数理解析研究所講究録
1519巻 2006年 38-58

COMPLEXFIED HOPF FIBERATION AND A PSEUDO-DIFFERENTIAL OPERATOR

東京理科大学理工学部数学科 古谷賢朗[&](KENRO FURUTANI)
DEAPRTMENT OF MATHEMATICS
FACULTY OF SCIENCE AND TECHNOLOGY
SCIENCE UNIVERSITY OF TOKYO
〒278-8510 千葉県野田市山崎 2641

CONTENTS

1. INTRODUCTION

We explain a complexfication of the Hopf bundle on the complex projective space. The base space is isomorphic to the punctured cotangent bundle of the complex projective space and the total space is an open subset of the cotangent bundle of the sphere, both of which are Kähler manifolds whose Kähler forms coincide with the natural symplectic forms respectively. Then we define Hilbert spaces on both of the base space and the total space consisting of L_2-holomorphic functions with respect to measures depending on certain parameters. The main purpose is to give a relation among the measures when the pull-back operator by the projection map of this principal bundle has a close image and show that the comparison operator of the quantization operator constructed by fiber integrations and that through the complexfied Hopf fiberation is a zeroth order pseudo-differential operator on the complex projective space.

2. HILBERT SPACE ON QUADRICS

Let $S^n = \{(x_0, \cdots, x_n) \in \mathbb{R}^{n+1} \mid \sum x_k^2 = 1\}$ be the n-dimensional sphere with the standard Riemann metric induced from the Euclidean metric on $\mathbb{R}^{n+1}$. We identify the

[&]Partially supported by the Grant-in-aid Scientific Research (C) No. 17540202, Japan Society for the Promotion of Science.

Mathematical Subject Classification 2000 : 53D50, 53D25, 32Q15.

Key words and phrases: geometric quantization, quantization operator, fiber integration, pairing of the polarizations, Kähler structure, sphere, complex projective space, quaternion projective space, geodesic flow, Hopf fiberation, reproducing kernel, Segal-Bargmann space, pseudo-differential operator.

tangent bundle $T(S^n)$ and the cotangent bundle $T^*(S^n)$ by this metric and realize them as a subspace in $\mathbb{R}^{n+1}\times\mathbb{R}^{n+1}$:

$$T(S^n)^* \cong T(S^n)$$
$$=\left\{(x,y)\in\mathbb{R}^{n+1}\times\mathbb{R}^{n+1}\,|\,\sqrt{\sum x_k{}^2}=\|x\|=1,\ (x,y)=\sum x_k y_k=0\right\}.$$

The symplectic form ω_S and the canonical one form θ_S on $T^*(S^n)$ are then the restrictions of the symplectic form $\sum_{k=0}^n dy_k\wedge dx_k$ and the canonical one form $\sum_{k=0}^n y_k dx_k$ on $T^*(\mathbb{R}^{n+1})\cong\mathbb{R}^{n+1}\times\mathbb{R}^{n+1}$ respectively.

Let $T_0^*(S^n)\cong\mathbf{E}_S=\{(x,y)\in T(S^n)\mid y\neq 0\}$ be the punctured cotangent bundle, and define the map

$$\tau_S:\mathbf{E}_S\longrightarrow\mathbb{C}^{n+1}\backslash\{0\},\ (x,y)\mapsto z=\|y\|x+\sqrt{-1}y.$$

The map τ_S gives a diffeomorphism with the space

$$\mathbf{X}_S=\{z\in\mathbb{C}^{n+1}\,|\,z^2=\sum z_k^2=0, z\neq 0\},$$

and through this map the canonical one form θ_S is expressed as

$$\theta_S=\sqrt{\frac{-1}{2}}\left(\partial\,\|z\|-\overline{\partial}\,\|z\|\right). \tag{2.1}$$

and so the symplectic form ω_S is expressed as

$$\omega_S=d\theta_S=\sqrt{-2}d\,\partial\|z\|=\sqrt{-2}\,\overline{\partial}\,\partial\|z\|.$$

Let $dvol_S$ be the Riemann volume form on the sphere, then it is expressed as

$$dvol_S=\sum_{k=0}^n(-1)^k x_k dx_0\wedge\cdots\wedge\widehat{dx_k}\wedge\cdots\wedge x_n.$$

Now we introduce inner products on the space of polynomials on $\mathbb{C}^{n+1}$ restricted to a quadric $\mathbf{X}_S$ depending on two parameters (h,N), $h>0$ and $N>-n$ as follows:

$$<p_1,p_2>_{(h,N)}=\int_{\mathbf{X}_S}p_1(z)\overline{p_2(z)}e^{-h\|z\|}\|z\|^N\Omega_S, \tag{2.2}$$

where $\Omega_S=\frac{(-1)^{n(n+1)/2}}{n!}\omega_S^n$ is the Liouville volume form.

When we decompose the quadric $\mathbf{X}_S$ as

$$\mathbf{X}_S\cong\mathbb{R}_+\times\Sigma(S),\ \Sigma(S)=\mathbf{X}_S\cap\{z\in\mathbb{C}^{n+1}\mid\|z\|=1\},$$

the Liouville volume form Ω_S is decomposed as

$$\Omega_S=t^{n-1}dt\wedge dv_{\Sigma(S)}(\sigma),\ (t,\sigma)\in\mathbb{R}_+\times\Sigma(S)$$

with a nowhere vanishing $(2n-1)$-form $dv_{\Sigma(S)}$ on $\Sigma(S)$ (volume form on $\Sigma(S)$) which is invariant under the action (lifted to $\mathbf{X}_S$) of the orthogonal group $O(n)$. By the condition $N>-n$, all polynomials (restricted to $\mathbf{X}_S$) are square-integrable with respect to the measure $dm_{(h,N)}(z)=e^{-h\|z\|}\|z\|^N\Omega_S(z)$.

Definition 2.1. We denote the Hilbert space taken the completion from the space of polynomials with respect to this inner product $<\bullet,\bullet>_{(h,N)}$ by $\mathcal{H}^2(\mathbf{X}_S,dm_{(h,N)}(z))$.

Next we introduce an operator $T^S_{(h,N)}$ in terms of the fiber integration from $\mathcal{H}^2(\mathbf{X}_S,dm_{(h,N)}(z))$ to $L_2(S^n)$:

Definition 2.2.

$$T^S_{(h,N)}(\varphi)(x)dvol_S(x) = (\pi_S \circ \tau_S^{-1})_*(\varphi(z)dm_{(h,N)}(z)), \ \pi_S \circ \tau^{-1}(z) = x.$$

Proposition 2.3. *Let $n \geq 2$, then there are no bounded holomorphic functions on the quadric $\mathbf{X}_S$.*

Also we note

Proposition 2.4. *The function $\varphi \in \mathcal{H}^2(\mathbf{X}_S, dm_{(h,N)}(z))$ has the expansion of the form $\varphi(z) = \sum_{k=0}^{\infty} \varphi_k(z)$ with k-th order polynomials φ_k.*

Then we have an expression of the norm of a function φ in $\mathcal{H}^2(\mathbf{X}_S, dm_{(h,N)}(z))$: let $\varphi(z) = \sum_{k=0}^{\infty} \varphi_k(z) \in \mathcal{H}^2(\mathbf{X}_S, dm_{(h,N)}(z))$, then

$$\begin{aligned}
\|\varphi\|^2_{(h,N)} &= \int_{\mathbf{X}_S} |\varphi(z)|^2 dm_{(h,N)}(z) \\
&= \sum_{k=0}^{\infty} \int_{\Sigma(S)} |\varphi_k(\sigma)|^2 dv_{\Sigma(S)}(\sigma) \times \int_0^{\infty} \|z\|^{2k+N+n-1} e^{-h\|z\|} d\|z\| \\
&= \sum_{k=0}^{\infty} \frac{\Gamma(2k+N+n)}{h^{2k+N+n}} \int_{\Sigma(S)} |\varphi_k(\sigma)|^2 dv_{\Sigma(S)}(\sigma)
\end{aligned}$$

By this expression we have

Proposition 2.5. *Let $-n < \widetilde{N} \leq N$, then*

$$\mathcal{H}^2(\mathbf{X}_S, dm_{(h,N)}(z)) \hookrightarrow \mathcal{H}^2(\mathbf{X}_S, dm_{(h,\widetilde{N})}(z)) \tag{2.3}$$

and the inclusion map is continuous.

Theorem 2.6. *Let $2h_0 = h$ and $N = 2N_0 + n/2$ then the operator*

$$T^S_{(h_0,N_0)} : \mathcal{H}^2(\mathbf{X}_S, dm_{(h,N)}(z)) \to L_2(S^n)$$

is an isomorphism.

Let $\mathcal{P}_k$ be the space of homogeneous polynomials of degree k on $\mathbb{C}^{n+1}$. If it is considered on $\mathbf{X}_S$, then we denote it by $\mathcal{P}_k(\mathbf{X}_S)$.

Let $\mathcal{S}_k = \mathcal{S}_k(\mathbb{R}^{n+1})$ be the space of harmonic polynomials of degree k on $\mathbb{R}^{n+1}$ and when we consider their restrictions to the sphere, it is well known that this space is an eigenspace of the Laplacian on the sphere with the eigenvalue $\lambda_k = k(k+n-1)$, $k = 0, 1, \cdots$ and the dimension is given by

$$\dim \mathcal{S}_k = \frac{\Gamma(k+n-1)(2k+n-1)}{k! \cdot (n-1)!}.$$

For each $k \in \mathbb{N}$, let $A^S_k : \mathcal{S}_k \to \mathcal{P}_k$ be a map defined by the integral

$$A^S_k(f)(z) = \int_{S^n} f(x) < x, z >^k dvol_S(x),$$

where $< x, z >= \sum_{i=0}^{n} x_i z_i$ is the bilinear form.

Also for each k, let $B^S_{(h,N;k)}$ be a map from $\mathcal{P}_k(\mathbf{X}_S)$ to $\mathcal{S}_k$ defined by

$$B^S_{(h,N;k)}(p)(x) = \int_{\mathbf{X}_S} p(z) < x, \overline{z} >^k e^{-h\|z\|}\|z\|^N \Omega_S(z).$$

Remark 2.7. Let Δ be the Laplacian on $\mathbb{R}^{n+1}$, then for $z \in \mathbf{X}_S$ we have $\Delta(< x, \overline{z} >^k) = 0$, which shows that the polynomial defined by the above integral is a harmonic polynomial.

