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The devil step and a strange slope

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Introduction

One of the properties of the chaotic theory is considered as non-differentiablity of functions which appear in that theory. The Cantor function $C(x)$ has such a property. Namely this function has a strange property in the following sense, and is sometimes called the Devil's step function.

(C-1) $C(x)$ is a monotone function of $[0,1]$ onto itself,

(C-2) $C'(x)=0$ for almost all x in $[0,1]$,

(C-3) $C(x)$ jumps on a set whose measure is 0.

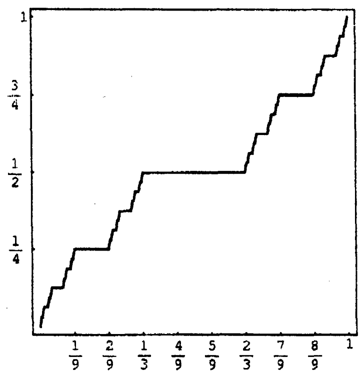
In the present note, we exhibit another strange function $h(x)$, whose property is

(h-1) $h(x)$ is a strictly monotone function of $[0,1]$ onto itself,

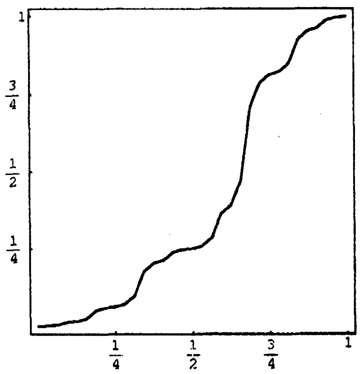
(h-2) $h'(x)=0$ for almost all x in $[0,1]$,

(h-3) $h(x)$ jumps on a set whose measure is 0 but which is dense in $[0,1]$.

The function $h(x)$ appears as a topological conjugacy between two tent maps on $[0,1]$. The following are graphs of two functions.



The grapf of $y=C(x)$



The grapf of $y=h(x)$

Tent maps are typical examples in the chaotic theory and the first author has shown convergent theorems about the Perron-Frobenius operator associated with chaotic map in the context of the theory of functional analysis (cf.[1],[3],[4]). After that, the role of topological conjugacy in the convergence theorem turned out to be clear and now we can show that every topological conjugacy between different two tent maps has no absolute-continuity on any interval in $[0,1]$.

In Section 1, we state three propositions concerning the uniform convergency of a orbit of probability density function by tent maps in the context of $L^1([0,1])$-space, the existence of topological conjugacies between two tent maps and non-absolute-continuity of those topological conjugacies.

In Section 2, we show our main result concerning the derivative h' of topological conjugacy h mentioned above and the detail of non-absolute-continuity of h with respect to the Lebesgue measure. Namely we show that

(1) $h'(x) = 0$ or ∞ if there exists a differential coefficient $h'(x)$.

(2) $h'(x) = 0$ on a dense set E_0 with $\mu(E_0) = 1$,

(3) $h'(x) = \infty$ on a dense set E_∞ with $\mu(E_\infty) = 0$,

(4) $\mu(h(E_0)) = 0$ and $\mu(h(E_\infty)) = 1$.

In our discussion, two sequences play an important role; one is the sequence denoting the orbit (itinerary) of a point under the tent map and the other is the sequence given by the infinite binary expansion of a number. The role of the former is very similar to that in the kneading theory (cf.[5]). In our theory, we show that the value $h'(x)$ is deeply related to the orbit of x. We here note that an element of x in $[0,1]$ is called a real number or a point by considering situation of discussion and that the symbol $\mathbf{N}$ means the set of all positive integers.

1. Convergence theorem for the Perron-Frobenius operator

In this paper, a *unimodal* map means a continuous map $\varphi : [0,1] \to [0,1]$ defined by

$$\varphi(x) = \begin{cases} \varphi_1(x) & \text{if } x \in [0,c), \\ \varphi_2(x) & \text{if } x \in [c,1], \end{cases}$$

where (1) $0 < c < 1$, (2) φ_1 and φ_2 are monotonically increasing and monotonically decreasing respectively, (3) φ and $\varphi_i^{-1} (i = 1,2)$ are absolutely continuous.

