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On Orlicz-Morrey spaces

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1. Introduction

In this paper we state basic properties of Orlicz-Morrey spaces and a definition of its predual without proofs. This is an announcement of my recent works.

Orlicz spaces are generalization of Lebesgue spaces L^p. They are useful tools to study harmonic analysis and its applications. For example, the Hardy-Littlewood maximal operator is bounded on L^p for $1 < p \leq \infty$, but not bounded on L^1. Using Orlicz spaces, we can investigate the boundedness of the operator near $p = 1$ precisely (see Kita [5, 6] and Cianchi [3]). It is known that fractional integral operators I_α is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ for $1 < p < q < \infty$ and $-n/p + \alpha = -n/q$ as the Hardy-Littlewood-Sobolev theorem. Trudinger [29] investigated the boundedness of I_α near $q = \infty$. The Hardy-Littlewood-Sobolev theorem and Trudinger's result are generalized by several authors, [20, 26, 27, 4, 3, 15, 16, 17], etc. For the theory of Orlicz spaces, see [10, 7, 24].

On the other hand Morrey spaces was introduced by [11] to estimate solutions of partial differential equations. After that there are many papers about Morrey spaces. For the boundedness of the Hardy-Littlewood maximal operator and fractional integral operators, see [23, 1, 2, 12].

The author introduced Orlicz-Morrey spaces in [18] to investigate the boundedness of generalized fractional integral operators. Orlicz-Morrey spaces unify Orlicz and Morrey spaces. Recently, using Orlicz-Morrey spaces, Sawano, Sobukawa and Tanaka [25] proved a Trudinger type inequality for Morrey spaces.

Our definition of Orlicz-Morrey space is different from one in Kokilashvili and Krbec [7, p.2].

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We recall the definitions and several properties of Orlicz and Morrey spaces in the next section. We state a definition of Orlicz-Morrey spaces in Section 3. In Section 4, we give generalized Hölder's inequality and inclusion relations for Orlicz-Morrey spaces. In Section 5 we give a definition of preduals of Orlicz-Morrey spaces.

2. Definitions and Properties of Orlicz and Morrey spaces

A function $\theta : (0,+\infty) \to (0,+\infty)$ is said to be almost increasing (almost decreasing) if there exists a constant $C > 0$ such that

$$\theta(r) \le C\theta(s) \quad (\theta(r) \ge C\theta(s)) \quad \text{for} \quad r \le s.$$

A function $\theta : (0,+\infty) \to (0,+\infty)$ is said to satisfy the doubling condition if there exists a constant $C > 0$ such that

$$C^{-1} \le \frac{\theta(r)}{\theta(s)} \le C \quad \text{for} \quad \frac{1}{2} \le \frac{r}{s} \le 2.$$

For functions $\theta, \kappa : (0,+\infty) \to (0,+\infty)$, we denote $\theta(r) \sim \kappa(r)$ ($\theta(r) \approx \kappa(r)$) if there exists a constant $C > 0$ such that

$$C^{-1}\theta(r) \le \kappa(r) \le C\theta(r) \quad (\theta(C^{-1}r) \le \kappa(r) \le \theta(Cr)) \quad \text{for} \quad r > 0.$$

First we recall the definition of Young functions. A function $\Phi : [0,+\infty] \to [0,+\infty]$ is called a Young function if Φ is convex, left-continuous, $\lim_{r\to+0} \Phi(r) = \Phi(0) = 0$ and $\lim_{r\to+\infty} \Phi(r) = \Phi(+\infty) = +\infty$. Any Young function is neither identically zero nor identically infinite on $(0,+\infty)$. From the convexity and $\Phi(0) = 0$ it follows that any Young function is increasing.

If there exists $s \in (0,+\infty)$ such that $\Phi(s) = +\infty$, then $\Phi(r) = +\infty$ for $r \ge s$. Let $r_0 = \inf\{s > 0 : \Phi(s) = +\infty\}$. Then $r_0 > 0$, since $\lim_{r\to+0} \Phi(r) = \Phi(0) = 0$. If $\Phi(r_0) < +\infty$, then Φ is absolutely continuous on $[0, r_0]$ by the convexity and increasingness. If $\Phi(r_0) = +\infty$, then Φ is absolutely continuous on any closed interval in $[0, r_0)$ and $\lim_{r\to r_0-0} \Phi(r) = +\infty$ by left-continuity.

