} \right\|_{\ell^q} < \infty$$

where χ_m denotes the characteristic function of the annulus A_m and ℓ^q stands for the q-summable sequence space. For $1 \leq p, q \leq \infty$, the space $\mathcal{K}^{p,\alpha}_q$ is a Banach space with the norm above. Also, we let $\mathcal{K}^{p,\alpha}_0$ be the subspace of $\mathcal{K}^{p,\alpha}_\infty$ consisting of all functions $f \in \mathcal{K}^{p,\alpha}_\infty$ such that $2^{-m\alpha}\|f\chi_m\|_{L^p} \to 0$ as $m \to \infty$. Note that $\mathcal{K}^{p,\alpha}_q \subset \mathcal{K}^{p,\alpha}_0$ for all $q < \infty$. Some basic properties of Herz spaces relevant to this note is collected in Section 2. For more information on Herz spaces, see [4] and references therein.

Boundedness of the Berezin transform is also studied in [2] on Herz spaces. However, the parameter range obtained there is shown to be be only sufficient. More precisely, the following is proved in [2, Proposition 3.8] for $1 \le p, q \le \infty$:

(1.3)
If $-n < \alpha + 1/p < 1$, *then the Berezin transform is bounded on* $\mathcal{K}_q^{p,\alpha}$.

In this note we show that the above parameter range is also necessary and, in addition, we extend the parameter q to the full range $0 \le q \le \infty$.

Theorem 1.1. *Let* $1 \le p \le \infty$ *and* α *be real. Given* $0 \le q \le \infty$, *the Berezin transform is bounded on* $\mathcal{K}_q^{p,\alpha}$ *if and only if* $-n < \alpha + 1/p < 1$.

For $1 \le p \le \infty$, all the parameters of Herz spaces that are contained in L^1, so that the Berezin transform is well defined on those spaces, is described in (2.3) in the next section. Our second result is the following growth estimate of the Berezin transform on such Herz spaces.

Theorem 1.2. *Let* $1 \le p \le \infty$ *and* α *be real. Assume* $\alpha + 1/p \le 1$. *Then there exist constants* $C = C(p, \alpha) > 0$ *such that the following inequalities hold for* $x \in B$ *and measurable functions* $f \ge 0$ *on* B.

(a) *If* $\alpha + 1/p < 1$, *then*

$$\widetilde{f}(x) \le C\|f\|_{\mathcal{K}_\infty^{p,\alpha}} \times \begin{cases} (1-|x|)^{-n/p-\alpha} & \text{if } \alpha > -n(1+1/p) \\ (1-|x|)^n \left(1 + \log \frac{1}{1-|x|}\right) & \text{if } \alpha = -n(1+1/p) \\ (1-|x|)^n & \text{if } \alpha < -n(1+1/p). \end{cases}$$

(b) *If* $\alpha + 1/p = 1$, *then*

$$\widetilde{f}(x) \le C\|f\|_{\mathcal{K}_1^{p,\alpha}}(1-|x|)^{-1-(n-1)/p}.$$

In Section 2 we briefly review some basic properties of Herz spaces, which we need in later sections. In Section 3 we prove the sufficiency of Theorem 1.1. In Section 4 we prove the necessity of Theorem 1.1, either by providing concrete counter-examples or proving some general fact. Finally, in Section 5, we prove Theorem 1.2.

Constants. Throughout the note the same letter C will denote various positive constants, which may change at each occurrence. The constant C may often depend on the dimension n and some other allowed parameters, but it will be always independent of particular functions, measures and points. We will often abbreviate inessential constants involved in inequalities by writing $X \lesssim Y$ or $Y \gtrsim X$ for positive quantities X and Y if the ratio X/Y has a positive upper bound. Also, we write $X \approx Y$ if $X \lesssim Y$ and $X \gtrsim Y$.

2. Preliminaries

In this section we recall some basic properties of Herz spaces. Given positive measurable functions f and g on B, note that an application of Hölder's inequality yields

$$\int_B f\bar{g}\,dV = \sum_m \int_{A_m} fg\,dV \leq \sum_m \|f\chi_m\|_{L^p}\|g\chi_m\|_{L^{p'}}$$

for $1 \leq p \leq \infty$. Here, and in what follows, p' denotes the conjugate index of p, i.e., $1/p + 1/p' = 1$. Now, another application of Hölder's inequality leads to Hölder's inequality for the Herz spaces as follows:

$$\int_B f\bar{g}\,dV \leq \|f\|_{\mathcal{K}_q^{p,\alpha}}\|g\|_{\mathcal{K}_{q'}^{p',-\alpha}} \tag{2.1}$$

for $1 \leq p, q \leq \infty$ and arbitrary α real. We remark in passing that this Hölder's inequality actually leads to dualities between Herz spaces; see [4, Theorem 2.1 and Corollary 2.7] for details.