Let $C^S_{(h,N;k)}$ be a map from $\mathcal{P}_k(\mathbf{X}_S)$ to $\mathcal{P}_k(\mathbf{X}_S)$ defined by the integral

$$C^S_{(h,N;k)}(p)(\lambda) = \int_{\mathbf{X}_S} p(z) < \overline{z}, \lambda >^k e^{-h\|z\|}\|z\|^N \Omega_S(z), \ (N > -n).$$

Among these operators we have the following relations:

$$\begin{aligned} B^S_{(h,N;k)} \circ A^S_k &= a^S_k(h, N), \\ T^S_{(h,N;k)} \circ A^S_k &= b^S_k(h_0, N_0), \\ C^S_{(h,N;k)} &= c^S_k(h, N) Id, \end{aligned}$$

especially, we have

$$T^S_{(h,N;k)} = \frac{b^S_k(h_0, N_0)}{a^S_k(h, N)} B^S_{(h,N;k)}.$$

The explicit values of the constants $a^S_k(h, N)$, $b^S_k(h_0, N_0)$ and $c^S_k(h, N)$ are given as

$$\begin{aligned} a^S_k(h, N) &= \frac{\Gamma(n)\mathrm{Vol}(\Sigma(S))\mathrm{Vol}(S^{n-2})}{h^{n+N}} \times \\ &\times \frac{\Gamma(k+1)^2\Gamma(2k+N+n)}{2^k \cdot h^{2k}(2k+n-1)\Gamma(k+n-1)\Gamma(k+(n+1)/2)}, \\ b^S_k(h_0, N_0) &= \frac{1}{\dim \mathcal{S}_k} \cdot Vol(\Sigma(S)) \frac{\Gamma(k+N_0+n)}{(\sqrt{2})^k \cdot h_0^{k+N_0+n}} \\ c^S_k(h, N) &= \frac{1}{\dim \mathcal{S}_k} \cdot \mathrm{Vol}(\Sigma(S)) \frac{\Gamma(2k+N+n)}{h^{2k+N+n}}, \end{aligned}$$

where the constant $\mathrm{Vol}(\Sigma(S))$ is the volume of $\Sigma(S)$ with respect to the volume form $dv_{\Sigma(S)}$.

Remark 2.8. All these operators $T^S_{(h_0,N_0)} : \mathcal{H}^2(\mathbf{X}_S, dm_{(2h_0, 2N_0+n/2)}) \to L_2(S^n)$ are not unitary, but each restriction on $\mathcal{P}_k(\mathbf{X}_S)$ is a constant multiple of a unitary operator.

Theorem 2.9. *Let $N \geq \widetilde{N} > -n$, and put $h_0 = h/2$, $N_0 = 1/2(N - n/2)$ and $\widetilde{N}_0 = 1/2(\widetilde{N} - n/2)$, then the operator $T^S_{(h_0,\widetilde{N}_0)} \circ {T^S_{(h_0,N_0)}}^{-1} : L_2(S^n) \to L_2(S^n)$, i.e.*

$$L_2(S^n) \xrightarrow{{T^S_{(h_0,N_0)}}^{-1}} \mathcal{H}^2(\mathbf{X}_S, dm_{(h,N)}(z)) \hookrightarrow \mathcal{H}^2(\mathbf{X}_S, dm_{(h,\widetilde{N})}(z)) \xrightarrow{T^S_{(h_0,\widetilde{N}_0)}} L_2(S^n)$$

is a pseudo-differential operator of order $\frac{1}{2}(\widetilde{N} - N)$.

Corollary 2.10. *Let $-n < \widetilde{N} < N$, then the inclusion map (2.3) is compact.*

Corollary 2.11. *Let $N - \widetilde{N} > 2n$, then the operator $T^S_{(h_0,\widetilde{N}_0)} \circ T^S_{(h_0,N_0)}{}^{-1}$ is a trace class operator and the trace is given by*

$$\mathbf{tr}(T^S_{(h_0,\widetilde{N}_0)} \circ T^S_{(h_0,N_0)}{}^{-1}) = (h/2)^{(N-\widetilde{N})/2} \sum_{k=0}^{\infty} \frac{\Gamma(k + \widetilde{N}/2 + (3n)/4)}{\Gamma(k + N/2 + (3n)/4)} \dim \mathcal{S}_k.$$

For each $N > \widetilde{N}$, the operator $T^S_{(h_0,\widetilde{N}_0)}$ maps the space $\mathcal{H}^2(\mathbf{X}_S, dm_{(h,N)}(z))$ onto the Sobolev space on S^n of order $(N - \widetilde{N})/2$, we have

Corollary 2.12. *The projective limits*

$$\varprojlim \mathcal{H}^2(\mathbf{X}_S, dm_{(h,N')}(z)) \cong \bigcap_{N'} \mathcal{H}^2(\mathbf{X}_S, dm_{(h,N')}(z))$$

is mapped onto the space $C^\infty(S^n)$ by the map $T^S_{(h_0,N_0)}$, where $h_0 > 0$ and $N_0 > -3n/4$.

For the proof of Theorem 2.9 we employ a criterion for an operator defined by functional calculus of a positive selfadjoint elliptic pseudo-differential operator to be a pseudo-differential operator. Here we state the theorem (cf. [Ta]):

Theorem 2.13. *Let A be a first order positive elliptic pseudo-differential operator of classical type defined on a closed manifold M and let*

$$A = \int_0^{+\infty} \lambda dE_A(\lambda)$$

be the spectral decomposition of A with the spectral measure $\{E_A(\lambda)\}$. Let $f \in S^m_{1,0}(\mathbb{R})$ ($m \in \mathbb{R}$) be in the symbol class of Hörmander, that is function f satisfies

$$\left|\frac{d^\ell}{dt^\ell}(f)(t)\right| \leq C_\ell(1 + |t|)^{m-\ell}$$

for any ℓ with constant $C_\ell > 0$. Then

$$f(A) = \int_0^{+\infty} f(\lambda) dE_A(\lambda) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} \widehat{f}(t) e^{\sqrt{-1}tA} dt \tag{2.4}$$

is a pseudo-differential operator in the class $L^m_{1,0}(M)$, where

$$\widehat{f}(t) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} e^{-\sqrt{-1}tx} f(x) dx$$

is the Fourier transformation of f.

3. Reproducing kernel

In this section we give a formula of the reproducing kernel of the Hilbert space $\mathcal{H}^2$ $(\mathbf{X}_S,\ dm_{(h,N)}(z))$ and state an asymptotic property of it. Then as an application of the operator $T^S_{(h,N)}$ and the existence of the reproducing kernel, we give an interpretation of a quantization of a free particle in the sense of classical mechanics.

By the properties listed in the preceding section we know that the Hilbert space $\mathcal{H}^2(\mathbf{X}_S, dm_{(h,N)}(z))$ has the reproducing kernel $\mathcal{K}^S_{(h,N)}(z,\lambda)$ $(=\mathcal{K}^S(z,\lambda)$, when the parameters (h, N) are fixed) which we can express in the form of power series:

$$\mathcal{K}^S(z,\lambda) = \sum_{k=0}^{\infty} \frac{1}{c^S_k(h,N)} < z, \overline{\lambda} >^k \tag{3.1}$$

$$= \sum_{k=0}^{\infty} \frac{1}{a_k^S(h,N)} \int_{S^n} < x, z >^k < x, \overline{\lambda} >^k \, dvol_S(x).$$

Note that for each k, we have

$$\frac{1}{a_k^S(h,N)} \int_{S^n} < x, z >^k < x, \overline{\lambda} >^k \, dvol_S(x) = \frac{1}{c_k^S(h,N)} < z, \overline{\lambda} >^k$$

and the series converges on the whole space $\mathbb{C}^{n+1} \times \mathbb{C}^{n+1}$.

We also give an integral representation of the reproducing kernel $\mathcal{K}^S_{(h,N)}(z,\lambda)$ for $N > 0$:

$$\begin{aligned}
&\mathcal{K}^S_{(h,N)}(z,\lambda) \\
&= C(h,n,N) \cdot 2 \cdot h^2 \cdot \frac{1}{2\sqrt{\pi}} \frac{1}{\Gamma(N+1)} \cdot \frac{1}{\Gamma(1/2)} \cdot < z, \overline{\lambda} > \times \\
&\times \int_0^1 \int_0^1 \int_0^1 \frac{s^{n-1}(1-s)^{N+1}(1-t)^N}{\sqrt{u(1-u)}} \cosh\left(\sqrt{s \cdot (1-s) \cdot t \cdot u \cdot (2h)^2 \cdot < z, \overline{\lambda} >}\right) dsdtdu \\
&+ C(h,n,N) \cdot (n-1) \cdot \frac{1}{\Gamma(N)} \cdot \frac{1}{2\sqrt{\pi}} \cdot \frac{1}{\Gamma(1/2)} \times \\
&\times \int_0^1 \int_0^1 \int_0^1 \frac{s^{n-2}(1-s)^{N}(1-t)^{N-1}}{\sqrt{u(1-u)}} \cosh\left(\sqrt{s \cdot (1-s) \cdot t \cdot u \cdot (2h)^2 \cdot < z, \overline{\lambda} >}\right) dsdtdu,
\end{aligned}$$

where the constant $C(h,n,N)$ is

$$C(h,n,N) = \frac{h^{n+N}}{Vol(\Sigma(S)) \cdot \Gamma(n)}.$$

Since cosh is an even function, the function $\cosh\left(\sqrt{s \cdot (1-s) \cdot t \cdot u \cdot (2h)^2 \cdot < z, \overline{\lambda} >}\right)$ can be defined for any $z \in \mathbb{C}$ and $\lambda \in \mathbb{C}$ without ambiguity.

As an application of the fact that the series (3.1) converges for any z and $\lambda \in \mathbb{C}^{n+1}$, every function in $\mathcal{H}^2(\mathbf{X}_S, dm_{(h,N)}(z))$ can be extended to the whole complex plane: let $Holo(\mathbb{C}^{n+1})$ be the space of holomorphic functions on $\mathbb{C}^{n+1}$ with the topology of the locally uniform convergence, then for $f \in \mathcal{H}^2(\mathbf{X}_S, dm_{(h,N)}(z))$, by the expression

$$f(\lambda) = < f(\bullet), \mathcal{K}^S(\bullet, \lambda) >_{(h,N)} = \sum_{k=0}^{\infty} \int_{\mathbf{X}_S} f(z) \frac{< \overline{z}, \lambda >^k}{c_k^S(h,N)} e^{-h\|z\|} \|z\|^N \Omega_S(z),$$

we have

Proposition 3.1.

$$\mathcal{H}^2(\mathbf{X}_S, dm_{(h,N)}(z)) \to Holo(\mathbb{C}^{n+1})$$

is continuous.

In the case of the Segal-Bargmann space $\mathcal{H}^2\left(\mathbb{C}^n, \frac{1}{\pi^n} e^{-\|z\|^2} \Omega_{\mathbb{R}}\right)$

$$\begin{aligned}
&\mathcal{H}^2\left(\mathbb{C}^n, \frac{1}{\pi^n} e^{-\|z\|^2} \Omega_{\mathbb{R}}\right) \\
&= \left\{ f \,\middle|\, f \text{ is holomorphic on } \mathbb{C}^n \text{ and } \|f\|^2 = \frac{1}{\pi^n} \int_{\mathbb{C}^n} |f(z)|^2 e^{-\|z\|^2} \Omega_{\mathbb{R}} < \infty \right\}
\end{aligned}$$

($\Omega_{\mathbb{R}}$ denotes the Liouville volume form on $T^*(\mathbb{R}^n) \cong \mathbb{C}^n$, or the Lebesgue measure on $\mathbb{R}^{2n}$), the reproducing kernel is $\mathcal{K}^{\mathbb{R}}(z,\lambda) = e^{<z,\lambda>}$ and the product

$$\mathcal{K}^{\mathbb{R}}(\lambda,\lambda)e^{-\|\lambda\|^2}\Omega_{\mathbb{R}} = \Omega_{\mathbb{R}}$$

returns to the Lebesgue measure on $\mathbb{R}^{2n}$.