The map φ canonically induces an isometric operator T_φ on $L^\infty([0,1])$. Namely

$$(T_\varphi g)(x) = g(\varphi(x)), \quad (g \in L^\infty([0,1])).$$

We denote by A_φ the dual operator T_φ^* with respect to a duality between two Banach spaces $(L^1([0,1]), L^\infty([0,1]))$ with the usual dual relation $< \cdot, \cdot >$:

$$< f, g >= \int_{[0,1]} f(x)g(x)d\mu, \quad (f \in L^1([0,1]), g \in L^\infty([0,1]),$$

where μ is the Lebesgue measure on $[0,1]$. Thus we have

$$\int_{[0,1]} (A_\varphi f)(x)g(x)d\mu =< A_\varphi f, g >=< f, T_\varphi g >= \int_{[0,1]} f(x)g(\varphi(x))d\mu.$$

Hence, it follows that

$$(A_\varphi f)(x) = \frac{d\mu\circ\varphi_1^{-1}}{d\mu}(x)f(\varphi_1^{-1}(x)) + \frac{d\mu\circ\varphi_2^{-1}}{d\mu}(x)f(\varphi_2^{-1}(x)),$$

where $\frac{d\mu\circ\varphi_i^{-1}}{d\mu}$ are the Radon-Nikodym derivatives of the measure $(\mu\circ\varphi_i^{-1})$ with respect to μ, $(i = 1, 2)$, where $(\mu\circ\varphi_i^{-1})(E) = \mu(\varphi_i^{-1}(E))$ for each measurable set E in $[0,1]$. Moreover we note that A_φ is a bounded linear operator on $L^1([0,1])$ into $L^1([0,1])$ and is called the Perron-Frobenius operator associated with φ.

In the following, we mention a convergence theorem of the iterations $\{A_\varphi^n\}$ in the case where φ is so-called a chaotic map. First, let us consider the logistic map λ defined by $\lambda(x) = 4x(1-x)$. In this case, we have

$$(A_\lambda f)(x) = \frac{1}{4\sqrt{1-x}}\left\{f\left(\frac{1-\sqrt{1-x}}{2}\right) + f\left(\frac{1+\sqrt{1-x}}{2}\right)\right\}.$$

Second we consider *the (generalized) tent maps* $\tau_c, (0 < c < 1)$ defined by

$$\tau_c(x) = \begin{cases} \frac{1}{c}x & \text{if } x \in [0,c), \\ \\ \frac{1}{c-1}(x-1) & \text{if } x \in [c,1]. \end{cases}$$

The Perron-Frobenius operator A_{τ_c} associated with τ_c is easily calculated as follows:

$$(A_{\tau_c} f)(x) = cf(cx) + (1-c)f(1-(1-c)x).$$

In particular, in the case of $c = 1/2$, we write $\tau = \tau_{1/2}$ and it follows that

$$(A_\tau f)(x) = \frac{1}{2}\left\{f\left(\frac{x}{2}\right) + f\left(1-\frac{x}{2}\right)\right\}.$$

In both cases where $\varphi = \lambda$ and $\varphi = \tau_c$, we have the following convergence theorem:

$$\lim_{n\to\infty} \|A_\lambda^n f - e\|_1 = 0, \qquad \lim_{n\to\infty} \|A_{\tau_c}^n f - g\|_1 = 0$$

for any probability density function f on $[0,1]$, where $e(x) = \frac{1}{\pi\sqrt{x(1-x)}}$ and $g(x) = \chi_{[0,1]}(x) = 1$. Needless to say, χ_E means the characteristic function of E. These results are derived by convergence theorems obtained by the first author (cf.[4:Theorem 2.4, Corollary 2.5, Example 2.9, Example 2.19), in which proofs are given in the context of operator algebras. In the present note, we need a concrete and precise discussion on convergence theorem in order to analyze differentiablity of the functions which are our main object. Here, we give only statements of the convergence theorems in the context of $L^1([0,1])$-space.

Proposition 1.1. *Suppose* $c \in (0,1)$. *Then it follows that*

$$\lim_{n\to\infty} \|A^n_{\tau_c} f - \chi_{[0,1]}\|_1 = 0$$

for any probability density function f *in* $L^1([0,1])$.