Let $\mathcal{Y}$ be the set of all Young functions Φ such that

$$0 < \Phi(r) < +\infty \quad \text{for} \quad 0 < r < +\infty. \tag{2.1}$$

If $\Phi \in \mathcal{Y}$, then Φ is absolutely continuous on any closed interval in $[0,+\infty)$ and bijective from $[0,+\infty)$ to itself.

A Young function Φ is said to satisfy the Δ_2-condition, denoted $\Phi \in \Delta_2$, if

$$\Phi(2r) \le C\Phi(r), \quad r \ge 0,$$

for some $C > 0$.

A Young function Φ is said to satisfy the ∇_2-condition, denoted $\Phi \in \nabla_2$, if

$$\Phi(r) \leq \frac{1}{2k}\Phi(kr), \quad r \geq 0,$$

for some $k > 1$. If $p > 1$, then $\Phi(r) = r^p$ satisfies both the Δ_2-condition and the ∇_2-condition. If $p = 1$, then $\Phi(r) = r^p$ satisfies the Δ_2-condition, but does not satisfy ∇_2-condition.

For a Young function Φ, the complementary function is defined by

$$\widetilde{\Phi}(r) = \sup\{rs - \Phi(s) : s \geq 0\}, \quad r \geq 0.$$

Then $\widetilde{\widetilde{\Phi}} = \Phi$, and, $\Phi \in \Delta_2$ if and only if $\widetilde{\Phi} \in \nabla_2$.

Example 2.1. (i) If $\Phi(r) = r^p/p$, $1 < p < \infty$, then $\widetilde{\Phi}(r) = r^{p'}/p'$, $1/p + 1/p' = 1$.
(ii) If $\Phi(r) = r$, then $\widetilde{\Phi}(r) = 0 (0 \leq r \leq 1), = +\infty (r > 1)$.
(iii) If $\Phi(r) = (r+1)\log(r+1) - r$, then $\widetilde{\Phi}(r) = e^r - r - 1$.

For a Young function Φ and for $0 \leq s \leq +\infty$, let

$$\Phi^{-1}(s) = \inf\{r \geq 0 : \Phi(r) > s\} \quad (\inf \emptyset = +\infty).$$

If $\Phi \in \mathcal{Y}$, then Φ^{-1} is the usual inverse function of Φ. We note that

$$\Phi(\Phi^{-1}(r)) \leq r \leq \Phi^{-1}(\Phi(r)), \quad \text{for} \quad 0 \leq r < +\infty.$$

For a Young function Φ and its complementary function $\widetilde{\Phi}$, we have

$$r \leq \Phi^{-1}(r)\widetilde{\Phi}^{-1}(r) \leq 2r, \quad 0 < r < +\infty.$$

Using this relation, we have the following.

Example 2.2. (i) Let $1 < p_i < \infty$, $1/p_i + 1/p_i' = 1$, $-\infty < \beta_i < +\infty$ $(i = 1, 2)$. If

$$\Phi(r) \approx \begin{cases} r^{p_1}(\log r)^{p_1\beta_1} & \text{for large } r, \\ r^{p_2}(1/\log(1/r))^{p_2\beta_2} & \text{for small } r, \end{cases}$$

then

$$\widetilde{\Phi}(r) \approx \begin{cases} r^{p_1'}(\log r)^{-p_1'\beta_1} & \text{for large } r, \\ r^{p_2'}(1/\log(1/r))^{-p_2'\beta_2} & \text{for small } r. \end{cases}$$

(ii) Let $0 < p_1, p_2 < \infty$. If

$$\Phi(r) \approx \begin{cases} r(\log r)^{1/p_1} & \text{for large } r, \\ r(1/\log(1/r))^{1/p_2} & \text{for small } r, \end{cases}$$

then

$$\widetilde{\Phi}(r) \approx \begin{cases} \exp(r^{p_1}) & \text{for large } r, \\ 1/\exp(1/r^{p_2}) & \text{for small } r. \end{cases}$$

