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Banach-Mazur distance and B-convex Banach spaces

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Abstract. A Banach space X is said to be B-convex if it is B_n-convex for some $n \geq 2$. As is well-known, B-convexity is an isomorphic invariant, but B_n-convexity is not so. In this short note, we are concerned with the stability of B_n-convexity under norm perturvations. It is known (cf.[7]) that X is B_n-convex $(n \geq 2)$ if and only if the n-th von Neumann-Jordan constant $C_{NJ}^{(n)}(X)$ is less than n. We show that for isomorphic Banach spaces X and Y it holds $C_{NJ}^{(n)}(Y) \leq C_{NJ}^{(n)}(X)d(X,Y)^2$, where $d(X,Y)$ denotes the *Banach-Mazur distance* between X and Y; and this implies that if X is B_n-convex, then there exists $\lambda_n > 1$ such that all Banach spaces Y satisfying $d(X,Y) < \lambda_n$ are B_n-convex. In the case $X = l_p$ or $L_p[0,1]$, $1 < p < \infty$, it is also shown that all Banach spaces Y satisfying $d(X,Y) < n^{1/r}$ are B_n-convex, where $r = \max\{p, p'\}$ and $1/p + 1/p' = 1$. Moreover, if $X = l_p^n$ or $L_p[0,1]$, $1 < p \leq 2$, then there exists a Banach space Y with $d(X,Y) = n^{1/p'}$ such that Y is not B_n-convex.

同型なバナッハ空間X, Yに対し，Banach-Mazur distance $d(X,Y)$はXとYの近さを表すと考えられる．X, Yが*isometric*であればXのもつ幾何学的性質(狭義凸性，一様凸性等)はすべてYに遺伝する．X, Yが*isometric*のとき$d(X,Y)=1$であるが，一般にその逆は成立しない．$d(X,Y)=1$のとき，狭義凸性は遺伝するとは限らないが，一様凸性等の超性質はすべて遺伝する．バナッハ空間論では局所的性質，とりわけ超性質(super property)の研究が重要である．一様凸性，一様平滑性，uniform non-squareness, type p, cotype q, B_n-convexity, J_n-convexity, 超回帰性などバナッハ空間の重要な性質の多くは超性質である．無限次元バナッハ空間に関する自明でない任意の超性質をPとするとき，無限次元ヒルベルト空間とisometricな空間は性質Pを有し，また，性質Pを有する任意の空間は有限のcotypeをもつ．つまり，ヒルベルト空間とisometricである

ことは最強の超性質であり，有限の cotype をもつことは最弱の超性質である．ここで素朴な疑問が生ずる : X, Y が近い ($d(X,Y)$ が小さい) とき，X の超性質は Y に遺伝するであろうか? 可分なヒルベルト空間 l_2 は，すべての超性質を有する．Y が l_2 と同型であれば，$1 \leq d(l_2, Y) < \infty$ である．$d(l_2, Y) = 1$ ならば．当然，Y はすべての超性質を有する．では，$d(l_2, Y) < \lambda$ となるすべての Y が超性質 P をもつような $\lambda > 1$ は存在するであろうか? 超回帰性あるいは B-convexity のような位相的性質については，当然，存在する ($\lambda > 1$ は任意でよい)．しかしながら，一様凸性あるいは一様平滑性のような幾何学的性質については事情が異なる．実際，任意の $\lambda > 1$ に対し，$d(l_2, Y) < \lambda$ となる Y で一様凸 (あるいは一様平滑) でないものがある．ところで，一様凸性 (あるいは一様平滑性) と超回帰性との間にある重要な概念として uniform non-squareness (B_2-convexity あるいは J_2-convexity と同値) がある．(超回帰的な空間は，一様凸空間が有するすべての位相的性質を共有することが知られている (Enflo [1]))．最近，uniformly non-square であるような空間は不動点性 (fixed point property) をもつことが示され，また，$d(l_2, Y) < \lambda$ であるようなすべての Y が不動点性をもつような最良の λ も研究されている (cf.[2], [8], [9])． ところで，$d(l_2, Y) < \lambda$ であるようなすべての Y が uniformly non-square となる λ の最大値は $\lambda = \sqrt{2}$ である (cf.[11])．

小論の目的は，uniform non-squareness あるいはより一般の B_n-convexity について，その性質の遺伝性を Banach-Mazur 距離との関係で考察すること，更に，B_n-convex であるような具体的な空間 X に対し，$d(X, Y) < \lambda_n$ であるすべての Y が B_n-convex となるような最良 (最大) の λ_n を決定することである．

1. Definitions (i) For isomorphic Banach spaces X and Y, the *Banach-Mazur distance* between X and Y, denoted by $d(X,Y)$, is defined to be the infimum of $\|T\|\|T^{-1}\|$ taken over all bicontinuous linear operators T of X onto Y.