Here we discuss the conjugate relation between two tent maps. Two unimodal maps ψ and φ are said to be *topologically conjugate* if there exists a homeomorphism h of $[0,1]$ onto itself such that $\varphi = h \circ \psi \circ h^{-1}$.

$$\begin{array}{ccc} & \psi & \\ [0,1] & \longrightarrow & [0,1] \\ h \downarrow & & \downarrow h \\ [0,1] & \longrightarrow & [0,1] \\ & \varphi & \end{array}$$

The homeomorphism h is called a *topological conjugacy* (cf. [2:Definition 7.4]). In the case where h and h^{-1} are absolutely continuous, we can define an isometric operator U_h on $L^1([0,1])$ as follows:

$$(U_h f)(x) = \frac{d\mu \circ h}{d\mu}(x) f(h(x))$$

and the equation $A_\varphi = U_h^{-1} A_\psi U_h$ holds. Moreover we have the following lemma.

Lemma 1.2. *Let* φ *and* ψ *be topologically conjugate unimodal maps on* $[0,1]$ *with* $\varphi = h \circ \psi \circ h^{-1}$. *If* h *and* h^{-1} *are absolutely continuous, then the following conditions* (A) *and* (B) *are equivalent.*

(A) $\lim_{n\to\infty} \|A^n_\psi f - e\|_1 = 0$ *for any probability density function* f *in* $L^1([0,1])$.

(B) $\lim_{n\to\infty} \|A^n_\varphi f - g\|_1 = 0$ *for any probability density function* f *in* $L^1([0,1])$, *where* $g = U_h^{-1} e$.

Remark 1.3. The logistic map λ is topologically conjugate to the tent map τ with $h(x) = \sin^2(\pi x/2)$. Namely $\lambda = h \circ \tau \circ h^{-1}$ and $\frac{d\mu \circ h^{-1}}{d\circ\mu}(x) = \frac{1}{\pi\sqrt{x(1-x)}}$. Since $A^n_\tau f$ converges to $\chi_{[0,1]}$, it follows that $A^n_\lambda f$ converges to the function $g(x) = \frac{d\mu \circ h^{-1}}{d\circ\mu}(x)\chi_{[0,1]}(h^{-1}(x)) = \frac{1}{\pi\sqrt{x(1-x)}}$.

In addition to τ and λ, there exists a topological conjugacy h between τ_c and τ_d. This is well-known and a general form in those kind of theorems concerning topological conjugacies is given by [5: Theorem 3.1 of Chapter II]. Here we note the statement of existence of h.

Proposition 1.4. *For any c and d in $(0,1)$, two tent maps τ_c and τ_d are topologically conjugate.*

As mentioned above, τ_c and τ_d are topologically conjugate with a unique topological conjugacy h. Now suppose that h and h^{-1} are absolutely continuous. Then, by Proposition 1.1, we have

$$\lim_{n\to\infty} A_{\tau_c}^n = \chi_{[0,1]} = \lim_{n\to\infty} A_{\tau_d}^n$$

for any probability density function f in $L^1([0,1])$. Hence, by Lemma 1.2, we have

$$\chi_{[0,1]} = U_h^{-1}\chi_{[0,1]} = \frac{d\mu \circ h^{-1}}{d \circ \mu}.$$

This implies that $\frac{d\mu\circ h^{-1}}{d\mu}(x) = 1$ for almost all x in $[0,1]$. Thus $h^{-1}(x) = h(x) = x$ for all x in $[0,1]$. Namely $c = d$. There we obtained the following proposition.

Proposition 1.5. *Let c and d be in $(0,1)$ and h the topological conjugacy between two tent maps τ_c and τ_d, ($\tau_c = h \circ \tau_d \circ h^{-1}$). If $c \neq d$, then h and h^{-1} are not absolutely continuous on $[0,1]$.*

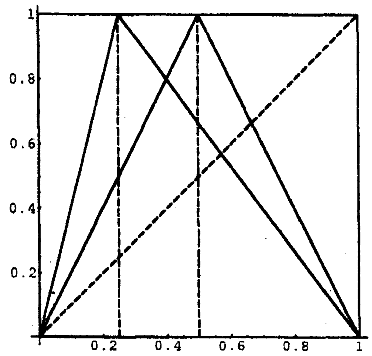
In the following section, we show the property of non-absolute-continuity of h.