(iii) Let $0 < p_1, p_2 < \infty$. If

$$\Phi(r) \approx \begin{cases} r(\log\log r)^{1/p_1} & \text{for large } r, \\ r(1/\log\log(1/r))^{1/p_2} & \text{for small } r, \end{cases}$$

then

$$\widetilde{\Phi}(r) \approx \begin{cases} \exp\exp(r^{p_1}) & \text{for large } r, \\ 1/\exp\exp(1/r^{p_2}) & \text{for small } r. \end{cases}$$

We note that, for Young functions Φ and Ψ, if there exist $C \geq 1$ and $R \geq 1$ such that

$$\Phi(C^{-1}r) \leq \Psi(r) \leq \Phi(Cr) \quad \text{for} \quad r \in (0, R^{-1}) \cup (R, +\infty),$$

then $\Phi \approx \Psi$.

Definition 2.1 (Orlicz space). For a Young function Φ, let

$$L^{\Phi}(\mathbb{R}^n) = \left\{ f \in L^1_{\mathrm{loc}}(\mathbb{R}^n) : \int_{\mathbb{R}^n} \Phi(k|f(x)|)\, dx < +\infty \text{ for some } k > 0 \right\},$$

$$\|f\|_{L^{\Phi}} = \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^n} \Phi\left(\frac{|f(x)|}{\lambda}\right) dx \leq 1 \right\}.$$

Let

$$M^{\Phi}(\mathbb{R}^n) = \left\{ f \in L^{\Phi}(\mathbb{R}^n) : \int_{\mathbb{R}^n} \Phi(\forall k|f(x)|)\, dx < +\infty \right\}.$$

Then $\|f\|_{L^{\Phi}}$ is a norm and $L^{\Phi}(\mathbb{R}^n)$ is a Banach space. This norm is introduced by Nakano [19] and Luxemburg [9]. $M^{\Phi}(\mathbb{R}^n)$ is a closed subspace of $L^{\Phi}(\mathbb{R}^n)$. If $\Phi(r) = r^p$, $1 \leq p < \infty$, then $L^{\Phi}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$. If $\Phi(r) = 0 (0 \leq r \leq 1), = +\infty (r > 1)$, then $L^{\Phi}(\mathbb{R}^n) = L^{\infty}(\mathbb{R}^n)$. If $\Phi \approx \Psi$, then $L^{\Phi}(\mathbb{R}^n) = L^{\Psi}(\mathbb{R}^n)$ with equivalent norms.

We note that

$$\int_{\mathbb{R}^n} \Phi\left(\frac{|f(x)|}{\|f\|_{L^{\Phi}}}\right) dx \leq 1.$$

Theorem 2.1 ([24, p. 77, Corollary 5 and Propositopn 6]). *Let* Φ *be a Young function. Then the following are equivalent.*

(1) $\Phi \in \Delta_2$.
(2) $L^{\Phi}(\mathbb{R}^n) = M^{\Phi}(\mathbb{R}^n)$.
(3) *For all* $f \in L^{\Phi}(\mathbb{R}^n)$ *with* $f \not\equiv 0$,

$$\int_{\mathbb{R}^n} \Phi\left(\frac{|f(x)|}{\|f\|_{L^\Phi}}\right) dx = 1.$$

Theorem 2.2 ([7, Theorem 1.2.1]). *Let* $\Phi \in \mathcal{Y}$. *Then the following are equivalent:*

(1) $\Phi \in \nabla_2$.
(2) *The Hardy-Littlewood maximal operator is bounded on* $L^{\Phi}(\mathbb{R}^n)$.

The Hölder's inequality is generalized to Orlicz spaces as follows.

Theorem 2.3 ([30]). *For a Young function* Φ *and its complementary function* $\widetilde{\Phi}$,

$$\int_{\mathbb{R}^n} |f(x)g(x)|\, dx \le 2\|f\|_{L^\Phi} \|g\|_{L^{\widetilde{\Phi}}}.$$

Theorem 2.4 ([20, Theorem 2.3]). *If there exists a constant* $c > 0$ *such that*

$$\Phi_1^{-1}(r)\Phi_3^{-1}(r) \le c\Phi_2^{-1}(r) \quad \textit{for all } r \ge 0,$$

then

$$\|fg\|_{L^{\Phi_2}} \le 2c\|f\|_{L^{\Phi_1}} \|g\|_{L^{\Phi_3}}.$$

Theorem 2.5 ([24, p. 110, Theorem 7]). *Let* $\Phi \in \mathcal{Y}$. *Then*

$$(M^{\Phi}(\mathbb{R}^n))^* = L^{\widetilde{\Phi}}(\mathbb{R}^n), \quad \|g\|_{(M^\Phi)^*} \sim \|g\|_{L^{\widetilde{\Phi}}}.$$

Next we recall the definition of Morrey spaces. Let $B(a,r)$ be the ball $\{x \in \mathbb{R}^n : |x-a| < r\}$ with center a and of radius $r > 0$.