(ii) A Banach space Y is called *finitely representable* (*f.r.*) in a Banach space X if for any finite dimensional subspace F of Y and for any $\epsilon > 0$ there exists a finite dimensional subspace E of X with dim E = dim F such that $d(E, F) < 1 + \epsilon$.

(iii) Let P be a property for Banach spaces. We say X has *super* P if any Banach space Y *f.r.* in X has P. P is called *super property* if P = *super* P. Of course, X is super-reflexive if any Banach space Y *f.r.* in X is reflexive.

2. Definitions (i) X is called *uniformly non-square* (James, 1964) if there exists $\delta > 0$ such that

$$\min(\|x+y\|, \|x-y\|) \leq 2(1-\delta) \quad \text{if} \quad \|x\| = \|y\| = 1.$$

(ii) The James constant of X is defined by

$$J(X) = \sup\{\min(\|x+y\|, \|x-y\|) : \|x\| = \|y\| = 1\}.$$

It is obvious that X is uniformly non-square if and only if $J(X) < 2$.

(iii) The von Neumann-Jordan constant of X is defined by

$$C_{NJ}(X) = \sup\left\{\frac{\|x+y\|^2+\|x-y\|^2}{2(\|x\|^2+\|y\|^2)} : x, y \in X, \text{ not both zero}\right\}.$$

It is known that X is uniformly non-square if and only if $C_{NJ}(X) < 2$ (cf.[5],[10]).

3. B-convexity and B_n-convexity X is said to be B_n*-convex* (or *uniformly non-*ℓ_1^n) provided there exists ε $(0 < \varepsilon < 1)$ such that for all $x_1, ..., x_n \in B_X$ there exist ε_j $(\varepsilon_j = \pm 1)$ satisfying

$$\|\varepsilon_1 x_1 + ... + \varepsilon_n x_n\| \le n(1-\varepsilon),$$

where B_X denotes the closed unit ball of X. X is called B*-convex* if X is B_n-convex for some $n \ge 2$. It is well-known that X is B*-convex* if and only if l_1 is not finitely representable in X; and if and only if X is of *type* p for some $p > 1$.

4. Theorem Let $1 < p < 2$. Suppose that there exists ε $(0 < \varepsilon < 1)$ such that for all $x_1, ..., x_n \in B_X$ there exist ε_j $(\varepsilon_j = \pm 1)$ satisfying

$$\|\varepsilon_1 x_1 + ... + \varepsilon_n x_n\| \le n^{1/p}(1-\varepsilon).$$

Then X is of type r for some $r > p$.

5. n-th von Neumann-Jordan constant In [7] the authors introduced the n-th von Neumann-Jordan constant $C_{NJ}^{(n)}(X)$, $n \ge 2$, by

$$C_{NJ}^{(n)}(X) := \sup\left\{ \sum_{\theta_j=\pm 1} \left\| \sum_{j=1}^n \theta_j x_j \right\|^2 \Big/ 2^n \sum_{j=1}^n \|x_j\|^2;\ \sum_{j=1}^n \|x_j\| \ne 0 \right\}.$$

It was shown in [7] that X is B_n-convex, $n \ge 2$, if and only if $C_{NJ}^{(n)}(X) < n$; and for $1 < p \le 2$, $C_{NJ}^{(n)}(l_p) = C_{NJ}^{(n)}(L_p) = n^{2/p-1}$ for all $n \ge 2$, where $\dim L_p = \infty$. Note that for $2 < p < \infty$, $C_{NJ}^{(2)}(l_p) = C_{NJ}^{(2)}(L_p) = 2^{2/p'-1}$, but $C_{NJ}^{(n)}(l_p) = C_{NJ}^{(n)}(L_p) < n^{2/p'-1}$ for some $n > 2$, where $l/p + 1/p' = 1$.

6. Remark Let $1 < p \le 2$ and $1/p + 1/p' = 1$. If (p, p')-Clarkson inequality holds in X, then $C_{NJ}^{(n)}(X) \le n^{2/p-1}$ for all $n \ge 2$; and if l_p is finitely representable in X, then $C_{NJ}^{(n)}(X) \ge n^{2/p-1}$ for all $n \ge 2$. In general, if Y is f.r. in X, then $C_{NJ}^{(n)}(Y) \le C_{NJ}^{(n)}(X)$.

The following result was proved in Kato-Maligranda-Takahashi [5].