2. Property of the topological conjugacy between two tent maps τ and τ_c

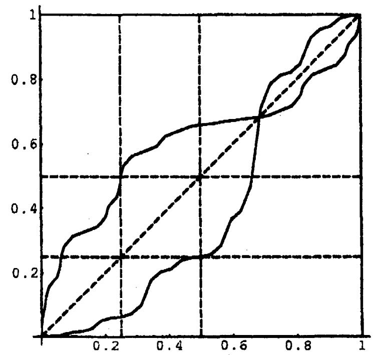
First we note that throughout this section τ denotes the tent map and h the topological conjugacy between τ and τ_c. Namely, τ and h mean

$$\tau = \tau_{\frac{1}{2}} \quad \text{and} \quad \tau_c = h \circ \tau \circ h^{-1}.$$

The graph of $y = \tau(x)$, $y = \tau_c(x)$ and that of $y = h(x)$, $y = h^{-1}(x)$ in the case $c = 1/4$ are shown at Graph[A] and Graph[B].



Graph[A]



Graph[B]

In the case $c = 1/2$, the topological conjugacy h is of course the identity map $h(x) = x$ and $h'(x) = 1$. Now we start calculations of coefficients $h'(x)$'s and, needless to say, we note that $h'(0)$ and $h'(1)$ mean $h'(0) = \lim_{\epsilon \to 0+0}(h(\epsilon) - h(0))/\epsilon$ and $h'(1) = \lim_{\epsilon \to 0+0}(h(1) - h(1-\epsilon))/\epsilon$.

Lemma 2.1. *Let x be in $[0,1]$.*

(1) *Suppose that there exist a sequences $\{y_i\}_{i=1}^{\infty}$ of points in $[0,1]$ and a sequence $\{n(i)\}_{i=1}^{\infty}$ of positive integers satisfying the following conditions.*

(1-1) $y_1 < y_2 < \cdots < y_i < \cdots < x$ *and* $\lim_{i\to\infty} y_i = x$.

(1-2) $n(1) < n(2) < \cdots < n(i) < \cdots$ *and there exists a positive integer K such that $n(i+1) - n(i) \leqq K$ for all i.*

(1-3) $|x - y_i| = 1/2^{n(i)}$ *for all i.*

(1-4) *There exists a limit $\omega_- = \lim_{i\to\infty} \frac{h(x)-h(y_i)}{x-y_i}$ and the limit ω_- is either 0 or ∞.*

Then it follows that $f'_-(x) = \omega_-$.

(2) *Suppose that there exist a sequences $\{z_i\}_{i=1}^{\infty}$ of points in $[0,1]$ and a sequence $\{m(i)\}_{i=1}^{\infty}$ of integers satisfying the following conditions.*

(2-1) $z_1 > z_2 > \cdots > z_i > \cdots > x$ *and* $\lim_{i\to\infty} z_i = x$.

(2-2) $m(1) < m(2) < \cdots < m(i) < \cdots$ *and there exists an integer L such that $m(i+1) - m(i) \leqq L$ for all i.*

(2-3) $|z_i - x| = 1/2^{m(i)}$ *for all i.*

(2-4) *There exists a limit $\omega_+ = \lim_{i\to\infty} \frac{h(z_i)-h(x)}{z_i-x}$ and the limit ω_+ is either 0 or ∞.*

Then it follows that $h'_+(x) = \omega_+$.

The following is a key lemma in our calculation of coefficients.

Lemma 2.2. *Suppose $c \in (0,1)$. Then the following equation holds.*

$$h(x) = \begin{cases} ch(\tau(x)) & \text{if } x \in [0,1/2), \\ (c-1)h(\tau(x)) + 1 & \text{if } x \in [1/2,1]. \end{cases}$$

In the following, we define two cardinal numbers related to the orbit of a point under the tent map τ, which play an important role in this paper.

$$N_0(x,n) = \sharp\{i | \tau^i(x) \in [0,1/2), 0 \leqq i \leqq n\},$$

$$N_1(x,n) = \sharp\{i | \tau^i(x) \in [1/2,1], 0 \leqq i \leqq n\},$$

where $\sharp$ means the number of a set. Then we have $N_0(x,n) + N_1(x,n) = n+1$ and, in case of no confusion, we use the notation $N_0(n)$ and $N_1(n)$ instead of $N_0(x,n)$ and $N_1(x,n)$.

Lemma 2.3. *Suppose* $c \in (0,1)$. *Let* x *be in* $[0,1]$. *Then, for each positive integer* n, *it follows that*

$$h(x) = \alpha_n h(\tau^n(x)) + \beta_n,$$

where $\alpha_n = c^{N_0(n-1)}(c-1)^{N_1(n-1)}$ *and* β_n *is a real number.*

The following is the first calculation of $h'(x)$.