Definition 2.2 (Morrey space). For $1 \le p < \infty$ and $0 \le \lambda \le n$, let

$$L^{p,\lambda}(\mathbb{R}^n) = \left\{f \in L^p_{\mathrm{loc}}(\mathbb{R}^n) : \|f\|_{L^{p,\lambda}} < +\infty\right\},$$

$$\|f\|_{L^{p,\lambda}} = \sup_{B=B(a,r)} \left(\frac{1}{r^\lambda}\int_B |f(x)|^p\, dx\right)^{1/p}.$$

Then $L^{p,\lambda}(\mathbb{R}^n)$ is a Banach space. If $\lambda = 0$, then $L^{p,\lambda}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$. If $\lambda = n$, then $L^{p,\lambda}(\mathbb{R}^n) = L^\infty(\mathbb{R}^n)$.

If $1/p_1 + 1/p_3 = 1/p_2$ and $\lambda_1/p_1 + \lambda_3/p_3 = \lambda_2/p_2$, then we can get by Hölder's inequality

$$\|fg\|_{L^{p_2,\lambda_2}} \le \|f\|_{L^{p_1,\lambda_1}} \|g\|_{L^{p_3,\lambda_3}}. \tag{2.2}$$

It is known that, if $1 \le p < q < \infty$ and $0 \le \lambda < n$, then there exists a function $f \in L^{p,\lambda}(\mathbb{R}^n)$ such that $f \notin L^{q,\mu}(\mathbb{R}^n)$ for all $0 \le \mu \le n$. For preduals of Morrey spaces, see [8].

Let $\mathcal{G}$ be the set of all functions $\phi : (0,+\infty) \to (0,+\infty)$ such that ϕ is almost decreasing and $\phi(r)r$ is almost increasing. If $\phi \in \mathcal{G}$, then ϕ satisfies doubling condition.

Proposition 2.6. *If $\phi \in \mathcal{G}$, then there exists $\bar{\phi} \in \mathcal{G}$ such that $\bar{\phi} \sim \phi$ and that $\bar{\phi}$ is continuous and strictly decreasing.*

Definition 2.3 (generalized Morrey space). For $1 \le p < \infty$ and $\phi \in \mathcal{G}$, let

$$L^{(p,\phi)}(\mathbb{R}^n) = \{f \in L^p_{\mathrm{loc}}(\mathbb{R}^n) : \|f\|_{L^{(p,\phi)}} < +\infty\},$$

$$\|f\|_{L^{(p,\phi)}} = \sup_B \left(\frac{1}{|B|\phi(|B|)} \int_B |f(x)|^p \, dx \right)^{1/p}.$$

Then $L^{(p,\phi)}(\mathbb{R}^n) = L^{p,\lambda}(\mathbb{R}^n)$ for $\phi(r) = r^{\lambda - n}$.

3. Orlicz-Morrey spaces

Now we define Orlicz-Morrey spaces. For $\Phi \in \mathcal{Y}$, $\phi \in \mathcal{G}$ and a ball B, let

$$\|f\|_{\Phi,\phi,B} = \inf\left\{ \lambda > 0 : \frac{1}{|B|\phi(|B|)} \int_B \Phi\left(\frac{|f(x)|}{\lambda}\right) dx \le 1 \right\}.$$

Definition 3.1 (Orlicz-Morrey space). For $\Phi \in \mathcal{Y}$ and $\phi \in \mathcal{G}$, let

$$L^{(\Phi,\phi)}(\mathbb{R}^n) = \{f \in L^1_{\mathrm{loc}}(\mathbb{R}^n) : \|f\|_{L^{(\Phi,\phi)}} < +\infty\},$$

$$\|f\|_{L^{(\Phi,\phi)}} = \sup_B \|f\|_{\Phi,\phi,B}.$$

Then $\|\cdot\|_{L^{(\Phi,\phi)}}$ is a norm and $L^{(\Phi,\phi)}(\mathbb{R}^n)$ is a Banach space, since $\|f\|_{\Phi,\phi,B} = \|f\|_{L^\Phi(B,\bar{d}x)}$ is a norm on the Orlicz space $L^\Phi(B,\bar{d}x)$ where $\bar{d}x = dx/(|B|\phi(|B|))$.