Given $0 < p \leq \infty$, note that we have $\|\chi_m\|_{L^p} \approx 2^{-m/p}$ for $m \geq 0$. It follows that the space $\mathcal{K}_q^{p,\alpha}$ contains constants if and only if

$$\text{either} \quad \alpha > -1/p \text{ and } q \text{ arbitrary;} \qquad \text{or} \quad \alpha = -1/p \text{ and } q = \infty.$$

Thus, if $1 \leq p \leq \infty$ and $\alpha < \frac{1}{p'}$, then $\mathcal{K}_\infty^{p,\alpha} \subset L^1$ by (2.1). Similarly, if $1 \leq p \leq \infty$ and $\alpha = \frac{1}{p'}$, then $\mathcal{K}_1^{p,\alpha} \subset L^1$. Consequently, if $1 \leq p \leq \infty$ and if

$$\begin{array}{ll} \text{either} & \alpha + 1/p < 1 \text{ and } q \text{ arbitrary;} \\ \text{or} & \alpha + 1/p = 1 \text{ and } 0 < q \leq 1, \end{array} \tag{2.2}$$

then the space $\mathcal{K}_q^{p,\alpha}$ is contained in L^1. It turns out that (2.2) is also necessary for the containment $\mathcal{K}_q^{p,\alpha} \subset L^1$ for the full range $0 < p \leq \infty$. To see this, consider the function $f_{\beta,\gamma}$ on B defined by

$$f_{\beta,\gamma}(x) = \frac{1}{(1-|x|)^\beta}\left(1 + \log\frac{1}{1-|x|}\right)^{-\gamma}$$

where β and γ are given real numbers. Note that

$$f_{\beta,\gamma}(x) \approx 2^{m\beta}(1+m)^{-\gamma}, \qquad x \in A_m$$

and thus

$$2^{-m\alpha}\|f_{\beta,\gamma}\chi_m\|_{L^p} \approx 2^{-m(\alpha+1/p-\beta)}(1+m)^{-\gamma}$$

for all $m \geq 0$. From this we immediately deduce the following for arbitrary parameters.

Lemma 2.1. *$f_{\beta,\gamma} \in \mathcal{K}_q^{p,\alpha}$ if and only if one of the following conditions holds:*

(i) $\alpha + 1/p > \beta$;
(ii) $\alpha + 1/p = \beta$ *and* $\gamma > 0 = q$;
(iii) $\alpha + 1/p = \beta$ *and* $0 < 1/\gamma < q < \infty$;
(iv) $\alpha + 1/p = \beta$, $\gamma \geq 0$ *and* $q = \infty$.

In particular, when $q < \infty$, we have $f_{\beta,0} \in \mathcal{K}_q^{p,\alpha}$ if and only if $\beta < \alpha + 1/p$. Also, $f_{\beta,0} \in \mathcal{K}_\infty^{p,\alpha}$ if and only if $\beta \leq \alpha + 1/p$. Thus, since $f_{\beta,0} \in L^1$ if and only if $\beta < 1$, we have $\mathcal{K}_q^{p,\alpha} \subset L^1$ for some q only when $\alpha + 1/p \leq 1$. Now, assume $\alpha + 1/p = 1$. Note $f_{1,\gamma} \in L^1$ if and only if $\gamma > 1$. Thus, $f_{1,1} \in \mathcal{K}_0^{p,\alpha} \subset \mathcal{K}_\infty^{p,\alpha}$ but $f_{1,1} \notin L^1$. Also, for $0 < q < \infty$, we have $\mathcal{K}_q^{p,\alpha} \subset L^1$ only when $q \leq 1$, because $f_{1,\gamma} \in \mathcal{K}_q^{p,\alpha}$ if and only if $\gamma > 1/q$.

Summarizing the above observations, we have

$$\mathcal{K}_q^{p,\alpha} \subset L^1 \iff (2.2) \tag{2.3}$$

for $1 \leq p \leq \infty$.

3. Sufficiency

This section is devoted to the proof of Theorem 1.1. The hard part is to extend (1.3) to $p = \infty$; the extension to $0 \leq q < 1$ is an easy modification of the proof given in [2]. To motivate the approach in this note for $p = \infty$, consider an arbitrary measurable function $f \geq 0$ on B. Since $f = \sum_{m=0}^{\infty} f\chi_m$, we have by the monotone convergence theorem

$$\tilde{f} = \sum_{m=0}^{\infty} \widetilde{f\chi_m} \leq \sum_{m=0}^{\infty} \tilde{\chi}_m \|f\chi_m\|_{L^\infty}. \tag{3.1}$$

This suggests that we need to estimate the Berezin transforms of the characteristic functions of annuli. To this end, we need some preliminary integral estimates involving the kernel function.

Let $S = \partial B$ be the unit sphere in $\mathbf{R}^n$. Note that each function $R(x, \cdot)$, $x \in B$, continuously extends to S. Given c real, let

$$J_c(x) = \int_S |R(x,\zeta)|^{1+(c-1)/n} \, d\sigma(\zeta), \qquad x \in B$$

where σ is the surface area measure on S. Note that we have

$$R(x, r\zeta) = R(rx, \zeta) \tag{3.2}$$

for $x \in B$, $0 \leq r \leq 1$ and $\zeta \in S$. Thus, when estimating integrals involving the kernel function by means of integration in polar coordinates, the next lemma is quite useful. In case $c > 0$, the upper estimate is contained in [3, Lemma 3.2(d)] and [5, Lemma 6].

Lemma 3.1. *Given c real, the following estimate holds for $x \in B$:*

$$J_c(x) \approx \begin{cases} 1 & \text{if } c < 0 \\ 1 + \log \frac{1}{1-|x|} & \text{if } c = 0 \\ (1-|x|)^{-c} & \text{if } c > 0. \end{cases}$$

The constants suppressed above depend only on n and c.

Before proceeding to the proof, we recall the slice integration formula (see, for example, Corollary A.5 of [1]):

$$\int_S h(\eta \cdot \zeta)\, d\sigma(\zeta) = c_n \int_{-1}^{1} h(r)(1-r^2)^{\frac{n-3}{2}}\, dr \tag{3.3}$$

for any $\eta \in S$ and measurable function $h \geq 0$ on $(-1, 1)$. The constant c_n above is determined by taking $h = 1$.