Theorem 2.4. (1) *Suppose* $c \in (0,1/2)$. *Then* $h'(0) = h'(1) = 0$.
(2) *Suppose* $c \in (1/2,1)$. *Then* $h'(0) = h'(1) = \infty$.

Here we divide the real all numbers in $[0,1]$ into two families of real numbers. *One is the set of those real numbers in* $[0,1]$ *which are fractions of the form* $q/2^p$ *and is denoted by* F_2. *The other one is the complement of* F_2 *in* $[0,1]$ *and is denoted by* NF_2. Concerning points in F_2, we have the following theorem.

Theorem 2.5. *Let* x *be a point in* F_2.
(1) *Suppose* $c \in (0,1/2)$. *Then* $h'(x) = 0$.
(2) *Suppose* $c \in (1/2,1)$. *Then* $h'(x) = \infty$.

Even in calculation of $f'(x)$ for x in NF_2, we need a sequence in $[0,1]$ which converges to x. Moreover we need two sequences consisting of 0 and 1 associated with a point in NF_2. First we note that, for x in NF_2, the following expression denotes the infinite binary expansion of x:

$$x = \sum_{n=1}^{\infty} \frac{b[x]_n}{2^n},$$

where each $b[x]_n$ is in $\{0,1\}$, and the sequence $\{b[x]_n\}_{n=1}^{\infty}$ is denoted by $B(x)$.

Next, for x in $[0,1]$, we define a sequence consisting of 0 and 1, which is associated with the orbit of x for τ. We set $I_0 = [0,1/2)$ and $I_1 = [1/2,1]$. For $x \in NF_2$, we denote by $O(x) = (x_n)_{n=0}^{\infty}$ the sequence defined by

$$x_n = i \text{ if } \tau^n(x) \text{ is in } I_i.$$

This means that

$$x \in I_{x_0},\ \tau(x) \in I_{x_1}, \ldots, \tau^n(x) \in I_{x_n}, \ldots.$$

Setting $i_j = x_j + 1$ $(j = 0, \ldots, n-1)$, we can see that the above relation is equivalent to

$$x \in \tau_{i_0}^{-1}(I_{x_1}),\ \tau(x) \in \tau_{i_1}^{-1}(I_{x_2}), \ldots, \tau^{n-1}(x) \in \tau_{i_{n-1}}^{-1}(I_{x_n}), \ldots,$$

and this is written by

$$x \in \tau_{i_0}^{-1}(I_{x_1}), \ldots, x \in \tau_{i_0}^{-1} \circ \tau_{i_1}^{-1} \cdots \circ \tau_{i_{n-1}}^{-1}(I_{x_n}), \ldots.$$

Now we set

$$K_n = \tau_{i_0}^{-1} \circ \tau_{i_1}^{-1} \circ \cdots \circ \tau_{i_{n-1}}^{-1}(I_{x_n}) \subset I, \ (n = 1, 2, \ldots).$$

Then K_n is an open or half open interval and it follows that

(1) $x \in K_n$,

(2) $I \supset K_1 \supset K_2 \supset \cdots \supset K_n \supset \cdots$,

(3) The length of K_n is $1/2^{n+1}$.

Thus it follows that $x \in \cup_{n=1}^{\infty} K_n = \{x\}$. Therefore the map

$$x \to O(x) = (x_n)_{n=0}^{\infty}$$

is an injective map of $[0,1]$ into the infinite product $\prod_{n=0}^{\infty}\{0,1\}$. We here remark that this map is not surjective. Indeed, the sequence

$$(0, 1, 0, 0, 0, \ldots)$$

does not correspond to $O(x)$ for any $x \in [0,1]$, although

$$(1, 1, 0, 0, 0, \ldots) = O(1/2).$$

Now we note a relationship between two sequences $O(x) = \{x_n\}_{n=0}^{\infty}$ and $B(x) = \{b[x]_n\}_{n=1}^{\infty}$ for a point x in NF_2.

(R1) $x_0 = b[x]_1$ and $x_n = b(\tau^n(x))_1$ for $n \geqq 1$.

(R2) For $x = \sum_{n=1}^{\infty} b[x]_n/2^n$, it follows that

$$\tau(x) = \begin{cases} \sum_{n=1}^{\infty} b[x]_{n+1}/2^n & \text{if } x \in I_0, \\ \sum_{n=1}^{\infty} (1 - b[x]_{n+1})/2^n & \text{if } x \in I_1. \end{cases}$$

In the following, we note some relationships between the periodicity of a point in $[0,1]$ under τ and two sequences $O(x)$, $B(x)$.