We state here a corresponding property of the product of the reproducing kernel restricted to the diagonal and the weight function of the volume form for defining the Hilbert space $\mathcal{H}^2\left(\mathbf{X}_S, dm_{(h,N)}(z)\right)$:

Theorem 3.2. *For $N > -n$, rational number (this assumption would be removed), $\mathcal{K}^S_{(h,N)}(\lambda,\lambda)e^{-h\|\lambda\|}\|\lambda\|^N = O(1)$.*

Let $\{g_t\}_{t\in\mathbb{R}}$ be the geodesic flow action on $\mathcal{H}^2(\mathbf{X}_S, dm_{(h,N)}(z))$, i.e.

$$\text{for } \varphi = \sum_{k=0}^{\infty}\varphi_k(z) \in \mathcal{H}^2(\mathbf{X}_S, dm_{(h,N)}(z)) \text{ with homogeneous } \varphi_k \text{ of degree } k,$$

$$g_t(\varphi)(z) = \varphi(e^{\sqrt{-1}t}\cdot z) = \sum e^{\sqrt{-1}tk}\varphi_k(z).$$

Then we have

Theorem 3.3.

$$T^S_{(h,N)} \circ g_t = e^{-t\sqrt{-1}(n-1)/2}e^{\sqrt{-1}t\sqrt{\Delta+((n-1)/2)^2}} \circ T^S_{(h,N)}. \tag{3.2}$$

Since the Hilbert space $\mathcal{H}^2(\mathbf{X}_S, dm_{(h,N)}(z))$ has the reproducing kernel, for each point $z \in \mathbf{X}_S$, we can assign a function $q_z(x) \in L_2(S^n)$ in such a way that

$$\begin{aligned} {T^S_{(h,N)}}^{-1}(f)(z) &= \int_{\mathbf{X}_S} {T^S_{(h,N)}}^{-1}(f)(\lambda)\cdot\overline{\mathcal{K}^S_{(h,N)}(z,\lambda)}dm_{(h,N)}(\lambda) \\ &=< f, q_z >_{(h,N)}= \int_{S^n} f(x)\overline{q_z(x)}dvol_S(x), \end{aligned}$$

and q_z is given by

$$q_z(x) = \sum_{k=0}^{\infty}\frac{1}{b^S_k(h,N)} < x, \overline{z} >^k .$$

Then the function q_z can be seen as a quatization of a classical free particle and we have a correspondence of classical and quantum paths:

Proposition 3.4.

$$q_{e^{\sqrt{-1}t}\cdot z} = e^{t\sqrt{-1}(n-1)/2}e^{-\sqrt{-1}t\sqrt{\Delta+((n-1)/2)^2}}(q_z).$$

4. Complex projective space

In this section we explain a corresponding Theorem to Theorem 2.6 and relating Theorems for the case of complex projective space $P^n\mathbb{C}$.

Let $\mathbf{X}_{\mathbb{C}}$ be the space in the $(n+1)\times(n+1)$ complex matrices defined by

$$\mathbf{X}_{\mathbb{C}} = \left\{A \in M(n+1;\mathbb{C}) \;\middle|\; A^2 = 0,\ \text{rank } A = 1\right\}. \tag{4.1}$$

In the paper [FT] we constructed an isomorphism between the space $\mathbf{X}_{\mathbb{C}}$ and the punctured (co)tangent bundle $T^*_0(P^n\mathbb{C})$ of the n-dimensional complex projective space.

First, we describe this isomorphism and the complexfied Hopf fiber bundle mostly by following [FY] to state our main theorems 4.8, 4.9 and 4.16 in this sction.

We identify $\mathbb{R}^{2n+2} \cong \mathbb{C}^{n+1}$ through the correspondence:

$$\begin{aligned}\mathbb{R}^{2n+2} \ni x &= (x_0, x_1, \cdots, x_n, x_{n+1}, \cdots, x_{2n+1}) \\ &= (x', x'') \longleftrightarrow x' + \sqrt{-1}x'' \\ &= (x_0 + \sqrt{-1}x_{n+1}, \cdots, x_n + \sqrt{-1}x_{2n+1}) \\ &= (p_0, \cdots, p_n) = p \in \mathbb{C}^{n+1}.\end{aligned}$$

$\mathbb{C}^{n+1}$ is equipped with the Hermitian inner product $< p, \bar{q} >= \sum p_i \bar{q}_i$ and the inner product on $\mathbb{R}^{2n+2}$ is defined by

$$Re < p, \bar{q} >= \sum_{i=0}^{2n+1} x_i y_i =< x, y >,$$

where $p_i = x_i + \sqrt{-1}x_{n+1+i}$, $q_i = y_i + \sqrt{-1}y_{n+1+i}$, $\bar{q} = (\overline{q_0}, \cdots, \overline{q_n})$.

Let $\pi_{\mathfrak{h}}$ be the projection map (Hopf fiberation):

$$\pi_{\mathfrak{h}} : S^{2n+1} \longrightarrow P^n\mathbb{C}.$$

We introduce the Riemann metric on $P^n\mathbb{C}$ by descending the standard Riemann metric on the unit sphere $S^{2n+1} = \{z \in \mathbb{C}^{n+1} \mid \|z\| = 1\}$ through this map $\pi_{\mathfrak{h}}$, and identify the tangent bundle $T(P^n\mathbb{C})$ and the cotangent bundle $T^*(P^n\mathbb{C})$ by this metric. Then the cotangent bundle $T^*(P^n\mathbb{C}) \cong T(P^n\mathbb{C})$ is realized in the matrix space as follows:

$$\begin{aligned}&T^*(P^n\mathbb{C}) \cong T(P^n\mathbb{C}) \\ &\cong \Big\{(P, Q) \in M(n+1, \mathbb{C}) \times M(n+1, \mathbb{C}) \;\Big|\; P^2 = P,\ \mathbf{tr}(P) = 1,\ PQ + QP = Q, \\ &\qquad\qquad P = P^*,\ Q = Q^*\Big\},\end{aligned}$$

where we put the inner product on the matrix space $M(n+1, \mathbb{C})$ by

$$\mathbf{tr}(A \cdot B^*).$$

We denote a subspace in this space with the condition $Q \neq 0$ by $\mathbf{E}_\mathbb{C}$ and let $\mathbf{E}_S \cong T_0(S^{2n+1}) \cong T_0^*(S^{2n+1})$ be the punctured (co)tangent bundle of the sphere

$$\begin{aligned}&T_0(S^{2n+1}) \\ &\cong \mathbf{E}_S = \Big\{(x', x''; y', y'') \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \;\Big| \\ &\qquad < x', x' > + < x'', x'' >= 1,\ < x', y' > + < x'', y'' >= 0,\ y' + \sqrt{-1}\,y'' = q \neq 0\Big\}.\end{aligned}$$

Also let $\mathbf{E}_S^0$ be a subspace in $\mathbf{E}_S$ such that

$$\begin{aligned}\mathbf{E}_S^0 = \Big\{(x', x''; y', y'') \in \mathbb{R}^{4n+4} \;\Big|\; &< x', x' > + < x'', x'' >= 1, \\ &< x', y' > + < x'', y'' >= 0,\ q + < p, \bar{q} > p \neq 0\Big\},\end{aligned}$$

The last condition says that the tangent vector $(x', x''; y', y'') \in T_{(x',x'')}(S^{2n+1})$ is not parallel to $(x', x''; -x'', x') \in T_{(x',x'')}(S^{2n+1})$.

The differential of the map $\pi_{\mathfrak{h}}$

$$d\pi_{\mathfrak{h}} : \mathbf{E}_S^0 \rightarrow \mathbf{E}_\mathbb{C}$$

gives

$$d\pi_{\mathfrak{h}}(x',x'';y',y'') = d\pi_{\mathfrak{h}}(p,q) = \Big(P,\ Q\Big),$$

$$P = \Big(p_i\overline{p_j}\Big),\ Q = \Big(q_i\overline{p}_j\Big) + \Big(p_i\overline{q}_j\Big),\ Q \neq 0.$$

We denote the image $\tau_S(x',x'';y',y'')$ as

$$\tau_S(x',x'';y',y'') = (\|q\|x' + \sqrt{-1}y', \|q\|x'' + \sqrt{-1}y'') = (u,v) \in \mathbb{C}^{n+1} \times \mathbb{C}^{n+1},$$

and put $\mathbf{X}_S^0 = \tau_S(\mathbf{E}_S^0)$.

Let α be the map

$$\alpha : \mathbf{X}_S^0 \longrightarrow \mathbf{X}_{\mathbb{C}},$$

$$\mathbf{X}_S^0 \ni (u,v) \mapsto A = \Big(a_{ij}\Big),\ a_{ij} = (u_i + \sqrt{-1}v_i)(u_j - \sqrt{-1}v_j),$$

then it will be easily seen that $\alpha \circ \tau_S(x',x'';y',y'') \in \mathbf{X}_{\mathbb{C}}$.

Now let γ be a map

$$\gamma : \mathbb{C}^{n+1} \times \mathbb{C}^{n+1} \longrightarrow \mathbb{C}^{n+1} \times \mathbb{C}^{n+1}$$

$$(u,v) \mapsto (z,w),\ z = u + \sqrt{-1}v,\ w = u - \sqrt{-1}v,$$

and $\beta : \mathbb{C}^{n+1} \times \mathbb{C}^{n+1} \to M(n+1,\mathbb{C})$

$$\beta(z,w) = \Big(a_{ij}\Big),\ a_{ij} = z_i w_j,$$

then,

$$\gamma(\mathbf{X}_S^0) = \Big\{(z,w) \in \mathbb{C}^{n+1} \times \mathbb{C}^{n+1}\ \Big|\ z \neq 0,\ w \neq 0,\ B(z,w) = \sum z_i w_i = 0\Big\}$$

and

$$\beta \circ \gamma = \alpha.$$

By this relation, we return back the obvious $\mathbb{C}^*$-action on $\gamma(\mathbf{X}_S^0)$ to the space $\mathbf{X}_S^0$, then the action is expressed as

$$(4.2)\qquad \left\{\begin{array}{l} \mathbf{X}_S^0 \times \mathbb{C}^* \to \mathbf{X}_S^0, \\ (u,v;\lambda) \mapsto (u,v)\cdot\lambda = (\widetilde{u},\widetilde{v}) \\ \widetilde{u} = \frac{1}{2}\left(\lambda + \frac{1}{\lambda}\right)u + \frac{\sqrt{-1}}{2}\left(\lambda - \frac{1}{\lambda}\right)v, \\ \widetilde{v} = \frac{-\sqrt{-1}}{2}\left(\lambda - \frac{1}{\lambda}\right)u + \frac{1}{2}\left(\lambda + \frac{1}{\lambda}\right)v, \end{array}\right.$$

and we have

Proposition 4.1. *$\alpha : \mathbf{X}_S^0 \to \mathbf{X}_{\mathbb{C}}$ is a $\mathbb{C}^*$-principal bundle.*

We call this the complexified Hopf bundle.