Proof. We provide a proof of the lower estimate for $c \geq 0$; the upper estimate is easier. Let $x \in B$. We may further assume $|x| \geq 1/2$. Let

$$\varphi(x, r) = (1-|x|)^2 + 2|x|(1-r)$$

for $x \in B$ and $r \in [-1, 1]$. Note, if $x = |x|\eta$ with $\eta \in S$, then

$$[x, \zeta]^2 = |x - \zeta|^2 = (1-|x|)^2 + 2|x|(1 - \eta \cdot \zeta) = \varphi(x, \eta \cdot \zeta)$$

for $\zeta \in S$. Thus, we have by (1.1) and (3.3)

(3.4)

$$J_c(x) = \frac{c_n}{\omega_n^{1+(c-1)/n}} \int_{-1}^{1} \frac{(1-r^2)^{\frac{n-3}{2}}}{\varphi(x,r)^{\frac{n-1+c}{2}}} \left| \frac{(1-|x|^2)^2}{\varphi(x,r)} - \frac{4|x|^2}{n} \right|^{1+(c-1)/n} dr.$$

Assume $|x|$ is sufficiently close to 1 and consider r such that $1 - r \geq (8n-1)(1-|x|)^2$. Then we have $\varphi(x, r) \geq 8n(1-|x|)^2 \geq 2n(1-|x|^2)^2$ and therefore

$$\frac{4|x|^2}{n} - \frac{(1-|x|^2)^2}{\varphi(x,r)} \geq \frac{1}{n} - \frac{1}{2n} = \frac{1}{2n}.$$

Thus, we see from the representation (3.4)

$$\begin{aligned} J_c(x) &\gtrsim \int_{\frac{1-r}{(1-|x|)^2} \geq 8n-1} \frac{(1-r)^{\frac{n-3}{2}}}{[(1-|x|)^2 + (1-r)]^{\frac{n-1+c}{2}}}\, dr \\ &= (1-|x|)^{-c} \int_{8n-1}^{2(1-|x|)^{-2}} \frac{t^{\frac{n-3}{2}}}{(1+t)^{\frac{n-1+c}{2}}}\, dt \\ &\approx (1-|x|)^{-c} \int_{8n-1}^{(1-|x|)^{-2}} t^{-\frac{c}{2}-1}\, dt. \end{aligned}$$

Now, the rest of the proof is an elementary calculation. The proof is complete. $\square$

Remark. Given $-1 < \alpha < \infty$ and c real, let

$$I_{\alpha,c}(x) = \int_B |R(x,y)|^{1+(\alpha+c)/n}(1-|y|)^\alpha \, dy$$

for $x \in B$. One may apply Lemma 3.1 to the representations of these integrals in polar coordinates and get the following estimate for $x \in B$:

$$I_{\alpha,c}(x) \approx \begin{cases} 1 & \text{if } c < 0 \\ 1 + \log \frac{1}{1-|x|} & \text{if } c = 0 \\ (1-|x|)^{-c} & \text{if } c > 0 \end{cases} \tag{3.5}$$

This estimate is already noticed in [2, Lemma 3.5] with a much more complicated proof.

Lemma 3.2. *Let $A = \{x \in B : 0 \le a < |x| < b \le 1\}$ be an annulus in B. Given $c > 0$, there exists a constant $C = C(n,c) > 0$ such that*

$$\frac{C^{-1}}{(1-a|x|)^c} \le \frac{1}{b-a} \int_A |R(x,y)|^{1+(c-1)/n} \, dy \le \frac{C}{(1-b|x|)^c}$$

for $x \in B$.

Proof. Let $x \in B$. By (3.2) and Lemma 3.1 we have

$$\int_A |R(x,y)|^{1+(c-1)/n} \, dy = \int_a^b r^{n-1} \int_S |R(rx,\zeta)|^{1+(c-1)/n} \, d\sigma(\zeta) \, dr$$
$$\approx \int_a^b \frac{r^{n-1}}{(1-r|x|)^c} \, dr,$$

which implies the lemma. This completes the proof. $\square$

We are now ready to prove the sufficiency of Theorem 1.1.

Theorem 3.3. *Let $1 \le p \le \infty$ and α be real. If $-n < \alpha + 1/p < 1$, then the Berezin transform is bounded on $\mathcal{K}_q^{p,\alpha}$ for all $0 \le q \le \infty$.*

Proof. Assume $-n-\alpha < 1/p < 1-\alpha$ and let $f \ge 0$ be a given measurable function. For $1 \le p < \infty$, it is shown in the proof of [2, Theorem 3.6] that there exist positive constants $\delta = \delta(p,\alpha)$ and $C = C(p,\alpha)$ such that

$$2^{-k\alpha} \|\tilde{f}\chi_k\|_{L^p} \le C \sum_{m=0}^{\infty} \frac{2^{-m\alpha}\|f\chi_m\|_{L^p}}{2^{\delta|m-k|}} \tag{3.6}$$

for all $k \ge 0$. Here, we show that this estimate still holds for $p = \infty$. So, assume $p = \infty$, in which case our parameter range reduces to $-n < \alpha < 1$.