Proposition 2.6. *Let x be a point in $[0,1]$. Then we have the following.*

(1) *x is a periodic point under τ with period p if and only if $O(x)$ is a periodic sequence with period p.*

(2) *If x is in NF_2 and a periodic point under τ with period p, then $B(x)$ is a periodic sequence with period $2p$.*

(3) *If x is in NF_2 and $B(x)$ is a periodic sequence with period p, then there exists a positive integer i such that $\tau^i(x)$ is a periodic point under τ with period $q \leqq p$.*

In order to show the difference between $O(x)$ and $B(x)$, we give an example of a periodic point $x = 2/3$ with period 1.

$O(2/3) = (1, 1, 1, 1, 1, \ldots)$,

$B(2/3) = (1, 0, 1, 0, 1, \ldots)$.

Moreover, before giving a sequence which converges to x, we note that, if x is in NF_2, $B(x)$ is uniquely determined and satisfies the following property.

Property (NF_2): *For any positive integer N, there exists positive integers $m, n \geqq N$ such that $b[x]_m = 0$ and $b[x]_n = 1$*

For $x = \sum_{n=1}^{\infty} \frac{b[x]_n}{2^n}$, we define $x(k)$ by

$$x(k) = \sum_{n=1}^{k-1} \frac{b[x]_n}{2^n} + \frac{1 - b[x]_k}{2^k} + \sum_{n=k+1}^{\infty} \frac{b[x]_n}{2^n} .$$

Then we have

$$x(k) = \begin{cases} x + (1/2^k) & \text{if } b[x]_k = 0, \\ x - (1/2^k) & \text{if } b[x]_k = 1. \end{cases}$$

Thus $|x(k) - x| = 1/2^k$ and $O(x(k))$ is given as follows.

Lemma 2.7. *Let x be a point in NF_2 with $O(x) = (x_n)_{n=0}^{\infty}$. Then it follows that*

$$O(x(k)) = (x_0, \ldots, x_{k-2}, z_{k-1}, z_k, x_{k+1}, \ldots),$$

where $|z_i - x_i| = 1$ for $i = k-1, k$.

Hereafter, in the calculation of $f'(x)$, we use following two notations.

$$H_k(x) = h(x(k)) - h(x) \quad \text{and} \quad D_k(x) = \frac{h(x(k)) - h(x)}{x(k) - x}.$$

Then $D_k(x) = |2^k H_k(x)|$. Now we introduce a concept concerning the behavior of the orbit of a point in NF_2. By virtue of the definition of NF_2, the set $\{\ell \geqq 1 | b[x]_n = b[x]_{n+\ell}\}$ is not empty. For a point x in NF_2, we set

$$p(n) = \min\{\ell \geqq 1 |\ b[x]_n = b[x]_{n+\ell}\}.$$

We say that the sequence $B(x) = \{b[x]_n\}_{n=1}^{\infty}$ is *quasi-periodic* if there exists a positive integer K such that $p(n) \leqq K$ for all n. Of course, if $B(x)$ is periodic with period p, then it is quasi-periodic with $K = p$. Moreover, it follows that, if a point x is periodic point or eventually periodic under τ, $B(x)$ is quasi-periodic. For a point x in NF_2 such that $B(x)$ is quasi-periodic, we have some lemmas.

Lemma 2.8. *Let x be a point in NF_2 such that $B(x)$ is quasi-periodic. Then there exists a limit $f'(x) = \omega$ if and only if there exists a limit $\omega = \lim_{n\to\infty} D_k$.*

Lemma 2.9. *Let x be a point in NF_2 with $O(x) = (x_n)_{n=0}^{\infty}$. Then it follows that*

$$\begin{array}{lll} \text{Case 1.} & H_k(x) = \alpha_{k-1}\cdot((1-2c)h(\tau^{k+1}(x))+c) & \text{if } (x_{k-1},x_k)=(0,0), \\ \text{Case 2.} & H_k(x) = \alpha_{k-1}\cdot(1-c) & \text{if } (x_{k-1},x_k)=(0,1), \\ \text{Case 3.} & H_k(x) = \alpha_{k-1}\cdot(c-1) & \text{if } (x_{k-1},x_k)=(1,0), \\ \text{Case 4.} & H_k(x) = \alpha_{k-1}\cdot((2c-1)h(\tau^{k+1}(x))-c) & \text{if } (x_{k-1},x_k)=(1,1). \end{array}$$