By the definition we have the following.

Proposition 3.1. *If $\phi(r) = 1/r$, then $L^{(\Phi,\phi)}(\mathbb{R}^n)$ coincides with the Orlicz space $L^\Phi(\mathbb{R}^n)$. If $\Phi(r) = r^p$ and $\phi(r) = r^{-1+\lambda/n}$ $(0 \le \lambda \le n)$, then $L^{(\Phi,\phi)}(\mathbb{R}^n)$ coincides with the Morrey space $L^{p,\lambda}(\mathbb{R}^n)$.*

Proposition 3.2. *Let Φ, Ψ be Young functions and let $\phi, \psi \in \mathcal{G}$.*

(1) *If $\Phi(r) \le \Psi(Cr)$, then*

$$L^{(\Phi,\phi)}(\mathbb{R}^n) \supset L^{(\Psi,\phi)}(\mathbb{R}^n), \qquad \|f\|_{L^{(\Phi,\phi)}} \le C\|f\|_{L^{(\Psi,\phi)}}.$$

(2) *If $\phi(r) \le C\psi(r)$, then*

$$L^{(\Phi,\phi)}(\mathbb{R}^n) \subset L^{(\Phi,\psi)}(\mathbb{R}^n), \qquad \max(1,C)\|f\|_{L^{(\Phi,\phi)}} \ge \|f\|_{L^{(\Phi,\psi)}}.$$

Therefore, if $\Phi \approx \Psi$ and $\phi \sim \psi$, then $L^{(\Phi,\phi)}(\mathbb{R}^n) = L^{(\Psi,\psi)}(\mathbb{R}^n)$.

Proposition 3.3. *Let $\Phi \in \mathcal{Y}$ and $\phi \in \mathcal{G}$.*

(1) *If $c_0 = \sup_{u>0} \phi(u) < +\infty$, then*

$$L^{(\Phi,\phi)}(\mathbb{R}^n) \subset L^{\infty}(\mathbb{R}^n) \quad \textit{and} \quad \|f\|_{L^\infty} \le \Phi^{-1}(c_0)\|f\|_{L^{(\Phi,\phi)}}.$$

(2) *If $c_1 = \inf_{u>0} \phi(u) > 0$, then*

$$L^{(\Phi,\phi)}(\mathbb{R}^n) \supset L^{\infty}(\mathbb{R}^n) \quad \textit{and} \quad \|f\|_{L^\infty} \ge \Phi^{-1}(c_1)\|f\|_{L^{(\Phi,\phi)}}.$$

Therefore, if $\phi \sim 1$, then $L^{(\Phi,\phi)}(\mathbb{R}^n) = L^{\infty}(\mathbb{R}^n)$ with equivalent norms.

4. Basic properties

Theorem 4.1. *Let Φ_i be Young functions and $\phi_i \in \mathcal{G}$, $i = 1,2,3$. Assume that there exists a constant $c > 0$ such that*

$$\Phi_1^{-1}(r\phi_1(s))\Phi_3^{-1}(r\phi_3(s)) \le c\,\Phi_2^{-1}(r\phi_2(s)) \quad \textit{for} \quad r,s > 0.$$

If $f \in L^{(\Phi_1,\phi_1)}(\mathbb{R}^n)$ and $g \in L^{(\Phi_3,\phi_3)}(\mathbb{R}^n)$, then $fg \in L^{(\Phi_2,\phi_2)}(\mathbb{R}^n)$ and

$$\|fg\|_{L^{(\Phi_2,\phi_2)}} \le 2c\|f\|_{L^{(\Phi_1,\phi_1)}}\|g\|_{L^{(\Phi_3,\phi_3)}}.$$