Next let $\tau_{\mathbb{C}}$ be a map

$$\tau_{\mathbb{C}}\ :\ \mathbf{E}_{\mathbb{C}} \longrightarrow \mathbf{X}_{\mathbb{C}}$$

$$(P,Q) \mapsto A \in \mathbf{X}_{\mathbb{C}}$$

$$A = \|Q\|^2 P - Q^2 + \sqrt{\frac{-1}{2}}\|Q\|Q,\ \|Q\| = \sqrt{\mathbf{tr}(QQ^*)},$$

and let $\mathbf{E}_S^H$ be a subspace in $\mathbf{E}_S^0$ such that

$$\mathbf{E}_S^H = \Big\{(p,q) \in \mathbf{E}_S^0\ |\ < p,\overline{q} >= 0\Big\},$$

then $d\pi_{\mathfrak{h}}(\mathbf{E}_S^H) = \mathbf{E}_{\mathbb{C}}$ and the following diagram is commutative:

$$\begin{array}{ccc} \mathbf{E}_S^H & \xrightarrow{\tau_S} & \mathbf{X}_S \\ {\scriptstyle d\pi_{\mathfrak{h}}}\downarrow & & \downarrow{\scriptstyle \alpha} \\ \mathbf{E}_{\mathbb{C}} & \xrightarrow[\tau_{\mathbb{C}}]{} & \mathbf{X}_{\mathbb{C}}, \end{array} \tag{4.3}$$

and we know that the map $\tau_{\mathbb{C}}$ is a diffeomorphism between $\mathbf{E}_{\mathbb{C}}$ and $\mathbf{X}_{\mathbb{C}}$.

Note that the diagram

$$\begin{array}{ccc} \mathbf{E}_S^0 & \xrightarrow{\tau_S} & \mathbf{X}_S^0 \\ {\scriptstyle d\pi_{\mathfrak{h}}}\downarrow & & \downarrow{\scriptstyle \alpha} \\ \mathbf{E}_{\mathbb{C}} & \xrightarrow[\tau_{\mathbb{C}}]{} & \mathbf{X}_{\mathbb{C}}, \end{array} \tag{4.4}$$

is not commutative.

The inverse map $\tau_{\mathbb{C}}^{-1}$ is given as

$$\tau_{\mathbb{C}}^{-1}(A) = (P,\ Q),$$
$$P = \frac{A + A^*}{2\|A\|} + \frac{A \cdot A^* + A^* \cdot A}{2\|A\|^2},$$
$$Q = \frac{A - A^*}{\sqrt{-2} \cdot \|A\|}.$$

Remark 4.2. Let $(p,q) \in \mathbf{E}_S^H$, i.e. $(p,q) \in T_0 S^{2n+1} \cong \mathbf{E}_S$ and assume $< p, \bar{q} >= 0$, then for $P = \left(p_i \bar{p}_j\right)$, $Q = \left(p_i \bar{q}_j\right) + \left(q_i \bar{p}_j\right)$ and $A = \tau_{\mathbb{C}}(P,Q)$

$$2\|q\|^2 = \|Q\|^2 = \|A\| = \sqrt{\mathbf{tr}(AA^*)}. \tag{4.5}$$

Proposition 4.3. *The canonical one form $\theta_{\mathbb{C}}$ and the symplectic form $\omega_{\mathbb{C}}$ on the cotangent bundle of the complex projective space is expressed on the space $\mathbf{X}_{\mathbb{C}}$ as follows ([FT]):*

$$\theta_{\mathbb{C}} = \sqrt{\frac{-1}{2}}(\partial \sqrt{\|A\|} - \bar{\partial}\sqrt{\|A\|}),$$
$$\omega_{\mathbb{C}} = d\theta_{\mathbb{C}} = \sqrt{-2}\,\bar{\partial}\partial\sqrt{\|A\|}.$$

We denote the Liouville volume form on $T_0^*(P^n\mathbb{C})$ with the same notation $\Omega_{\mathbb{C}}$ ($\Omega_{\mathbb{C}}(P,Q)$, or $\Omega_{\mathbb{C}}(A)$, $A \in \mathbf{X}_{\mathbb{C}}$) under the identification $T_0^* P^n\mathbb{C} \cong \mathbf{E}_{\mathbb{C}} \overset{\tau_{\mathbb{C}}}{\cong} \mathbf{X}_{\mathbb{C}}$ and by the decomposition

$$\begin{aligned} \mathbf{E}_{\mathbb{C}} &\cong \mathbb{R}_+ \times \left(\mathbf{E}_{\mathbb{C}} \bigcap \{(P,Q) \mid P,\ Q \in M(n+1,\mathbb{C}),\ \|Q\| = 1\}\right) \\ &\cong \mathbf{X}_{\mathbb{C}} \cong \mathbb{R}_+ \times \left(\left\{A \in M(n+1,\mathbb{C}) \mid \|A\| = 1\right\} \bigcap \mathbf{X}_{\mathbb{C}}\right) = \mathbb{R}_+ \times \Sigma(\mathbb{C}), \\ (t, P, Q/\|Q\|) &\longleftrightarrow (s, A/\|A\|) \in \mathbb{R}_+ \times \Sigma(\mathbb{C}) \\ s &= t^2,\ A = \tau^{\mathbb{C}}(P,Q), \end{aligned}$$

we can decompose $\Omega_{\mathbb{C}}$ as

$$\Omega_{\mathbb{C}}(A) = \frac{1}{2} s^{n-1} ds \wedge dv_{\Sigma(\mathbb{C})}(\sigma),\ A = (\|A\|, A/\|A\|) = (s,\sigma) \in \mathbb{R}_+ \times \Sigma(\mathbb{C}) \tag{4.6}$$

with a (nowhere vanishing) $(4n-1)$-form $dv_{\Sigma(\mathbb{C})}(\sigma)$ on $\left\{A \in M(n+1,\mathbb{C}) \mid \|A\| = 1\right\} \bigcap \mathbf{X}_{\mathbb{C}}$.

Let $\mathcal{P}_k = \mathcal{P}_k(M(n+1,\mathbb{C}))$ be the space of homogeneous polynomials on the matrix space $M(n+1,\mathbb{C})$ of degree k, and denote the space of their restrictions to $\mathbf{X}_\mathbb{C}$ by $\mathcal{P}_k(\mathbf{X}_\mathbb{C})$.

Let $h > 0$ and $N > -n$. For each fixed (h, N), let

$$dm_{(h,N)}(A) = e^{-h\sqrt{\|A\|}}\|A\|^N\Omega_\mathbb{C}(A) \tag{4.7}$$

be a volume form on the space $\mathbf{X}_\mathbb{C}$ and we define an inner product on the space $\sum \oplus\mathcal{P}_k(\mathbf{X}_\mathbb{C})$ by the integral

$$< \varphi, \overline{\psi} >_{(h,N)}= \int_{\mathbf{X}_\mathbb{C}} \varphi(A)\overline{\psi(A)}dm_{(h,N)}(A). \tag{4.8}$$

Then for each $k \neq \ell$, the spaces $\mathcal{P}_k(\mathbf{X}_\mathbb{C})$ and $\mathcal{P}_\ell(\mathbf{X}_\mathbb{C})$ are orthogonal. We denote the completion of $\sum \oplus\mathcal{P}_k(\mathbf{X}_\mathbb{C})$ with respect to the norm $\|\bullet\|_{(h,N)}$ defined by this inner product by

$$\mathcal{H}^2(\mathbf{X}_\mathbb{C}, dm_{(h,N)}(A)). \tag{4.9}$$

Proposition 4.4. *There are no bounded holomorphic functions on $\mathbf{X}_\mathbb{C}$ $(n \geq 1)$.*

Proposition 4.5. *The function $\varphi(A) \in \mathcal{H}^2(\mathbf{X}_\mathbb{C}, dm_{(h,N)}(A))$ has the expansion of the form $\varphi(A) = \sum_{k=0}^{\infty} \varphi_k(A)$ with k-th order homogeneous polynomials $\varphi_k(A)$ on $M(n+1,\mathbb{C})$.*

Then we have an expression of the norm of a function $\varphi(A)$ in $\mathcal{H}^2(\mathbf{X}_\mathbb{C}, dm_{(h,N)}(A))$: let $\varphi(A) = \sum_{k=0}^{\infty} \varphi_k(A) \in \mathcal{H}^2(\mathbf{X}_\mathbb{C}, dm_{(h,N)}(A))$, then

$$\begin{aligned}
\|\varphi\|^2_{(h,N)} &= \int_{\mathbf{X}_\mathbb{C}} |\varphi(A)|^2 dm_{(h,N)}(A) \\
&= \sum_{k=0}^{\infty} \int_{\Sigma(\mathbb{C})} |\varphi_k(\sigma)|^2 dv_{\Sigma(\mathbb{C})}(\sigma) \times \int_0^\infty \frac{1}{2}\cdot \|A\|^{2k+N+n-1}e^{-h\sqrt{\|A\|}}d\|A\| \\
&= \sum_{k=0}^{\infty} \frac{\Gamma(4k+2N+2n)}{h^{4k+2N+2n}} \int_{\Sigma(\mathbb{C})} |\varphi_k(\sigma)|^2 dv_{\Sigma(\mathbb{C})}(\sigma)
\end{aligned}$$

Proposition 4.6. *Let $\tilde{N} \leq N$, then*

$$\mathcal{H}^2(\mathbf{X}_\mathbb{C}, dm_{(h,N)}(A)) \hookrightarrow \mathcal{H}^2(\mathbf{X}_\mathbb{C}, dm_{(h,\tilde{N})}(A)) \tag{4.10}$$

and the inclusion map is continuous.

For a positive integer $k \geq 0$, let $\mathcal{S}^\mathbb{C}_{2k}$ be the space of harmonic polynomials on $\mathbb{R}^{2n+2} \cong \mathbb{C}^{n+1}$ which are invariant under the action of $U(1)$ (and necessarily of even degree), that is, harmonic polynomials of the variables $p_i\overline{p}_j$, and we denote the space of their descends to the complex projective space by $E^\mathbb{C}_k$. Then

Proposition 4.7. *$E^\mathbb{C}_k$ is an eigenspace of the Laplacian Δ on $P^n\mathbb{C}$ with the eigenvalue*

$$\lambda_k = 4k(n+k) \tag{4.11}$$

and its dimension is given by

$$\dim E^\mathbb{C}_k = n(n+2k)\left(\frac{n(n+1)\cdots(n+k-1)}{k!}\right)^2 = n(n+2k)\frac{\Gamma(n+k)^2}{\Gamma(n)^2\Gamma(k+1)^2}, \tag{4.12}$$

(*cf.* [BGM]).