Lemma 2.10. *Let x be a point in NF_2.*
(1) *Suppose $c \in (0,1/2)$. Then we have the following inequality.*

$$2^k c|\alpha_{k-1}| \leqq D_k(x) \leqq 2^k(1-c)|\alpha_{k-1}|$$

(2) *Suppose $c \in (1/2,1)$. Then we have the following inequality.*

$$2^k(1-c)|\alpha_{k-1}| \leqq D_k(x) \leqq 2^k c|\alpha_{k-1}|$$

Using Lemma 2.3. we express Lemma 2.10 as follows.

Lemma 2.11. *Let x be a point in NF_2.*
(1) *Suppose $c \in (0,1/2)$. Then we have the following inequality.*

$$2^k c^{N_0(k-2)+1}(1-c)^{N_1(k-2)} \leqq D_k(x) \leqq 2^k c^{N_0(k-2)}(1-c)^{N_1(k-2)+1}.$$

(2) *Suppose $c \in (1/2,1)$. Then we have the following inequality.*

$$2^k c^{N_0(k-2)}(1-c)^{N_1(k-2)+1} \leqq D_k(x) \leqq 2^k c^{N_0(k-2)+1}(1-c)^{N_1(k-2)}.$$

Lemma 2.11 immediately implies the following.

Lemma 2.12. *Suppose $c \in (0,1/2)\cup(1/2,1)$. Let x be a point in NF_2 such that $B(x)$ is quasi-periodic. Then we have the following.*
(a) $\lim_{k\to\infty} 2^k c^{N_0(k)}(1-c)^{N_1(k)} = 0$ *if and only if* $\lim_{k\to\infty} D_k(x) = 0$.
(b) $\lim_{k\to\infty} 2^k c^{N_0(k)}(1-c)^{N_1(k)} = \infty$ *if and only if* $\lim_{k\to\infty} D_k(x) = \infty$.

We have discussed the existence of $h'(x)$ and now we show that the possibility of taking the value of $f'(x)$ is only 0 and ∞.

Lemma 2.13. *Let x be a point in NF_2. If there exists $\omega = \lim_{k\to\infty} D_k(x)$, then $\omega = 0$ or $\omega = \infty$.*

Immediately, by Theorem 2.5 and Lemma 2.13, we have the following proposition.

Proposition 2.14. *If there exists $f'(x)$, then $f'(x) = 0$ or $f'(x) = \infty$.*

By Lemma 2.9, 2.12 and Proposition 2.14, we have the following proposition.

Proposition 2.15. *Let x be a point in NF_2 such that $B(x)$ is quasi-periodic. Then it follows that*

$$h'(x) = \begin{cases} 0 & \text{if } \lim_{n\to\infty} 2^n c^{N_0(n)}(1-c)^{N_1(n)} = 0 \quad \cdots(1) \\ \infty & \text{if } \lim_{n\to\infty} 2^n c^{N_0(n)}(1-c)^{N_1(n)} = \infty \quad \cdots(2) \\ \text{does not exist} & \text{otherwise.} \quad \cdots(3) \end{cases}$$

Now let x be a periodic point with period p. Then $O(x) = (x_n)_{n=0}^{\infty}$ is a periodic sequence with period p. Here we set $r_p(x) = 2^p c^{N_0(p-1)}(1-c)^{N_1(p-1)}$. Then, since $N_i(mp-1) = mN_i(p-1)$ $(i = 0, 1)$, it follows that $r_{mp}(x) = (r_p(x))^m$. Hence we have the following.

Proposition 2.16. *If x is a periodic point in $[0,1]$ with period p under τ, then it follows that*

$$h'(x) = \begin{cases} 0 & \text{if } r_p(x) < 1, \\ \infty & \text{if } r_p(x) > 1. \end{cases}$$

Using Proposition 2.16, we have the following proposition.

Proposition 2.17. *Suppose $c \in (0,1/2) \cup (1/2,1)$. Let $E_0 = \{x \in [0,1] | h'(x) = 0\}$ and $E_\infty = \{x \in [0,1] | h'(x) = \infty\}$. Then*
(1) E_0 and E_∞ are dense in $[0,1]$,
(2) $\mu(E_0) = 1$ and $\mu(E_\infty) = 0$,
(3) $\mu(h(E_0)) = 0$ and $\mu(h(E_\infty)) = 1$.