Corollary 4.2. *Let Φ_i be Young functions, $i = 1,2,3$, and $\phi \in \mathcal{G}$. Assume that there exists a constant $c > 0$ such that*

$$\Phi_1^{-1}(r)\Phi_3^{-1}(r) \le c\,\Phi_2^{-1}(r) \quad \textit{for} \quad r > 0.$$

If $f \in L^{(\Phi_1,\phi)}(\mathbb{R}^n)$ and $g \in L^{(\Phi_3,\phi)}(\mathbb{R}^n)$, then $fg \in L^{(\Phi_2,\phi)}(\mathbb{R}^n)$ and

$$\|fg\|_{L^{(\Phi_2,\phi)}} \le 2c\|f\|_{L^{(\Phi_1,\phi)}}\|g\|_{L^{(\Phi_3,\phi)}}.$$

Corollary 4.3 ([13, 14]). *Let $1 \le p_i < \infty$ and $\phi_i \in \mathcal{G}$, $i = 1,2,3$. Assume that $1/p_1 + 1/p_3 = 1/p_2$ and that there exists a constant $c > 0$ such that*

$$\phi_1^{1/p_1}(r)\phi_3^{1/p_3}(r) \le c\phi_2^{1/p_2}(r) \quad \text{for} \quad r > 0.$$

If $f \in L^{(p_1,\phi_1)}(\mathbb{R}^n)$ and $g \in L^{(p_3,\phi_3)}(\mathbb{R}^n)$, then $fg \in L^{(p_2,\phi_2)}(\mathbb{R}^n)$ and

$$\|fg\|_{L^{(p_2,\phi_2)}} \le 2c\|f\|_{L^{(p_1,\phi_1)}}\|g\|_{L^{(p_3,\phi_3)}}.$$

Theorem 4.4. *Let $\Phi_i \in \mathcal{Y}$ and $\phi_i \in \mathcal{G}$, $i = 1,2$. Assume that*

$$\Phi_2(r)\Phi_2(s) \le c_0\Phi_2(rs) \quad \text{for} \quad r > 0,\ s > 0,$$

and that there exists $\Phi_3 \in \mathcal{Y}$ such that

$$\Phi_1^{-1}(r)\Phi_3^{-1}(r) \le c_1\Phi_2^{-1}(r) \quad \text{and} \quad \phi_1(r)/\Phi_2(\Phi_3^{-1}(\phi_1(r))) \le c_2\phi_2(r) \quad \text{for} \quad r > 0.$$

Then

$$L^{(\Phi_1,\phi_1)}(\mathbb{R}^n) \subset L^{(\Phi_2,\phi_2)}(\mathbb{R}^n) \quad \text{and}$$
$$\|f\|_{L^{(\Phi_2,\phi_2)}} \le 2\max(1,c_0)\,c_1\max(1,c_2)\|f\|_{L^{(\Phi_1,\phi_1)}}.$$

Corollary 4.5. *Let $1 \le q \le p < \infty$ and $\phi \in \mathcal{G}$. Then*

$$L^{(p,\phi)}(\mathbb{R}^n) \subset L^{(q,\phi^{q/p})}(\mathbb{R}^n) \quad \text{and} \quad \|f\|_{L^{(q,\phi^{q/p})}} \le \|f\|_{L^{(p,\phi)}}.$$

Corollary 4.6. *Let $\Phi \in \mathcal{Y}$ and $\phi \in \mathcal{G}$. Then $\Phi^{-1}(\phi) \in \mathcal{G}$ and*

$$L^{(\Phi,\phi)}(\mathbb{R}^n) \subset L^{(1,\Phi^{-1}(\phi))}(\mathbb{R}^n) \quad \text{and} \quad \|f\|_{L^{(1,\Phi^{-1}(\phi))}} \le \|f\|_{L^{(\Phi,\phi)}}.$$

Corollary 4.7 ([17]). *Let $\Phi \in \mathcal{Y}$ and $\phi(r) = \Phi^{-1}(1/r)$. Then $\phi \in \mathcal{G}$ and*

$$L^{\Phi}(\mathbb{R}^n) \subset L^{(1,\phi)}(\mathbb{R}^n) \quad \text{and} \quad \|f\|_{L^{(1,\phi)}} \le C\|f\|_{L^{\Phi}}.$$