Next we define an operator $T^{\mathbb{C}}_{h,N}$ from functions on $\mathbf{X}_{\mathbb{C}}$ with a suitable integrability condition to functions on $P^n\mathbb{C}$.

For a function $\psi(A)$ on $\mathbf{X}_{\mathbb{C}}$, we denote the distribution on $P^n\mathbb{C}$

$$C^{\infty}(P^n\mathbb{C}) \ni f \mapsto \int_{\mathbf{X}_{\mathbb{C}}} \left(\pi_{\mathbb{C}} \circ \tau_{\mathbb{C}}^{-1}\right)^{*}(f)(A) \cdot \psi(A) e^{-h\sqrt{\|A\|}} \|A\|^{N} \Omega_{\mathbb{C}}(A).$$

by

$$\left(\pi_{\mathbb{C}} \circ \tau_{\mathbb{C}}^{-1}\right)_{*} \left(\psi(\bullet) \cdot e^{-h\sqrt{\|\bullet\|}} \| \bullet \|^{N} \Omega_{\mathbb{C}}(\bullet)\right),$$

i.e. a distribution defined by the fiber integration of a $4n$-degree form

$$\psi(A) \cdot e^{-h\sqrt{\|A\|}} \|A\|^{N} \Omega_{\mathbb{C}}(A)$$

along the fiber of the map $\pi_{\mathbb{C}} \circ \tau_{\mathbb{C}}^{-1} : \mathbf{X}_{\mathbb{C}} \to P^n\mathbb{C}$. The resulting form is a smooth $2n$-degree form on $P^n\mathbb{C}$. We denote this form as

$$T^{\mathbb{C}}_{h,N}(\psi)(P) dvol_{\mathbb{C}}(P), \tag{4.13}$$

where $dvol_{\mathbb{C}}(P)$ is the Riemann volume form on $P^n\mathbb{C}$. So we define the operator

$$T^{\mathbb{C}}_{h,N}(A) : \sum \oplus \mathcal{P}_k(\mathbf{X}_{\mathbb{C}}) \longrightarrow C^{\infty}(P^n\mathbb{C}),$$

and consider their extensions to $\mathcal{H}^2(\mathbf{X}_{\mathbb{C}}, dm_{(h',N')}(A))$.

Theorem 4.8. *Let $2h_0 = h$ and $2N_0 + \dfrac{n}{2} = N > -n$, then*

$$T^{\mathbb{C}}_{h_0,N_0}(A) : \mathcal{H}^2(\mathbf{X}_{\mathbb{C}}, dm_{(h,N)}) \longrightarrow L_2(P^n\mathbb{C})$$

is an isomorphism (but not unitary).

Theorem 4.9. *Let $-n < \widetilde{N} \leq N$ and put $N_0 = (N - n/2)/2$ and $\widetilde{N}_0 = (\widetilde{N} - n/2)/2$ then*

$$T^{\mathbb{C}}_{(h/2,\widetilde{N}_0)} \circ \left(T^{\mathbb{C}}_{(h/2,N_0)}\right)^{-1} : L_2(P^n\mathbb{C}) \longrightarrow L_2(P^n\mathbb{C}),$$

is an pseudo-differential operator $\in L^{\widetilde{N}-N}_{1,0}(P^n\mathbb{C})$, i.e.

$$L_2(P^n\mathbb{C}) \overset{\left(T^{\mathbb{C}}_{(h/2,N_0)}\right)^{-1}}{\longrightarrow} \mathcal{H}^2(\mathbf{X}_{\mathbb{C}}, dm_{(h,N)}(A)) \hookrightarrow \mathcal{H}^2(\mathbf{X}_{\mathbb{C}}, dm_{(h,\widetilde{N})}(A)) \overset{T^{\mathbb{C}}_{(h/2,\widetilde{N}_0)}}{\longrightarrow} L_2(P^n\mathbb{C})$$

is a pseudo-differential operator of order $\widetilde{N} - N$ on the complex projective space.

Corollary 4.10. *For $-n < \widetilde{N} < N$, the inclusion map (4.10) is a compact operator.*

We introduced the family of the volume form $dm_{(h,N)}(A)$ on $\mathbf{X}_{\mathbb{C}}$ with parameters (h, N) as we did for the sphere case. The form of the weight function is taken from that in the paper [FY] *(see also* [Ful]*) where we determined it by means of the pairing polarizations on $\mathbf{X}_{\mathbb{C}}$ (vertical polarization on $\mathbf{X}_{\mathbb{C}}$ and the Kähler polarization given by the realization of the punctured cotangent bundle $T^*_0(P^n\mathbb{C})$ as the space $\mathbf{X}_{\mathbb{C}}$). Here we allow, as for the case of the sphere, all positive values of h and the exponent $N > -n$. The proof of this theorem 4.8 is done along the same line with that given in the papers* [Ra2], [Ful] *and* [FY].

Although we do not give the proofs here (cf. [BF2]), we introduce the operators $A^{\mathbb{C}}_k$, $B^{\mathbb{C}}_{(h,N;k)}$ and $C^{\mathbb{C}}_{(h,N;k)}$ corresponding to the operators A^S_k, $B^S_{(h,N;k)}$ and $C^S_{(h,N;k)}$ for the sphere case and state their relations, some of which we need in the last section.

Definition 4.11.

$$\begin{aligned}
A_k^{\mathbb{C}} :& E_k^{\mathbb{C}} \longrightarrow \mathcal{P}_k,\\
& f :\mapsto A_k^{\mathbb{C}}(f)(A) = \int_{P^n\mathbb{C}} f(P)\Big(\mathbf{tr}(P\cdot A)\Big)^k dvol_{\mathbb{C}}(P),\\
B_{(h,N;k)}^{\mathbb{C}} :& \mathcal{P}_k(\mathbf{X}_{\mathbb{C}}) \longrightarrow E_k^{\mathbb{C}},\\
& \psi :\mapsto B_{(h,N;k)}^{\mathbb{C}}(\psi)(P) = \int_{\mathcal{P}_{\mathbb{C}}} \psi(A)\Big(\mathbf{tr}(P\cdot A^*)\Big)^k e^{-h\sqrt{\|A\|}}\|A\|^N\Omega_{\mathbb{C}}(A),\\
C_{(h,N;k)}^{\mathbb{C}} :& \mathcal{P}_k(\mathbf{X}_{\mathbb{C}}) \longrightarrow \mathcal{P}_k(\mathbf{X}_{\mathbb{C}}),\\
& \psi :\mapsto C_{(h,N;k)}^{\mathbb{C}}(\psi)(A) = \int_{\mathbf{X}_{\mathbb{C}}} \psi(B)\Big(\mathbf{tr}(A\cdot B^*)\Big)^k e^{-h\sqrt{\|B\|}}\|B\|^N\Omega_{\mathbb{C}}(B).
\end{aligned}$$

Let's $T_{(h,N;k)}^{\mathbb{C}}$ denote the restriction of the operator $T_{(h,N)}^{\mathbb{C}}$ to the space $\mathcal{P}_k(\mathbf{X}_{\mathbb{C}})$, then

Proposition 4.12.

$$B_{(h,N;k)}^{\mathbb{C}} \circ A_k^{\mathbb{C}} = a_k^{\mathbb{C}}(h,N)Id, \tag{4.14}$$

and

$$T_{(h,N;k)}^{\mathbb{C}} \circ A_k^{\mathbb{C}} = b_k^{\mathbb{C}}(h,N)Id, \tag{4.15}$$

with the constants $a_k^{\mathbb{C}}(h,N) > 0$ *and* $b_k^{\mathbb{C}}(h,N) > 0$.

Proposition 4.13. *The operator* $C_{(h,N;k)}^{\mathbb{C}}$ *is a constant multiple of the identity operator with a constant* $c_k^{\mathbb{C}}(h,N)$.

The constants $a_k^{\mathbb{C}}(h,N)$, $b_k^{\mathbb{C}}(h,N)$ and $c_k^{\mathbb{C}}(h,N)$ are given as follows:

$$\begin{aligned}
a_k^{\mathbb{C}}(h,N)\cdot \dim E_k^{\mathbb{C}} &= \frac{\pi}{2}\frac{\Gamma(n-1)\Gamma(k+1)^2}{\Gamma(2k+n+1)}Vol(S^{2n-3})\cdot\frac{\Gamma(2N+4k+2n)}{h^{2N+4k+2n}}\cdot Vol(\Sigma(\mathbb{C})),\\
b_k^{\mathbb{C}}(h,N)\cdot \dim E_k^{\mathbb{C}} &= \frac{\Gamma(2N+2k+2n)}{2^k h^{2N+2k+2n}}\cdot Vol(\Sigma(\mathbb{C})),\\
c_k^{\mathbb{C}}(h,N)\cdot \dim E_k^{\mathbb{C}} &= Vol(\Sigma(\mathbb{C}))\cdot\frac{\Gamma(2N+4k+2n)}{h^{2N+4k+2n}}.
\end{aligned}$$

Next we represent the reproducing kernel $\mathcal{K}_{(h,N)}^{\mathbb{C}}(A,B)$ of the Hilbert space $\mathcal{H}^2(\mathbf{X}_{\mathbb{C}}, dm_{(h,N)}(A))$ and state the asymptotic behaviour $\lim_{\|A\|\to\infty}\mathcal{K}_{(h,N)}^{\mathbb{C}}(A,A)$. Since both of the operators $\frac{1}{a_k^{\mathbb{C}}(h,N)}A_k^{\mathbb{C}}\circ B_{(h,N;k)}^{\mathbb{C}}$ and $\frac{1}{c_k^{\mathbb{C}}(h,N)}C_{(h,N;k)}^{\mathbb{C}}$ are the identity operator on the space $\mathcal{P}_k^{\mathbb{C}}(\mathbf{X}_{\mathbb{C}})$, their kernel functions coincide:

Proposition 4.14.

$$\frac{1}{a_k^{\mathbb{C}}(h,N)}\int_{P^n\mathbb{C}}\Big(\mathbf{tr}(P\cdot A)\cdot\mathbf{tr}(P\cdot B^*)\Big)^k dvol_{\mathbb{C}}(P) = \frac{1}{c_k^{\mathbb{C}}(h,N)}\Big(\mathbf{tr}(A\cdot B^*)\Big)^k.$$