By (3.1) we have

$$2^{-k\alpha}\|\widetilde{f}\chi_k\|_{L^\infty} \leq \sum_{m=0}^{\infty} 2^{(m-k)\alpha}\|\widetilde{\chi}_m\chi_k\|_{L^\infty} \cdot 2^{-m\alpha}\|f\chi_m\|_{L^\infty}$$

for each k. Meanwhile, we have by Lemma 3.2

$$\widetilde{\chi}_m(x) \approx (1-|x|)^n \frac{2^{-m}}{(1-|x|+2^{-m}|x|)^{n+1}}, \qquad x \in B$$

and thus

$$2^{(m-k)\alpha}\|\widetilde{\chi}_m\chi_k\|_{L^\infty} \lesssim 2^{(m-k)\alpha}\frac{2^{-m-kn}}{(2^{-k}+2^{-m})^{n+1}} = \frac{2^{(1-\alpha)(k-m)}}{(1+2^{k-m})^{n+1}}$$

for all m and k. If $m \geq k$, then

$$\frac{2^{(1-\alpha)(m-k)}}{(1+2^{m-k})^{n+1}} \approx \frac{2^{(1-\alpha)(m-k)}}{2^{(n+1)(m-k)}} = \frac{1}{2^{(n+\alpha)|m-k|}}.$$

If $m < k$, then

$$\frac{2^{(1-\alpha)(m-k)}}{(1+2^{m-k})^{n+1}} \approx 2^{(1-\alpha)(m-k)} = \frac{1}{2^{(1-\alpha)|m-k|}}.$$

So, taking $\delta = \min\{1-\alpha, n+\alpha\} > 0$ and combining all these estimates, we conclude (3.6) for $p = \infty$.

Now, we have by (3.6) and Young's inequality

$$\|\widetilde{f}\|_{\mathcal{K}_q^{p,\alpha}} \leq C\left(\sum_{k=-\infty}^{\infty} 2^{-\delta|k|}\right)\|f\|_{\mathcal{K}_q^{p,\alpha}}$$

for $1 \leq q < \infty$. Meanwhile, for $0 < q \leq 1$, we have by (3.6)

$$\begin{aligned}
\|\widetilde{f}\|^q_{\mathcal{K}_q^{p,\alpha}} &\leq C^q \sum_{k=0}^{\infty}\left(\sum_{m=0}^{\infty}\frac{2^{-m\alpha}\|f\chi_m\|_{L^p}}{2^{\delta|m-k|}}\right)^q \\
&\leq C^q \sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\frac{2^{-qm\alpha}\|f\chi_m\|^q_{L^p}}{2^{q\delta|m-k|}} \\
&\leq C^q\left(\sum_{k=-\infty}^{\infty} 2^{-q\delta|k|}\right)\|f\|^q_{\mathcal{K}_q^{p,\alpha}}
\end{aligned}$$

and therefore

$$\|\widetilde{f}\|_{\mathcal{K}_q^{p,\alpha}} \leq C\left(\sum_{k=-\infty}^{\infty} 2^{-q\delta|k|}\right)^{1/q}\|f\|_{\mathcal{K}_q^{p,\alpha}}.$$

Finally, we consider the case $q = 0$. Assume $f \in \mathcal{K}_0^{p,\alpha}$ and let an integer $k \geq 0$ be given. Then we have by (3.6)

$$\begin{aligned} 2^{-k\alpha}\|\widetilde{f}\chi_k\|_{L^p} &\lesssim \sum_{m=0}^{\infty} \frac{2^{-m\alpha}\|f\chi_m\|_{L^p}}{2^{\delta|m-k|}} \\ &= \sum_{m>d} + \sum_{m\leq d} \\ &\lesssim \sup_{m>d} 2^{-m\alpha}\|f\chi_m\|_{L^p} + \|f\|_{\mathcal{K}_\infty^{p,\alpha}} \sum_{m\leq d} \frac{1}{2^{\delta|m-k|}} \end{aligned}$$

for each integer $d \geq 1$. Now, taking the limit $k \to \infty$ (with d fixed), we obtain

$$\limsup_{k\to\infty} 2^{-k\alpha}\|\widetilde{f}\chi_k\|_{L^p} \lesssim \sup_{m>d} 2^{-m\alpha}\|f\chi_m\|_{L^p}$$

for all d. So, taking another limit $d \to \infty$, we conclude $\widetilde{f} \in \mathcal{K}_0^{p,\alpha}$. The proof is complete. $\square$

4. Necessity

Throughout this section we consider parameters $0 < p \leq \infty$and α real such that

$$\text{(4.1)} \qquad \text{either} \quad \alpha + 1/p \leq -n; \qquad \text{or} \quad \alpha + 1/p \geq 1.$$

In order to prove the necessity of Theorem 1.1, we need to prove:

Given $0 \leq q \leq \infty$, *there exists some* $f \in \mathcal{K}_q^{p,\alpha}$ *but* $\widetilde{f} \notin \mathcal{K}_q^{p,\alpha}$.

We will prove this for general p, which is not necessarily greater than or equal to 1.

The source of our examples is the collection of functions $f_{\beta,\gamma}$ introduced in Section 2. So, before proceeding, we introduce some notation for simplicity. Given β and γ real, let

$$h_{\beta,\gamma}(r) = \frac{1}{(1-r)^\beta}\left(1 + \log\frac{1}{1-r}\right)^{-\gamma}, \qquad 0 \leq r < 1$$

so that

$$f_{\beta,\gamma}(x) = h_{\beta,\gamma}(|x|), \qquad x \in B.$$

We separately consider two cases in (4.1) for convenience.

4.1. **The Case $\alpha + 1/p \leq -n$.** We further split this case into the following four subcases:

(1) $\alpha + 1/p < -n$ with q arbitrary;
(2) $\alpha + 1/p = -n$ with $q = 0$;
(3) $\alpha + 1/p = -n$ with $0 < q < \infty$;
(4) $\alpha + 1/p = -n$ with $q = \infty$.

Note that the characteristic function of a compact set belong to all Herz spaces. Thus the next example covers the subcases (1) and (2).