7. Theorem Let X and Y be isomorphic Banach spaces. Then:

$$J(X)/d(X,Y) \le J(Y) \le J(X)d(X,Y) \tag{1}$$

$$C_{NJ}(X)/d(X,Y)^2 \le C_{NJ}(Y) \le C_{NJ}(X)d(X,Y)^2 \tag{2}$$

8. Remark There exist Banach spaces X and Y such that

$$J(Y) = J(X)d(X,Y) \text{ and } C_{NJ}(Y) = C_{NJ}(X)d(X,Y)^2.$$

Of course, if both X and Y are not uniformly non-square, then equalities hold if and only if $d(X,Y) = 1$. On the other hand, if $X = l_2^2$ and $Y = l_p^2$, $1 \leq p \leq \infty$, then both equalities hold (cf.[12]). Let us mention that there are infinite dimensional uniformly non-square Banach spaces X and Y such that both equalities hold. Hence the inequalities (1) and (2) in Theorem 7 are sharp.

We shall extend the inequalities (2) in Theorem 7 to n-th von Neumann-Jordan constants.

9. Theorem Let X and Y be isomorphic Banach spaces. Then for all $n \geq 2$, we have

$$C_{NJ}^{(n)}(X)/d(X,Y)^2 \leq C_{NJ}^{(n)}(Y) \leq C_{NJ}^{(n)}(X)d(X,Y)^2$$

10. Corollary (cf.[12]) Let $1 \leq p \leq q \leq \infty$. If $1 \leq p \leq q \leq 2$ or $2 \leq p \leq q \leq \infty$, then $d(l_p^n, l_q^n) = n^{1/p-1/q}$.

Using Theorem 9, we easily have

11. Proposition For each B_n-convex Banach space X, there exists $\lambda_n > 1$ such that all Banach spaces Y satisfying $d(X,Y) < \lambda_n$ are B_n-convex.

12. Theorem Let $1 < p < \infty$, $1/p + 1/p' = 1$ and $r = \max\{p, p'\}$. Then all Banach spaces Y satisfying $d(l_p^n, Y) < n^{1/r}$ are B_n-convex. In the case that $X = l_p$ or L_p (dim $L_p = \infty$), all Banach spaces Y satisfying $d(X,Y) < n^{1/r}$ are B_n-convex. (For $n = 2$, if X is one of the spaces l_p^2, l_p and $L_p[0,1]$, then there is a Banach space Y with $d(X,Y) = 2^{1/r}$ such that Y is not B_2-convex.)

For a B_n-convex Banach space X, we denote by $\lambda_n(X)$ the best value of λ_n in Proposition 11, that is, all Banach spaces Y satisfying $d(X,Y) < \lambda_n(X)$ are B_n-convex. whereas there exists a Banach space Z with $d(X,Z) = \lambda_n(X)$ such that Z is not B_n-convex.

Now we shall consider the best values λ_n for some B_n-convex spaces X. Let $1 < p \leq 2$ and $1/p + 1/p' = 1$. If $X = l_p^n$, then by Theorem 12 we haxe $\lambda_n(X) \geq n^{1/p'}$, and so $\lambda_n(l_p^n) = n^{1/p'}$ since $d(l_p^n, l_1^n) = n^{1/p'}$ and l_1^n is not B_n-convex (cf.[12], see also Corollary 10).

The next example shows that if $X = L_p[0,1]$, $1 < p \leq 2$, then the best value $\lambda_n = \lambda_n(X) = n^{1/p'}$.

13. Example For $1 \leq p \leq 2$ and $\lambda \geq 1$ let $Y_{\lambda,p}$ be the space $L_p[0,1]$ with the norm $\|x\|_{\lambda,p} = \max\{\|x\|_p, \lambda\|x\|_1\}$. Then $C_{NJ}^{(n)}(Y_{\lambda,p}) = \min\{n, \lambda^2 n^{2/p-1}\}$ and $d(L_p, Y_{\lambda,p}) = \lambda$. Hence $Y_{\lambda,p}$ is B_n-convex if and only if $\lambda < n^{1/p'}$; and if $\lambda = n^{1/p'}$, then $Y_{\lambda,p}$ is not B_n-convex and $d(L_p, Y_{\lambda,p}) = n^{1/p'}$. (Note that $Y_{\lambda,p} = L_p[0,1]$ if $\lambda = 1$.)

14. Theorem Let $1 < p \leq 2$. Then, $\lambda_n(l_p^n) = \lambda_n(L_p[0,1]) = n^{1/p'}$. In particular, $\lambda_n(l_2) = \sqrt{n}$.

Let X be a Banach space with dim $X \geq n$ and $1 < p < \infty$. Define the constant $d_p^n(X)$ by

$$d_p^n(X) = \sup\{d(l_p^n, E) : E \subset X,\ \dim E = n\}.$$

15. Theorem Let X be a Banach space with dim $X \geq n$. Let $1 < p < \infty$, $1/p + 1/p' = 1$ and $r = \max\{p, p'\}$. If $d_p^n(X) < n^{1/r}$, then X is B_n-convex. In particular, if $d_p^2(X) < 2^{1/r}$, then X is uniformly non-square.

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