Then the power series

$$\sum_{k=0}^{\infty} n(n+2k)\cdot\Big(\frac{n(n+1)\cdots(n+k-1)}{k!}\Big)^2\frac{\lambda^k}{\Gamma(2N+4k+2n)}$$

converges for any $\lambda \in \mathbb{C}$, we can express the reproducing kernel $\mathcal{K}^{\mathbb{C}}_{(h,N)}(A,B)$ of the Hilbert space $\mathcal{H}^2(\mathbf{X}_{\mathbb{C}}, dm_{(h,N)}(A))$ in the form of infinite series with the explicit expression of the constants $\dim E^{\mathbb{C}}_k$, $a^{\mathbb{C}}_k(h,N)$ and $c^{\mathbb{C}}_k(h,N)$:

$$\begin{aligned}\mathcal{K}^{\mathbb{C}}_{(h,N)}(A,B) &= \sum_{k=0}^{\infty} \frac{1}{c^{\mathbb{C}}_k(h,N)} \Big(\mathbf{tr}(A\cdot B^*)\Big)^k \\ &= \frac{h^{2N+2n}}{Vol(\Sigma(\mathbb{C}))} \cdot \sum_{k=0}^{\infty} n(n+2k) \cdot \Big(\frac{\Gamma(n+k)}{\Gamma(n)\Gamma(k+1)}\Big)^2 \cdot \frac{h^{4k}}{\Gamma(2N+4k+2n)} \Big(\mathbf{tr}(A\cdot B^*)\Big)^k \\ &= \sum_{k=0}^{\infty} \frac{1}{a^{\mathbb{C}}_k(h,N)} \int_{P^n\mathbb{C}} \Big(\mathbf{tr}(P\cdot A)\Big)^k \Big(\mathbf{tr}(P\cdot B^*)\Big)^k dvol_{\mathbb{C}}(P).\end{aligned}$$

Note that the reproducing kernel $\mathcal{K}^{\mathbb{C}}_{(h,N)}(A,B)$ is defined for any $A, B \in M(n+1,\mathbb{C})$, holomorphic with respect to the variable $A \in M(n+1,\mathbb{C})$ and anti-holomorphic with respect to the variable $B \in M(n+1,\mathbb{C})$. Especially

$$\mathcal{K}^{\mathbb{C}}_{(h,N)}(A,0) \equiv \frac{1}{c^{\mathbb{C}}_k} = \frac{h^{2N+2n}}{Vol(\Sigma(\mathbb{C}))\Gamma(2N+2n)} \equiv \mathcal{K}^{\mathbb{C}}_{(h,N)}(0,B).$$

Proposition 4.15. *Any function in $\mathcal{H}^2(\mathbf{X}_{\mathbb{C}}, dm_{(h,N)}(A))$ can be extended to a holomorphic function on $M(n+1,\mathbb{C})$.*

prop

Theorem 4.16. *Let $N > -n$ be rational number, then*

$$\begin{aligned}&\mathcal{K}^{\mathbb{C}}_{(h,N)}(A,A)e^{-h\sqrt{\|A\|}}\|A\|^N \\ &= \frac{h^{2N+2n}\cdot n}{Vol(\Sigma(\mathbb{C}))\Gamma(n)^2} \times \\ &\times \sum \frac{(n+2k)\Gamma(n+k)^2}{\Gamma(4k+2N+2n)} \cdot \frac{\Big(h^2\|A\|\Big)^{2k}}{(k!)^2} \cdot e^{-h\sqrt{\|A\|}}\|A\|^N = O(1).\end{aligned}$$

Remark 4.17. This property gives us the Hilbert-Schmidt-ness of Hankel operator on $\mathbf{X}_{\mathbb{C}}$ (and also $\mathbf{X}_S$) following a theory in the paper [Bau] and [BF1]. These aspects will be discussed more precisely in the paper [BF3] together with a study of invariant functions under Berezin transformation.

Finally in this section we state a corresponding theorem to Theorem 3.3.

Let $\{g^{\mathbb{C}}_t\}_{t\in\mathbb{R}}$ be the geodesic flow action (cf. [FT]) on $\mathcal{H}^2(\mathbf{X}_{\mathbb{C}}, dm_{(h,N)}(A))$, i.e.

for $\varphi = \sum_{k=0}^{\infty} \varphi_k(A) \in \mathcal{H}^2(\mathbf{X}_{\mathbb{C}}, dm_{(h,N)}(A))$ with homogeneous φ_k of degree k,

$$g^{\mathbb{C}}_t(\varphi)(A) = \varphi(e^{\sqrt{-1}t}\cdot A) = \sum e^{\sqrt{-1}tk}\varphi_k(A).$$

Then we have

Theorem 4.18.

$$T^{\mathbb{C}}_{(h,N)} \circ g_t = e^{-(t\sqrt{-1}n)/2} e^{(\sqrt{-1}t)/2\sqrt{\Delta_{P^n\mathbb{C}}+n^2}} \circ T^{\mathbb{C}}_{(h,N)}. \tag{4.16}$$

Also for each point $A \in \mathbf{X}_{\mathbb{C}}$, let $q_A^{\mathbb{C}}(P) \in L_2(P^n\mathbb{C})$ be the function given by the Riesz' representation theorem:

$$T_{(h,N)}^{\mathbb{C}}{}^{-1}(f)(A) = \int_{\mathbf{X}_{\mathbb{C}}} T_{(h,N)}^{\mathbb{C}}{}^{-1}(f)(B) \cdot \overline{\mathcal{K}^{\mathbb{C}}(A,B)} \cdot dm_{(h,N)}(B)$$
$$=< f, q_A^{\mathbb{C}} >_{(h,N)} = \int_{P^n\mathbb{C}} f(P)\overline{q_A^{\mathbb{C}}(P)} dvol_{\mathbb{C}}(P),$$

then,

$$q_A(P) = \sum_{k=0}^{\infty} \frac{1}{b_k^{\mathbb{C}}(h,N)} \mathbf{tr}\left(A^* \cdot P\right)^k,$$

where $P = \left(p_i\overline{p}_j\right)$, $p \in \mathbb{C}^{n+1}$, $\|p\| = 1$. Now we have

Proposition 4.19.

$$q^{\mathbb{C}}_{e^{\sqrt{-1}t}\cdot A} = e^{(t\sqrt{-1}n)/2} e^{-\sqrt{-1}t/2\sqrt{\Delta_{P^n\mathbb{C}}+n^2}}(q_A^{\mathbb{C}}).$$

5. Complexfied Hopf fiberation and quantization operator

Recall the $\mathbb{C}^*$-principal bundle

$$\alpha : \mathbf{X}_S^0 \to \mathbf{X}_{\mathbb{C}}.$$

We called it complexfied Hopf bundle. In this section first we determine the fiber integral of the measure

$$\alpha_* \left(e^{-h\sqrt{\|u\|^2+\|v\|^2}} \cdot \left(\|u\|^2 + \|v\|^2\right)^{\frac{N}{2}} \Omega_S\right),$$

and then we give a sufficient condition under which the image of the operator

$$\alpha^* : \mathcal{H}^2\left(\mathbf{X}_{\mathbb{C}}, dm_{(h_{\mathbb{C}},N_{\mathbb{C}})}(A)\right) \to \mathcal{H}^2\left(\mathbf{X}_S, dm_{(h_S,N_S)}(u,v)\right).$$

is a close subspace in $\mathcal{H}^2\left(\mathbf{X}_S, dm_{(h_S,N_S)}(u,v)\right)$.

We work on

$$\beta : \gamma(\mathbf{E}_S^0) = \left\{(z,w) \in \mathbb{C}^{n+1} \times \mathbb{C}^{n+1} \mid z \neq 0,\ w \neq 0, \sum z_i w_i = 0\right\} \to \mathbf{X}_{\mathbb{C}},$$
$$\beta : (z,w) \mapsto \left(\ z_i w_j\ \right).$$

Proposition 5.1.

$$\beta_* \left(e^{-h/\sqrt{2}\sqrt{\|z\|^2+\|w\|^2}} \cdot \left(\frac{\|z\|^2+\|w\|^2}{2}\right)^{\frac{N}{2}} \Omega_S\right)(A) \tag{5.1}$$
$$= \frac{\pi \cdot 2^{n+1}}{2^{N/2}} \|A\|^n I(\|A\|) \Omega_{\mathbb{C}}(A),$$

where

$$I(\|A\|) = \int_0^{\infty} e^{-\frac{h}{\sqrt{2}}\sqrt{r^2+(\|A\|/r)^2}} \left(r^2 + (\|A\|/r)^2\right)^{\frac{N+1}{2}-n} \frac{dr}{r}.$$

The formula (5.1) above is given in the paper [FY]. Here we give it again in a simple way for the sake of the completeness of this note. We denote the Liouville volume form Ω_S (resp. $\Omega_{\mathbb{C}}$) and its pull-back by the map $\left(\tau_S \circ \gamma\right)^{-1}$ (resp. $\left(\tau_{\mathbb{C}}\right)^{-1}$) with the same notation always. Also note that $2(\|u\|^2 + \|v\|^2) = \|z\|^2 + \|w\|^2$ for $\gamma(u,v) = (z,w)$.

Proof. Since the unitary group $U(n+1)$ acts on $\mathbf{X}_{\mathbb{C}} \bigcap \{A \in M(n+1,\mathbb{C}) | \, \mathbf{tr}(A \cdot A^*) = 1\}$ transitively (adjoint action : $A \mapsto UAU^*$, $U \in U(n+1)$) and the Liouville volume form $\Omega_{\mathbb{C}}$ is invariant under this action together with the equivariance between the Liouville volume form on $\mathbf{X}_S$ under the action of the unitary group $U(n+1)$ on $\mathbb{C}^{n+1} \times \mathbb{C}^{n+1}$ by $(z,w) \mapsto (U(z), \overline{U}(w))$, it is enough to determine the fiber integration on the points

$$A_t = \begin{pmatrix} 0 & t & 0 & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 \end{pmatrix} \in \mathbf{X}_{\mathbb{C}}, \ t > 0.$$

Note that the space $\mathbf{X}_S^0$ is invariant under the action $(z,w) \mapsto (U(z), \overline{U}(w))$.

Then we take a local coordinate system on $\mathbf{X}_{\mathbb{C}}$ which together gives a local trivialization of β :

$$\begin{array}{ccc} \mathbb{C}^* \times \mathbb{C}^n \times \mathbb{C}^* \times \mathbb{C}^{n-1} & \longrightarrow & \mathbf{X}_S^0 \\ \downarrow & & \downarrow{\scriptstyle \beta} \\ \mathbb{C}^n \times \mathbb{C}^* \times \mathbb{C}^{n-1} & \longrightarrow & \mathbf{X}_{\mathbb{C}}, \end{array}$$

such that

$$\begin{array}{ccc} (\lambda, z, w) & \longrightarrow & \left(\lambda, \lambda z_1, \cdots, \lambda z_n, -\frac{<z,w>}{\lambda}, \frac{w_1}{\lambda}, \cdots, \frac{w_n}{\lambda}\right) \\ \downarrow & & \downarrow{\scriptstyle \beta} \\ (z,w) & \longrightarrow & A \in \mathbf{X}_{\mathbb{C}}, \end{array}$$

where $< z, w >= \sum z_i w_i$ and

$$A = \begin{pmatrix} -<z,w> & w_1 & w_2 & \cdots & w_n \\ -z_1 <z,w> & z_1 w_1 & z_1 w_2 & \cdots & z_1 w_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -z_n <z,w> & z_n w_1 & z_n w_2 & \cdots & z_n w_n \end{pmatrix}.$$