Example 4.1. *If (1) or (2) holds, then $\widetilde{\chi}_A \notin \mathcal{K}_q^{p,\alpha}$ for any compact annular region $A \subset B$.*

Proof. Assume that (1) or (2) holds and let $A \subset B$ be an arbitrary compact annular region. By Lemma 3.2 we have

$$\widetilde{\chi}_A(x) \gtrsim (1 - |x|)^n = f_{-n,0}(x)$$

for $x \in B$. Under the condition (1) or (2), we have $f_{-n,0} \notin \mathcal{K}_q^{p,\alpha}$ by Lemma 2.1 and thus $\widetilde{\chi}_A \notin \mathcal{K}_\infty^{p,\alpha}$. □

For the subcase (3) we have the following example.

Example 4.2. *If (3) holds, then $f_{-n,\gamma} \in \mathcal{K}_q^{p,\alpha}$ but $\widetilde{f}_{-n,\gamma} \notin \mathcal{K}_q^{p,\alpha}$ for all γ with $\gamma > 1/q$.*

Proof. Assume that (3) holds and let $\gamma > 1/q$. Then we have $f_{-n,\gamma} \in \mathcal{K}_q^{p,\alpha}$ by Lemma 2.1.

Now, we estimate $\widetilde{f}_{-n,\gamma}$. Let $x \in B$ and assume $|x| \geq 1/2$. Note that

$$1 - r|x| = (1 - |x|) + |x|(1 - r) \approx (1 - |x|) + (1 - r) \tag{4.2}$$

for $|x| \geq 1/2$ and $0 \leq r < 1$.

Thus, integrating in polar coordinates, we have by (3.2) and Lemma 3.1

$$\begin{aligned}
\widetilde{f}_{-n,\gamma}(x) &\approx (1 - |x|)^n \int_0^1 \frac{h_{-n,\gamma}(r)}{(1 - r|x|)^{n+1}}\, dr \\
&\approx (1 - |x|)^n \int_0^1 \frac{h_{-n,\gamma}(r)}{[(1 - |x|) + (1 - r)]^{n+1}}\, dr \\
&= (1 - |x|)^n \int_0^1 \frac{t^n}{(1 - |x| + t)^{n+1}} (1 - \log t)^{-\gamma}\, dt \\
&\geq (1 - |x|)^n \int_{1/2}^1 \frac{(1 - \log t)^{-\gamma}}{t}\, dt \\
&\gtrsim (1 - |x|)^n.
\end{aligned}$$

That is, we have

$$\widetilde{f}_{-n,\gamma}(x) \gtrsim f_{-n,0}(x)$$

for $|x| \geq 1/2$ and thus for all $x \in B$. Since we have $\widetilde{f}_{-n,0} \notin \mathcal{K}_q^{p,\alpha}$ by (c) and Lemma 2.1, we conclude $\widetilde{f} \notin \mathcal{K}_q^{p,\alpha}$. The proof is complete. $\square$

Finally, for the subcase (4), we have the following example.

Example 4.3. *If (4) holds, then $f_{-n,0} \in \mathcal{K}_\infty^{p,\alpha}$ but $\widetilde{f}_{-n,0} \notin \mathcal{K}_\infty^{p,\alpha}$.*

Proof. Assume that (4) holds. Then we have $f_{-n,0} \in \mathcal{K}_\infty^{p,\alpha}$ by Lemma 2.1. Also, we have by (3.5)

$$\widetilde{f}_{-n,0}(x) \approx (1 - |x|)^n \left(1 + \log \frac{1}{1 - |x|}\right) = f_{-n,-1}(x)$$

for $x \in B$. Since we have $\widetilde{f}_{-n,-1} \notin \mathcal{K}_\infty^{p,\alpha}$ by (4) and Lemma 2.1, we conclude $\widetilde{f}_{-n,0} \notin \mathcal{K}_\infty^{p,\alpha}$. $\square$

While explicit examples are provided for the subcases (1)-(3), those examples are actually special cases of a general fact, Theorem 4.6 below. With that in mind we introduce more notation. Given $\epsilon > 0$ and $\delta > 1/2$, let

$$\begin{aligned} Q_\epsilon(a) &= \left\{x \in B : |a - x| < \epsilon(1 - |a|^2) \text{ and } |x| \leq |a|\right\}, \\ \Gamma_\delta(a) &= \left\{x \in B : |a||x|[a, x] < \delta(1 - |a|^2|x|^2)\right\}, \end{aligned}$$

and

$$E_\delta(a) = \{\zeta \in S : |a||a - \zeta| < \delta(1 - |a|^2)\}$$

for $a \in B$. For $a = |a|\zeta$, we always have $\zeta \in E_\delta(a)$ and $r\zeta \in \Gamma_\delta(a)$ for $0 \leq r < 1$. In particular, $\Gamma_\delta(a)$ and $E_\delta(a)$ are nonempty open subsets of B and S, respectively; this is the reason why we make the restriction on the parameter δ to be greater than 1/2. The regions $\Gamma_\delta(a)$ and $E_\delta(a)$ are closely related in the sense that $r\zeta \in \Gamma_\delta(a)$ if and only if $\zeta \in E_\delta(ra)$. We need a couple of lemmas concerning these regions.