Theorem 4.8. *Let $\Phi, \Psi \in \mathcal{Y}$, $\phi \in \mathcal{G}$ and $\phi(r) \to +\infty$ as $r \to 0$. If $\lim_{r\to+\infty} \Phi^{-1}(r)/\Psi^{-1}(r) = +\infty$, then there exists a function $f \in L^{(\Phi,\phi)}(\mathbb{R}^n)$ with compact support such that $f \notin L^{(\Psi,\psi)}(\mathbb{R}^n)$ for all $\psi \in \mathcal{G}$.*

Corollary 4.9. *Let $1 \le p < q < \infty$, $\phi \in \mathcal{G}$ and $\phi(r) \to +\infty$ as $r \to 0$. Then there exists a function $f \in L^{p,\phi}(\mathbb{R}^n)$ with compact support such that $f \notin L^{q,\psi}(\mathbb{R}^n)$ for all $\psi \in \mathcal{G}$.*

5. Predual

Definition 5.1. Let $\Phi \in \mathcal{Y}$ and $\phi \in \mathcal{G}$. A function b on $\mathbb{R}^n$ is called a (Φ,ϕ)-block if there exists a ball B such that

$$\begin{cases} (i) & \operatorname{supp} b \subset \overline{B}, \\ (ii) & \displaystyle\int_B \Phi(k|b(x)|)\,dx < +\infty \quad \text{for all } k > 0, \\ (iii) & \|b\|_{\Phi,\phi,B} \le \dfrac{1}{|B|\phi(|B|)}, \end{cases}$$

where $\overline{B}$ is the closure of B.

Let $\mathfrak{D}'$ be the space of distributions on $\mathbb{R}^n$.

Definition 5.2. Let $\Phi \in \mathcal{Y}$ and $\phi \in \mathcal{G}$. We define the space $B_{(\Phi,\phi)}(\mathbb{R}^n) \subset \mathfrak{D}'$ as follows:

$f \in B_{(\Phi,\phi)}(\mathbb{R}^n)$ if and only if there exist sequences (Φ,ϕ)-blocks $\{b_j\}$ and positive numbers $\{\lambda_j\}$ such that

$$f = \sum_j \lambda_j b_j \text{ in } \mathfrak{D}' \quad \text{and} \quad \sum_j \lambda_j < +\infty. \tag{5.1}$$

$$\|f\|_{B_{(\Phi,\phi)}} = \inf\left\{\sum_j \lambda_j : f = \sum_j \lambda_j b_j \text{ in } \mathfrak{D}'\right\},$$

Then $B_{(\Phi,\phi)}(\mathbb{R}^n)$ is a Banach space.

Theorem 5.1. *Let $\Phi \in \mathcal{Y}$ and $\phi \in \mathcal{G}$. Assume that $\widetilde{\Phi} \in \mathcal{Y}$. Then*

$$(B_{(\Phi,\phi)}(\mathbb{R}^n))^* = L^{(\widetilde{\Phi},\phi)}(\mathbb{R}^n).$$

More precisely, for $g \in L^{(\widetilde{\Phi},\phi)}(\mathbb{R}^n)$, there exists a linear functional L given by

$$L(f) = \int_{\mathbb{R}^n} f(x)g(x)\,dx \quad \text{for } f \in M^{\Phi}_{\mathrm{comp}}(\mathbb{R}^n), \tag{5.2}$$

and satisfies

$$\|L\| \le c\|g\|_{L^{(\widetilde{\Phi},\phi)}}.$$

Conversely, every linear functional L on $B_{(\Phi,\phi)}(\mathbb{R}^n)$ can be realized as (5.2), *with $g \in L^{(\widetilde{\Phi},\phi)}(\mathbb{R}^n)$, and with*

$$\|g\|_{L^{(\widetilde{\Phi},\phi)}} \le c'\|L\|.$$

Lemma 5.2. *Let* $f \in M^{\Phi}_{\mathrm{comp}}(\mathbb{R}^n)$ *and* $f = \sum_j \lambda_j b_j$ *be any decomposition in* $B_{(\Phi,\phi)}(\mathbb{R}^n)$. *Then*

$$\int_{\mathbb{R}^n} f(x)g(x)\,dx = \sum_j \lambda_j \int_{\mathbb{R}^n} b_j(x)g(x)\,dx, \tag{5.3}$$

for all $g \in L^{(\widetilde{\Phi},\phi)}(\mathbb{R}^n)$

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