In terms of these coordinates

$$\omega_S = \sqrt{-1}\,\overline{\partial}\partial\left(\sqrt{|\lambda|^2(1+\|z\|^2) + \frac{|<z,w>|^2 + \|w\|^2}{|\lambda|^2}}\right)$$

and at the point $(\lambda, z, w_1, w_2, \cdots, w_n) = (\lambda, 0, t, 0, \cdots, 0)$, if we put

$$\begin{aligned} &4 \cdot \left(|\lambda|^2 + \frac{t^2}{|\lambda|^2}\right)^{3/2} \cdot \omega_S \\ &= a_{0\,0} d\overline{\lambda} \wedge d\lambda + \sum a_{0,k} d\overline{z}_k \wedge d\lambda + \sum b_{0,k} d\overline{w}_k \wedge d\lambda \\ &+ \sum \overline{a}_{0\,j} d\overline{\lambda} \wedge dz_j + \sum a_{j\,k} d\overline{z}_k \wedge dz_j + \sum b_{j\,k} d\overline{w}_k \wedge dz_j \\ &+ \sum \overline{b}_{0\,j} d\overline{\lambda} \wedge dw_j + \sum \overline{b}_{k\,j} d\overline{z}_k \wedge dw_j + \sum c_{j\,k} d\overline{w}_k \wedge dw_j. \end{aligned}$$

then the values $a_{k,j}$, $b_{k,j}$ and $c_{k,j}$ are given as follows:

$$a_{0\,0} = |\lambda|^2 + \frac{6t^2}{|\lambda|^2} + \frac{t^4}{|\lambda|^6}, \ a_{0\,k} = 0 \ (k \geq 1),$$

$$b_{0\,1} = -\frac{t}{\lambda|\lambda|^2}\left(3|\lambda|^2 + \frac{t^2}{|\lambda|^2}\right), \ b_{0\,k} = 0 \ (k \geq 2),$$

$$a_{11} = 2\Big(|\lambda|^2 + \frac{t^2}{|\lambda|^2}\Big)^2, \; a_{kk} = 2\Big(|\lambda|^2 + \frac{t^2}{|\lambda|^2}\Big)|\lambda|^2 \; (k \geq 2), \; a_{jk} = 0 \; (k \neq j),$$

$$b_{kj} = 0 \; (\forall k \geq 1, \forall j \geq 1), \; c_{11} = 2 + \frac{t^2}{|\lambda|^4}, \; c_{kk} = 2\Big(|\lambda|^2 + \frac{t^2}{|\lambda|^2}\Big) \cdot \frac{1}{|\lambda|^2} \; (k \geq 2),$$

$$c_{jk} = 0 \; (k \neq j).$$

Put

$$D(\lambda, t) = \begin{pmatrix} a_{00} & a_{01} & \cdots & a_{0n} & b_{01} & \cdots & b_{0n} \\ \overline{a}_{01} & a_{11} & \cdots & a_{1n} & b_{11} & \cdots & b_{1n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \overline{a}_{0n} & \overline{a}_{1n} & \cdots & a_{nn} & b_{n1} & \cdots & b_{nn} \\ \overline{b}_{01} & \overline{b}_{11} & \cdots & \overline{b}_{n1} & c_{11} & \cdots & c_{1n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \overline{b}_{0n} & \overline{b}_{1n} & \cdots & \overline{b}_{nn} & \overline{c}_{1n} & \cdots & c_{nn} \end{pmatrix}.$$

Now we have an expression of the Liouville volume form Ω_S by the coordinates (λ, z, w):

$$\Omega_S = \frac{(-1)^{n(2n+1)}}{(2n+1)!}\,\omega_S^{2n+1}$$

$$= \frac{(-1)^{n(2n+1)}}{(2n+1)!} \cdot (-1)^n\sqrt{-1} \cdot \left(\frac{1}{4\Big(|\lambda|^2 + \frac{t^2}{|\lambda|^2}\Big)^{3/2}}\right)^{2n+1} \cdot (2n+1)! \cdot \det D(\lambda, t)$$

$$d\overline{\lambda} \wedge d\lambda \wedge d\overline{z}_1 \wedge dz_1 \wedge \cdots \wedge d\overline{z}_n \wedge dz_n \wedge d\overline{w}_1 \wedge dw_1 \wedge \cdots \wedge d\overline{w}_n \wedge dw_n$$

$$= \sqrt{-1} \cdot \left(\frac{1}{4\Big(|\lambda|^2 + \frac{t^2}{|\lambda|^2}\Big)^{3/2}}\right)^{2n+1} \times$$

$$\times \left(\left(|\lambda|^2 + \frac{6t^2}{|\lambda|^2} + \frac{t^4}{|\lambda|^6}\right) \cdot \left(2 + \frac{t^2}{|\lambda|^4}\right) - \frac{t^2}{|\lambda|^6}\left(3|\lambda|^2 + \frac{t^2}{|\lambda|^2}\right)^2\right) \times$$

$$\times 2\left(|\lambda|^2 + \frac{t^2}{|\lambda|^2}\right)^2 \cdot \left(2\left(|\lambda|^2 + \frac{t^2}{|\lambda|^2}\right)|\lambda|^2\right)^{n-1} \cdot \left(2\left(|\lambda|^2 + \frac{t^2}{|\lambda|^2}\right)\frac{1}{|\lambda|^2}\right)^{n-1}$$

$$d\overline{\lambda} \wedge d\lambda \wedge d\overline{z}_1 \wedge dz_1 \wedge \cdots \wedge d\overline{z}_n \wedge dz_n \wedge d\overline{w}_1 \wedge dw_1 \wedge \cdots \wedge d\overline{w}_n \wedge dw_n$$

$$= \frac{\sqrt{-1}}{2^{2n+2}|\lambda|^2} \cdot \left(\frac{1}{|\lambda|^2 + \frac{t^2}{|\lambda|^2}}\right)^{n-1/2}$$

$$d\overline{\lambda} \wedge d\lambda \wedge d\overline{z}_1 \wedge dz_1 \wedge \cdots \wedge d\overline{z}_n \wedge dz_n \wedge d\overline{w}_1 \wedge dw_1 \wedge \cdots \wedge d\overline{w}_n \wedge dw_n.$$

Hence at the point

$$A_t = \begin{pmatrix} 0 & t & 0 & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 \end{pmatrix} \in \mathbf{X}_{\mathbf{C}}, \; t > 0.$$

the push-forward of the $(2n+1)$-form $e^{-h\sqrt{\|u\|^2+\|v\|^2}}\left(\|u\|^2+\|v\|^2\right)^{N/2}\Omega_S$ by the map β is

$$
\begin{aligned}
&\beta_*\left(e^{-h\sqrt{\|u\|^2+\|v\|^2}}\left(\|u\|^2+\|v\|^2\right)^{N/2}\Omega_S\right)(A_t)\\
(5.2)\qquad &= \frac{\pi}{2^{2n}}\cdot\int_0^\infty \frac{1}{r}\left(\frac{1}{r^2+(t/r)^2}\right)^{n-1/2} e^{-h/\sqrt{2}\sqrt{r^2+(t/r)^2}}\left(\frac{r^2+(t/r)^2}{2}\right)^{\frac{N}{2}} dr\\
&\qquad d\overline{z}_1\wedge dz_1\wedge\cdots\wedge d\overline{z}_n\wedge dz_n\wedge d\overline{w}_1\wedge dw_1\wedge\cdots\wedge d\overline{w}_n\wedge dw_n.
\end{aligned}
$$

On the other hand, the Liouville volume form

$$
\Omega_{\mathbf{C}}=\frac{(-1)^{n(2n-1)}}{(2n)!}\left(\sqrt{-2}\,\overline{\partial}\,\partial\sqrt{\|A\|}\right)^{2n}
$$

is expressed around the point $(0,\cdots,0,w_1,0\cdots,0)$ (we put $w_1=t>0$) in the local coordinates (z,w) as

$$
\begin{aligned}
&\Omega_{\mathbf{C}}(A_t)\\
(5.3)\qquad &= \frac{1}{2^{3n+1}\cdot t^n}d\overline{z}_1\wedge dz_1\wedge\cdots\wedge d\overline{z}_n\wedge dz_n\wedge d\overline{w}_1\wedge dw_1\wedge\cdots\wedge d\overline{w}_n\wedge dw_n.
\end{aligned}
$$

By comparing (5.2) and (5.3) we have (5.1). □

Let $f\in\mathcal{P}_k(\mathbf{X}_{\mathbf{C}})$, a polynomial of degree k of the variable $A\in M(n+1,\mathbf{C})$, then $\alpha^*(f)(u,v)$ is a $2k$-degree polynomial on $\mathbf{C}^{n+1}\times\mathbf{C}^{n+1}$. The L_2-norm of $\alpha^*(f)(u,v)=\gamma^*(\beta^*(f))(u,v)$ with respect to a measure $e^{-h\sqrt{\|u\|^2+\|v\|^2}}\left(\|u\|^2+\|v\|^2\right)^{\frac{N}{2}}\Omega_S$ is

$$
\begin{aligned}
&\int_{\mathbf{X}_S^0}|\alpha^*(f)|^2e^{-h\sqrt{\|u\|^2+\|v\|^2}}\left(\|u\|^2+\|v\|^2\right)^{\frac{N}{2}}\Omega_S\\
&=\int_{\mathbf{X}_S}|\alpha^*(f)|^2e^{-h\sqrt{\|u\|^2+\|v\|^2}}\left(\|u\|^2+\|v\|^2\right)^{\frac{N}{2}}\Omega_S\\
&=F(n,N)\int_{\mathbf{X}_{\mathbf{C}}}\int_0^\infty|f(A)|^2\|A\|^n\cdot e^{-\frac{hs}{\sqrt{2}}\sqrt{r^2+(\|A\|/r)^2}}\left(r^2+(\|A\|/r)^2\right)^{\frac{N+1}{2}-n}\frac{dr}{r}\wedge\Omega_{\mathbf{C}}(A)\\
&=F(n,N)\int_{\Sigma(\mathbf{C})}\left|f(A/\|A\|)\right|^2dv_{\Sigma(\mathbf{C})}(\sigma)\times\\
&\qquad\times\int_0^\infty\int_0^\infty t^{2k}\cdot t^n\cdot e^{-\frac{h}{\sqrt{2}}\sqrt{r^2+(t/r)^2}}\cdot\left(r^2+(t/r)^2\right)^{\frac{N+1}{2}-n}\cdot t^{n-1}\frac{dr\wedge dt}{r}\\
&=F(n,N)\cdot\int_{\Sigma(\mathbf{C})}\left|f(A/\|A\|)\right|^2dv_{\Sigma(\mathbf{C})}(\sigma)\times\\
&\qquad\times\frac{2^{n+(N-1)/2}}{h^{2n+N+1}}\cdot\frac{2^{2k}}{h^{4k}}\cdot\frac{\Gamma(4k+2n+N+1)\cdot\Gamma(k+n)^2}{\Gamma(2k+2n)}\\
(5.4)\qquad&\\
&=G(n,N)\cdot\int_{\Sigma(\mathbf{C})}\left|f(A/\|A\|)\right|^2dv_{\Sigma(\mathbf{C})}(\sigma)\cdot\frac{2^{2k}}{h^{4k}}\cdot\frac{\Gamma(4k+2n+N+1)\cdot\Gamma(k+n)^2}{\Gamma(2k+2n)},
\end{aligned}
$$

where we put $F(n,N)=\dfrac{\pi\cdot 2^n}{2^{N/2}}$ and $G(n,N)=F(n,N)\cdot\dfrac{2^{n+(N-1)/2}}{h^{2n+N+1}}$.