Lemma 4.4. *Let $\epsilon > 0$ and $\delta > 1/2$. Then $Q_\epsilon(a) \subset \Gamma_{\epsilon+\delta}(x)$ for $a \in B$ and $x \in \Gamma_\delta(a)$.*

Proof. Using the identity $[x, y] = \big|x/|x| - y|x|\big|$, one can easily verify the inequality

$$\big|[x, z] - [x, y]\big| \leq |x||y - z| \leq |y - z|$$

for all $x, y, z \in B$. Now, let $a \in B$, $x \in \Gamma_\delta(a)$ and $y \in Q_\epsilon(a)$. Since $|y| \le |a|$, we have by the above inequality

$$\begin{aligned} |x||y|[x,y] &\le |x||a|[x,a] + |a-y| \\ &< \delta(1-|a|^2|x|^2) + \epsilon(1-|a|^2) \\ &\le \delta(1-|x|^2|y|^2) + \epsilon(1-|y|^2) \\ &\le (\epsilon+\delta)(1-|x|^2|y|^2), \end{aligned}$$

which completes the proof. □

Note that if $|a| < \delta/(1+\delta)$, then $E_\delta(a) = S$ and thus $\sigma[E_\delta(a)] = \sigma(S)$. As $|a| \to 1$, one can expect $\sigma[E_\delta(a)] \approx (1-|a|^2)^{n-1}$, because $E_\delta(a)$ gets close roughly to a ball in S of radius proportional to $\delta(1-|a|^2)$. The following lemma shows this expectation by an accurate computation.

Lemma 4.5. *Given $\delta > 1/2$, the ratio $\sigma[E_\delta(a)]/(1-|a|^2)^{n-1}$ converges to a finite positive limit as $|a| \to 1$.*

Proof. Let $\delta > 1/2$ be given. Let $a = |a|\eta$ where $\eta \in S$ and assume that $|a|$ is sufficiently close to 1. Note that $\zeta \in E_\delta(a)$ if and only if

$$\begin{aligned} \delta(1-|a|^2) &> |a|\sqrt{1-2a\cdot\zeta+|a|^2} \\ &= |a|\sqrt{(1-|a|)^2+2|a|(1-\eta\cdot\zeta)}, \end{aligned}$$

or equivalently,

$$1-\eta\cdot\zeta < u(|a|)$$

where

$$u(t) = \frac{(1-t)^2}{2t}\left\{\left[\frac{\delta(1+t)}{t}\right]^2 - 1\right\}$$

for $0 < t < 1$. Note $u(t) > 0$ for all t, because $\delta > 1/2$. Also, note $u(t) < 1$ for t sufficiently close to 1. Thus, by the slice integration formula (3.3), we have

$$\begin{aligned} c_n^{-1}\sigma[E_\delta(a)] &= \int_{1-u(|a|)}^{1} (1-r^2)^{\frac{(n-3)}{2}}\,dr \\ &= \int_0^{u(|a|)} t^{\frac{(n-3)}{2}}(2-t)^{\frac{(n-3)}{2}}\,dt \\ &= u(|a|)^{\frac{(n-1)}{2}} \int_0^1 s^{\frac{(n-3)}{2}}\left[2-u(|a|)s\right]^{\frac{(n-3)}{2}}\,ds. \end{aligned}$$

Now, a little manipulation yields

$$\lim_{|a|\to 1} \frac{\sigma[E_\delta(a)]}{(1-|a^2|)^{n-1}} = \frac{4(2\delta-1)^{\frac{(n-1)}{2}}}{n-1}c_n,$$

which completes the proof. $\square$

We are now ready to prove the next theorem describing a pathological behavior of the Berezin transform.

Theorem 4.6. *Let $\mu \geq 0$ and $\mu(B) > 0$. Then the following statements hold:*

(a) *If $p \leq -(\alpha+1)/n$, then $\widetilde{\mu} \notin L^p_\alpha$.*
(b) *If $\alpha + 1/p < -n$, then $\widetilde{\mu} \notin \mathcal{K}^{p,\alpha}_\infty$.*
(c) *If $\alpha + 1/p = -n$, then $\widetilde{\mu} \notin \mathcal{K}^{p,\alpha}_0$.*

Proof. Let μ_1 be the restriction of μ to $\frac{1}{3}B$. First, consider the case $\mu_1(B) > 0$. Note that if $|y| < 1/3$, then

$$\left(\frac{1-|x|^2|y|^2}{[x,y]}\right)^2 - \frac{4|x|^2|y|^2}{n} \geq (1-|x||y|)^2 - \frac{4|x|^2|y|^2}{n} > \frac{4}{9} - \frac{4}{9n} > 0$$

for all $x \in B$. It follows from this and (1.1) that

$$\widetilde{\mu_1}(x) \gtrsim (1-|x|)^n \int_{\frac{1}{3}B} |R(x,y)|^2\, d\mu_1(y) \gtrsim (1-|x|)^n \mu_1(B) \tag{4.3}$$

for all $x \in B$. This yields (a). Also, we have (b) and (c) by Lemma 2.1.

Next, consider the case $\mu_1(B) = 0$. Fix $\epsilon > 0$ and $\delta > 1/2$ such that $\epsilon + \delta < \sqrt{n}/2$. Since $\mu(B) > 0$ by assumption, we have $\mu[Q_\epsilon(a)] > 0$ for some $a \in B$. Let $x \in \Gamma_\delta(a)$ and assume $|x| \geq 1/2$. We have by (1.1)

$$R(x,y) \gtrsim \frac{|x|^2|y|^2}{[x,y]^n}\left(\frac{1}{(\epsilon+\delta)^2} - \frac{4}{n}\right) \gtrsim 1$$

for $y \in \Gamma_{2\delta}(x)$ with $|y| \geq 1/3$. Since μ is supported on $B \setminus \frac{1}{3}B$, we deduce from the above that

$$\widetilde{\mu}(x) \gtrsim (1-|x|)^n \mu[\Gamma_{2\delta}(x)] \geq (1-|x|)^n \mu[Q_\delta(a)] \tag{4.4}$$

for $x \in \Gamma_\delta(a)$ with $|x| \geq 1/2$. We may assume $\mu[Q_\delta(a)] = 1$ for simplicity.