Remark 2.18. We note that $r(x) \neq 1$ does not necessarily hold though it holds if c is a rational number. Indeed, it does not hold if $c = \frac{3-\sqrt{5}}{4}$. In the following, we show that $h'(x)$ does not exist for a τ-periodic point with period 3.

Example 2.19. Let $c = \frac{3-\sqrt{5}}{4} \in (0,1/2)$ and $x = 2/7$. Then $8c(c-1)^2 = 1$ and x is a τ-periodic point with period 3 with the following orbit:

$$O(x) = (\overline{0,1,1}) \quad \text{and} \quad B(x) = (\overline{0,1,0}).$$

Thus $r_3(2/7) = 2^3c(c-1)^2 = 1$. Here we calculate $D_k(x)'s$. Using $8c(1-c)^2 = 1$, we have

$\alpha_{3\ell} = \{c(1-c)^2\}^\ell = 1/2^{3\ell}$,

$\alpha_{3\ell+1} = (c-1)\alpha_{3\ell} = (c-1)/2^{3\ell}$,

$\alpha_{3\ell+2} = (c-1)\alpha_{3\ell+1} = (c-1)^2/2^{3\ell}$.

Thus, since $(x_{3(\ell-1)+2}, x_{3\ell}) = (1,0)$, we have

$D_{3\ell}(x) = 2^{3\ell}|H_{3\ell}| = 2^{3\ell}|\alpha_{3\ell-1}(c-1)| = 2^{3\ell}(1-c)^3/2^{3(\ell-1)} = 8(1-c)^3 = 2+\sqrt{5}$.

Moreover, since $(x_{3\ell}, x_{3\ell+1}) = (0,1)$, we have

$D_{3\ell+1}(x) = 2^{3\ell+1}|H_{3\ell+1}| = 2^{3\ell+1}|\alpha_{3\ell}(1-c)| = 2^{3\ell+1}(1-c)/2^{3\ell} = 2(1-c) = \frac{1+\sqrt{5}}{2}$.

Therefore the sequence $\{D_k(x)\}_{k=1}^\infty$ does not converge, that is, $h'(x)$ does not exist.

Now we reach at our conclusion.

Theorem 2.20. *Suppose* $c \in (0,1/2) \cup (1/2,1)$. *Then*

$$h'(x) = \begin{cases} 0 & \text{if } x \in E_0\ , \\ \infty & \text{if } x \in E_\infty, \\ \text{does not exist} & \text{if } x \in F, \end{cases}$$

where $\{E_0, E_\infty, F\}$ *are measurable sets satisfying the following conditions.*

(1) $\{E_0, E_\infty, F\}$ *are mutually disjoint,*

(2) $\mu(E_0) = 1, \mu(E_\infty) = 0$ *and* $\mu(F) = 0$,

(3) E_0 *and* E_∞ *are dense in* $[0,1]$,

(4) $\mu(h(E_0)) = 0, \mu(h(E_\infty)) = 1$ *and* $\mu(h(F)) = 0$.

Moreover it follows that

(5) *the set* F_2 *is included in* E_0 *(resp.* E_∞*) if* $c \in (0,1/2)$ *(resp.*$c \in (1/2,1)$*).*

Finally we note that the condition of quasi-periodicity in Lemma 2.8 may be deleted, though we cannot prove the lemma without that condition in the present paper and have no counter example.

References

[1] P. Ahmed, S. Kawamura and S. Sasaki, Banach lattices and the Perron-Frobenius operator associated with chaotic map, Far East Journal of Dynamical Systems, 8-1(2006), 1-25.

[2] R. L. Devaney, *An introduction to chaotic dynamical systems*, Second Edition, Addision-Wesley, Redwood City, 1989.

[3] S. Kawamura, Covariant representations associated with chaotic dynamical systems, Tokyo Jour. Math., 20-1(1997), 205-217.

[4] S. Kawamura, Chaotic maps on a measure space and the behavior of the orbit of a state, Tokyo Jour. Math., 24-2(2001), 509-533.

[5] W. de, Melo and S. van Strien, *One-Dimensional Dynamics,* Ergebnisse 3 Folge Band 25, Springer 1993.

[6] I.P. Natanson, *Theory of functions of a real variable,* Frederic Ungar Publishing Co. New York 1964