Next we give an expression of the L_2-norm of a degree k polynomial $f \in \mathcal{P}_k(M(n+1,\mathbb{C})$ with respect to a measure $e^{-h\sqrt{\|A\|}}\|A\|^N\Omega_{\mathbb{C}}$:

$$\int_{\mathbf{X}_{\mathbb{C}}} |f(A)|^2 e^{-h\sqrt{\|A\|}}\|A\|^N\Omega_{\mathbb{C}} \tag{5.5}$$
$$= \frac{2}{h^{2n+2N}} \cdot \int_{\Sigma(\mathbb{C})} |f(\sigma)|^2 dv_{\Sigma(\mathbb{C})}(\sigma) \cdot \frac{\Gamma(4k+2n+2N)}{h^{4k}}.$$

From these we have

Theorem 5.2. *Let $h_S = h_{\mathbb{C}} > 0$ and $N_S = 2N_{\mathbb{C}} - \frac{1}{2}$ with $N_{\mathbb{C}} > -n,\ N_S > -2n-1$, then the map*

$$\alpha^* : \mathcal{H}^2(\mathbf{X}_{\mathbb{C}}, dm_{(h_{\mathbb{C}},N_{\mathbb{C}})}(A)) \to \mathcal{H}^2(\mathbf{X}_S, dm_{(h_S,N_S)}(u,v))$$

is continuous, and the image $\alpha^*\big(\mathcal{H}^2(\mathbf{X}_{\mathbb{C}}, dm_{(h_{\mathbb{C}},N_{\mathbb{C}})})\big)$ is a closed subspace of $\mathcal{H}^2(\mathbf{X}_S, dm_{(h_S,N_S)}(u,v))$.

Proof. The square root of the ratio of (5.4) by (5.5) does not depend on the polynomials $f \in \mathcal{P}_k(\mathbf{X}_{\mathbb{C}})$, and so it equal to the norm $\|\alpha^*_{|\mathcal{P}_k(\mathbf{X}_{\mathbb{C}})}\|$ of the operator α^* restricted on each subspace $\mathcal{P}_k(\mathbf{X}_{\mathbb{C}})$, which maps $\mathcal{P}_k(\mathbf{X}_{\mathbb{C}})$ onto the subspace $\mathcal{P}_{2k}(\mathbf{X}_S) \subset \mathcal{H}^2(\mathbf{X}_S, dm_{(h_S,N_S)}(u,v))$, we know that $\lim_{k\to\infty} \|\alpha^*_{|\mathcal{P}_k(\mathbf{X}_{\mathbb{C}})}\|$ is finite and non-zero by applying Stirling formula of the Gamma function, when $h_S = h_{\mathbb{C}}$ and $N_S = 2N_{\mathbb{C}} - \frac{1}{2}$. □

Again let $f \in \mathcal{P}_k(M(n+1,\mathbb{C}))$, then

Proposition 5.3. *The pull-back of the function f by the map α, $\alpha^*(f)(u,v)$, is a $2k$-degree polynomial on $\mathbb{C}^{n+1}\times\mathbb{C}^{n+1}$, and the function*

$$B^S_{(h,N;2k)}(\alpha^*(f))(x',x'')$$
$$= \int_{\mathbf{X}_S} \alpha^*(f)(u,v)\Big(< x',\overline{u} > + < x'',\overline{v} >\Big)^{2k} e^{-h\sqrt{\|u\|^2+\|v\|^2}}\Big(\|u\|^2+\|v\|^2\Big)^{\frac{N}{2}}\Omega_S(u,v)$$

is a $U(1)$-invariant harmonic function. So that, its descend to $P^n\mathbb{C}$ (we denote its by $B^S_{(h,N;2k)}(f)$) is an eigenfunction $\in E^{\mathbb{C}}_k$ of the Laplacian on $P^n\mathbb{C}$. (It does not coincides with $B^{\mathbb{C}}_{(h',N';k)}(f)$.)

Conversely, let $g(x) \in \mathcal{S}_{2k}(\mathbb{C}^{n+1})$ be a harmonic polynomial of the variables $p_i\overline{p}_j$, i.e. it can be descended to the complex projective space, then

Proposition 5.4.

$$A^S_{2k}(g)(u,v) = \int_{S^{2n+1}} g(x)\Big(< x',u > + < x'',v >\Big)^{2k} dvol_S$$

can be descended to $\mathbf{X}_{\mathbb{C}}$, that is, $A^S_{2k}(g)$ is invariant under the action (4.2).

Theorem 5.5. *Let $N_S = 2N_{\mathbb{C}} - 1/2,\ N_{\mathbb{C}} > -n$, then the operator*

$$T^S_{(h/2,N_{\mathbb{C}}-(n+1)/4)} \circ \alpha^* : \mathcal{H}^2(\mathbf{X}_{\mathbb{C}}, dm_{(h,N_{\mathbb{C}})}(A)) \longrightarrow L_2(P^n\mathbb{C})$$

is an isomorphism and the operator

$$T^S_{(h/2,N_{\mathbb{C}}-(n+1)/4)} \circ \alpha^* \circ {T^{\mathbb{C}}_{(h/2,N_{\mathbb{C}}/2-n/4)}}^{-1}$$

is a zeroth order elliptic pseudo-differential operator in $L^0_{1,0}(P^n\mathbb{C})$.

Here the function $T^S_{(h/2,N_\mathbb{C}-(n+1)/4)} \circ \alpha^*(f)$ should be understood as a function descended to complex projective space according to Proposition 5.3.

Remark 5.6. Although the determinations of the values N_S and $N_\mathbb{C}$ (also h_S and $h_\mathbb{C}$) by pairing polarizations (cf. [Ra2] and [FY]) are done independently in each case, they satisfy the relation above.

References

[AAR] G. E. Andrews, R. Askey and R. Roy, *Special functions, Encyclopedia of mathematics and its applications*, **Vol. 71**(1999), Cambridge Univ. Press

[Bar] V. Bargmann, *Irreducible unitary representations of the Lorentz group*, Ann. Math. **48**(1947), 568-640.

[Bau] Wolfram Bauer, *Hilbert-Schmidt Hankel Operators on the Segal-Bagmann Space*, Proc. AMS, **132**, No. 10(2004), 2989-2996.

[Be] A. L. Besse, *Manifolds all of whose Geodesics are Closed*, Springer-Verlag. 1978.

[BGM] M. Berger, P. Gauduchon and E. Mazet, *Le Spectre dúne Variété Riemannienne*, Springer Lecture Notes in Mathematics **194**, 1971.

[BF1] W. Bauer and K. Furutani, *Hilbert Schmidt Hankel Operators and Berezin Iteration*, in preparation.

[BF2] ———, ——— *Quantization operators on quadrics*, in preparation

[BF3] ———, ——— *Quantization operator on quaternion projective space and Cayley projective space*, in preparation

[DG] J. J. Duistermaat and V. Guillemin, *The spectrum of positive elliptic operators and periodic bicharacteristics*, Inventiones Math., **29**(1975), 39-79.

[FT] K. Furutani and R. Tanaka, *A Kähler structure on the punctured cotangent bundle of complex and quaternion projective spaces and its application to geometric quantization I*, J. Math. Kyoto Univ. **34**(1994), 719–737.

[FY] K. Furutani and S. Yoshizawa, *A Kähler structure on the punctured cotangent bundle of complex and quaternion projective spaces and its application to geometric quantization II*, Japan. J. Math. **21** (1995), 355–392.

[Fu1] K. Furutani, *Quantization of the geodesic flow on quaternion projective spaces*, Annals of Global Analysis and geometry, Kluwer Academic Publishers.

[Fu2] ———, *A Kähler structure on the punctured cotangent bundle of the Cayley projective plane*, Math. Phy. Studies **Vol.24**(2003), pp. 163-182, Jean Leray '99 Conference Proceedings, ed. by M. de Gosson (Kluwer Academic Publishers).

[Fu3] ———, *Pseudo-differential operator and Reproducing kernels arrising in Geometric quantization*, Operator theory Advances and Application (Birkhäuser verlag) **Vol. 164** (2006), pp. 133-151.

[Go] M. de Gosson, *On the Leray-Maslov quantization of Lagrangian manifold*, J. Geom. Phys. **13**(1994), 155-168.

[GS] V. Guillemin and S. Sternberg, *Geometric Asymtotics*, Math. Surveys Monographs, **vol. 14**, (1977), Amer. Math. Soc..

[Ho1] L. Hörmander, *Pseudo-differential operators and non-elliptic boundary problems*, Ann. of Math. **83** (1966), 129–209.

[Ho2] ———, *Fourier integral operators I*, Acta Math, **127**(1971), 79-183.

[Ho3] ———, *The Analysis of Linear Partial Differential Operators III, Springer*, Berlin, 1985.

[Ii1] K. Ii, *On a Bargmann-type transform and a Hilbert space of holomorphic functions*, Tôhoku Math. J. (1) **38**(1986), 57–69.

[Ii2] K. Ii and T. Morikawa, *Kähler structures on the tangent bundle of Riemannian manifolds of constant positive curvature*, Bull. Yamagata Univ. Natur. Sci. **14** (1999), 141–154.

[Li] W. Lichtenstein, *A system of quadrics describing the orbit of the highest weight vector*, Proc. Amer. Math. Soc. **84**(1982), 605–608.

[Ra1] J. H. Rawnsley, *Coherent states and Kähler manifolds*, Quart. J. Math. Oxford Ser. **28**(1977), 403–415.

[Ra2] ———, *A non-unitary pairing of polarization for the Kepler problem*, Trans. Amer. Math. Soc. **250** (1979), 167–180.

[So1] J. M. Souriau, *Structure des Systèmes Dynamiques*, Dunod, Paris 1970.

[So2] ———, *Sur la variété de Kepler*, Symposia Math. **14** (1974), 343–360.

[Sz1] R. Szőke, *Complex structures on the tangent bundle of Riemannian manifolds*, Math. Ann. **291** (1991), 409–428.

[Sz2] ———, *Adapted complex structures and geometric quantization*, Nagoya J. Math. **154** (1999), 171–183.

[Sz3] ———, *Involutive structures on tangent bundles of symmetric spaces*, Math. Ann. **319** (2001), 319-348.

[Ta] M. Taylor, *Pseudo-differential operators*, Princeton Mathematical Series **34**(1981), Princeton Univ. Press.

[We] A. Weinstein, *Fourier Integral Operators, Quantization and the Spectra of Riemannian Manifolds*, In Colloque International de Géométrie Symplectique et Physique Mathématique CNRS Aix (June 1974).

[Wh] N. M. J. Woodhouse, *Geometric Quantization*, 2nd ed., Oxford Mathematical Monographs. Oxford: Clarendon Press. 1997.