Let $0 < p < \infty$. Using (4.4) and integrating in polar coordinates, we have

$$\|\widetilde{\mu}\|^p_{L^p_\alpha} \gtrsim \int_{\Gamma_\delta(a),|x|\geq 1/2} (1-|x|)^{pn+\alpha}\, dx \approx \int_{1/2}^1 (1-r)^{pn+\alpha}\sigma[E_\delta(ra)]\, dr.$$

Meanwhile, by Lemma 4.5, we have $\sigma[E_\delta(ra)] \approx (1-r|a|)^{n-1} \geq (1-|a|)^{n-1}$ for all $0 \leq r < 1$. Combining these observations, we have

$$\|\widetilde{\mu}\|^p_{L^p_\alpha} \gtrsim N \int_{1/2}^1 (1-r)^{pn+\alpha}\, dr = \infty$$

for $pn + \alpha \leq -1$ where $N = (1-|a|)^{n-1}$. Thus (a) holds.

In order to prove (b) and (c), we first estimate $\|\widetilde{\mu}\chi_m\|_{L^p}$. Let $m \geq 1$ so that $r_m \geq 1/2$. Proceeding as above, we have

$$\|\widetilde{\mu}\chi_m\|_{L^p}^p \gtrsim N \int_{r_m}^{r_{m+1}} (1-r)^{np}\,dr \approx N 2^{-m(1+np)}$$

so that

$$2^{-\alpha m}\|\widetilde{\mu}\chi_m\|_{L^p} \approx N^{1/p} 2^{-m(\alpha+1/p+n)}$$

for all $m \geq 1$. Moreover, this estimate remains valid for $p = \infty$ by (4.4). This yields (b) and (c). The proof is complete. $\square$

Since $\mathcal{K}_q^{p,\alpha} \subset \mathcal{K}_0^{p,\alpha} \subset \mathcal{K}_\infty^{p,\alpha}$ for any $0 \leq q < \infty$, we have the following consequence of Theorem 4.6(b), which also takes care of the subcases (1)-(3).

Corollary 4.7. *If either one of (1)-(3) holds, then $\widetilde{f} \notin \mathcal{K}_q^{p,\alpha}$ for any $f \in \mathcal{K}_q^{p,\alpha}$ with $f \geq 0$ and $f \not\equiv 0$.*

4.2. **The Case $\alpha + 1/p \geq 1$.** We further split this case into the following three subcases:

(5) $\alpha + 1/p > 1$ with q arbitrary;
(6) $\alpha + 1/p = 1$ with $1 < q \leq \infty$ or $q = 0$;
(7) $\alpha + 1/p = 1$ with $0 < q \leq 1$.

The next example covers the subcases (5) and (6).

Example 4.8. *If (5) or (6) holds, then $f_{1,1} \in \mathcal{K}_q^{p,\alpha}$ but $\widetilde{f}_{1,1} = \infty$ on B.*

Proof. Under the condition (5) or (6), we have $f_{1,1} \in \mathcal{K}_q^{p,\alpha}$ by Lemma 2.1. Note that $h_{1,1}$ is not integrable near $r = 1$. Thus, following the proof of Example 4.2, we have by Lemma 3.1

$$\widetilde{f}_{1,1}(x) \approx (1-|x|)^n \int_0^1 \frac{h_{1,1}(r)}{(1-r|x|)^{n+1}}\,dr \geq (1-|x|)^n \int_0^1 h_1(r)\,dr = \infty$$

for all $x \in B$, as desired. $\square$

In case of (7), we have $\mathcal{K}_q^{p,\alpha} \subset L^1$ by (2.3) so that Berezin transform is well defined on that space. So, one cannot expect an example whose Berezin transform blows up as in Example 4.8.

Example 4.9. *If (7) holds, then $f_{1,\gamma} \in \mathcal{K}_q^{p,\alpha}$ but $\widetilde{f}_{1,\gamma} \notin \mathcal{K}_q^{p,\alpha}$ for all γ with $1/q < \gamma \leq 1/q + 1$.*

Proof. Assume that (7) holds and let $1/q < \gamma \leq 1/q + 1$. Since $1/q < \gamma$, we have $f_{1,\gamma} \in \mathcal{K}_q^{p,\alpha}$ by Lemma 2.1.

Now, we estimate $\widetilde{f}_{1,\gamma}$. Let $x \in B$ and assume $|x| \geq 1/2$. Following the proof of Example 4.2, we have

$$\begin{aligned}
\widetilde{f}_{1,\gamma}(x) &\approx (1-|x|)^n \int_0^1 \frac{h_{1,\gamma}(r)}{[(1-|x|)+(1-r)]^{n+1}}\, dr \\
&= (1-|x|)^n \int_0^1 \frac{(1-\log t)^{-\gamma}}{(1-|x|+t)^{n+1}} \frac{dt}{t} \\
&\geq (1-|x|)^n \int_0^{1-|x|} \frac{(1-\log t)^{-\gamma}}{(1-|x|+t)^{n+1}} \frac{dt}{t} \\
&\gtrsim \frac{1}{1-|x|} \int_0^{1-|x|} (1-\log t)^{-\gamma} \frac{dt}{t}.
\end{aligned}$$

Evaluating the integral above, we obtain

$$\widetilde{f}_{1,\gamma}(x) \gtrsim \frac{1}{1-|x|}\left(1+\log\frac{1}{1-|x|}\right)^{1-\gamma} = f_{1,\gamma-1}(x)$$

for $|x| \geq 1/2$ and thus for all $x \in B$. Note $0 \leq 1/q - 1 < \gamma - 1 \leq 1/q$. Thus, by (7) and Lemma 2.1, we have $\widetilde{f}_{1,\gamma-1} \notin \mathcal{K}_q^{p,\alpha}$ and thus $\widetilde{f}_{1,\gamma} \notin \mathcal{K}_q^{p,\alpha}$. The proof is complete. □

5. Growth Estimates

Throughout this section we restrict the range of p to $1 \leq p \leq \infty$. Suppose parameters p, q and α satisfy (2.3) so that the Berezin transform is well defines on Herz spaces with such parameters. In this section we prove pointwise growth estimates of Berezin transforms in two cases: (i) $\alpha + 1/p < 1$ and $q = 1$ and (ii) $\alpha + 1/p = 1$ and $q = \infty$. In what follows, we let $R_x = R(x, \cdot)$.

The starting point is Hölder's inequality. Namely, given a measurable function $f \geq 0$ on B, we have by (2.1)

$$\widetilde{f}(x) \lesssim (1-|x|)^n \|f\|_{\mathcal{K}_q^{p,\alpha}} \|R_x^2\|_{\mathcal{K}_{q'}^{p',-\alpha}}, \qquad x \in B \tag{5.1}$$

for $1 \leq q \leq \infty$. Recall that p' denotes the conjugate index of p. So, we need to estimate the growth rate of Herz norms of R_x^2.

Lemma 5.1. *Let $1 \leq p \leq \infty$ and α be real. Assume $\alpha \geq -1/p$. Then there exist constants $C = C(p, \alpha) > 0$ such that the following inequalities hold for $x \in B$.*

(a) *If* $\alpha > -1/p$, *then*

$$\|R_x^2\|_{\mathcal{K}_1^{p,\alpha}} \leq C \times \begin{cases} 1 & \text{if} \quad \alpha > n(2-1/p) \\ 1 + \log \frac{1}{1-|x|} & \text{if} \quad \alpha = n(2-1/p) \\ (1-|x|)^{-2n+n/p+\alpha} & \text{if} \quad \alpha < n(2-1/p). \end{cases}$$

(b) *If* $\alpha = -1/p$, *then*

$$\|R_x^2\|_{\mathcal{K}_\infty^{p,\alpha}} \leq C(1-|x|)^{-2n+(n-1)/p}.$$

Proof. Fix $x \in B$. We may further assume $|x| \geq 1/2$. We first estimate $\|R_x^2 \chi_m\|_{L^p}$. For $p < \infty$, we have by Lemma 3.2 and (4.2)

$$\|R_x^2 \chi_m\|_{L^p}^p \lesssim \frac{r_{m+1} - r_m}{(1 - r_{m+1}|x|)^{2pn-n+1}} \approx \frac{2^{-m}}{(1-|x|+2^{-m})^{2pn-n+1}}$$

and therefore

$$2^{-m\alpha}\|R_x^2\chi_m\|_{L^p} \lesssim \frac{2^{-m(\alpha+1/p)}}{(1-|x|+2^{-m})^{2n-(n-1)/p}} \tag{5.2}$$

for all m. Since

$$|R(x,y)| \lesssim \frac{1}{[x,y]^n} \leq \frac{1}{(1-|x||y|)^n} \approx \frac{1}{[(1-|x|)+(1-|y|)]^n}$$

for $y \in B$ by (1.1) and (4.2), the estimate (5.2) remains valid even for $p = \infty$.

Clearly, (b) holds by (5.2). We now prove (a). So, assume $\alpha > -1/p$. If $1 - |x| \leq 2^{-m}$, then we have by (5.2)

$$2^{-m\alpha}\|R_x^2\chi_m\|_{L^p} \lesssim \frac{2^{-m(\alpha+1/p)}}{2^{-m(2n-(n-1)/p)}} = 2^{m(2n-n/p-\alpha)}.$$

Meanwhile, if $1 - |x| > 2^{-m}$, then we have by (5.2)

$$2^{-m\alpha}\|R_x^2\chi_m\|_{L^p} \lesssim \frac{2^{-m(\alpha+1/p)}}{(1-|x|)^{2n-(n-1)/p}}.$$

It follows from these estimates that

$$\begin{aligned} \|R_x^2\|_{\mathcal{K}_1^{p,\alpha}} &\lesssim \sum_{m \leq \log_2(1-|x|)^{-1}} 2^{m(2n-n/p-\alpha)} \\ &\quad + \frac{1}{(1-|x|)^{2n-(n-1)/p}} \sum_{m > \log_2(1-|x|)^{-1}} 2^{-m(\alpha+1/p)} \\ &:= I + II \end{aligned}$$

It is not hard to verify the following estimate:

$$I \approx \begin{cases} 1 & \text{if} \quad \alpha > n(2-1/p) \\ 1 + \log \frac{1}{1-|x|} & \text{if} \quad \alpha = n(2-1/p) \\ (1-|x|)^{-2n+n/p+\alpha} & \text{if} \quad \alpha < n(2-1/p). \end{cases}$$

Also, since $\alpha + 1/p > 0$, we have

$$\sum_{m > \log_2 (1-|x|)^{-1}} 2^{-m(\alpha+1/p)} \approx (1-|x|)^{\alpha+1/p}$$

and thus

$$II \approx (1-|x|)^{-2n+n/p+\alpha}.$$

Now, putting the estimates of I and II together, we conclude (a). The proof is complete. □

Proof of Theorem 1.2. The theorem follows from (5.1) and Lemma 5.1. □

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