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The Tamagawa Number Conjecture of Bloch-Kato for Dirichlet Motives at the prime 2

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This survey article is oriented to introducing the Tamagawa number conjecture along the line of the author's talk in Kyoto 2005, December, with special emphasis on the cohomological side.

It is commonly believed that the Tamagawa Number Conjecture and the Iwasawa Main Conjecture in the sense of Kato are incarnations of the same mathematical content, though the author only explains the Tamagawa number conjecture side in this article. Note that the conjecture exposed in this article is non-equivariant i.e. original one by Bloch-Kato, and the one generalized by Fontaine-Perrin-Riou.

The author wants to appologize that he gave the talk under too much assumption on the subject, so he prepared this article for the beginners on the conjecture. When writing this article, he learned a lot from the beautiful survey article due to O. Venjakob [V]. The author thanks K. Nakamura for pointing out many mistakes in the previous version of this article.

1 Notations and Definitions.

1.1 Notaitions.

In this paper, $E/\mathbb{Q}$ is a coefficient number field of motives, and put $\mathcal{O} := \mathcal{O}_E$, the integer ring of E. For a rational prime p, let us denote $\mathcal{O}_p := \mathcal{O} \otimes_{\mathbb{Z}} \mathbb{Z}_p$, and for a rational place v, $E_v := E \otimes_{\mathbb{Q}} \mathbb{Q}_v$. Let us denote $\mathrm{G}_F = \mathrm{Gal}(\bar{F}/F)$ for a field F. We denote by c the complex conjugation, in $\mathrm{G}_{\mathbb{Q}}$. Frobenii are chosen to be geometric, and denote them by Fr_v for a finite place v. In this paper, the reciprocity isomorphism is fixed as follows :

$$\mathrm{rec} : \mathrm{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{Q}) \simeq (\mathbb{Z}/N\mathbb{Z})^{\times};\ \mathrm{Fr}_p \mapsto p \mod N.$$

If an $\mathcal{O}_p[[\mathrm{G}_{\mathbb{Q}}]]$-module M has a $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$-action, M^+ always means $H^0(\mathbb{R}, M) = \{m \in M|\ \ c\cdot m = m\}$, and it does not mean $\frac{1-c}{2}M$. These two are in general different if 2 is not invertible in $\mathcal{O}_p$. For an E-motif M over rational numbers $\mathbb{Q}$, we will abbreviate the statement of the Tamagawa number conjecture for the motif M, by TNC_M. If we consider a continuous E_p-linear $\mathrm{G}_{\mathbb{Q}}$-module M_p, fix a finite closed subset of $\mathrm{Spec}\,\mathbb{Z}$, which includes the ramified primes of M_p. Fix such one S. Then, we can regard the Galois module M_p as the étale sheaf on $\mathrm{Spec}\,\mathbb{Q}$. Let us denote the open immersion of generic point by $j{:}\,\mathrm{Spec}\,\mathbb{Q} \hookrightarrow \mathrm{Spec}\,\mathbb{Z}[1/Sp]$. Then, we denote in the bounded derived category of E_p-modules,

$$\mathbb{R}\Gamma(\mathbb{Z}[1/Sp], M_p) := \mathbb{R}\Gamma(\mathrm{Spec}\,\mathbb{Z}[1/Sp], j_*M_p),$$
$$\mathbb{R}\Gamma_c(\mathbb{Z}[1/Sp], M_p) := \mathrm{Cone}[\ \mathbb{R}\Gamma(\mathbb{Z}[1/Sp], M_p) \xrightarrow{\mathrm{res}} \bigoplus_{v|Sp\infty} \mathbb{R}\Gamma(\mathbb{Q}_v, M_p)\ ].$$

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For the $\mathcal{O}_p$-coefficient case, we also define it in the same way. That is, for $\mathcal{O}_p$-lattice T_p of M_p, we also define the compact supported cohomology functor $\mathbb{R}\Gamma_c(\mathbb{Z}[1/Sp], T_p)$ in the derived category of $\mathcal{O}_p$-modules bounded below, by

$$\mathbb{R}\Gamma_c(\mathbb{Z}[1/Sp], T_p) := \mathrm{Cone}[\ \mathbb{R}\Gamma(\mathrm{Spec}\,\mathbb{Z}[1/Sp], T_p) \xrightarrow{\mathrm{res}} \bigoplus_{v|Sp\infty} \mathbb{R}\Gamma(\mathbb{Q}_v, T_p)\ ].$$

Note that for the case $p = 2$, we have the different definition from the Milne's one of $\mathbb{R}\Gamma_c$, because we need to compute $H^i(\mathbb{R}, T_p)$, $i = 1, 2$. And by this consideration, this complex $\mathbb{R}\Gamma_c(\mathbb{Z}[1/Sp], T_p)$ is bounded both. We consider the determinant functor of Knudsen-Mumford up to sign (cf. [K][Section 2.1]). Finally, the *K-groups* are always Quillen's ones.

1.2 Motives.

We present the definitions enough to formulate TNC for the case of pure (Chow) motives. Readers can assume the motives always to be pure, which are explained below. See [FP] for the mixed case.

Definition 1.1 (Pure Chow Motives, cf [Schl]). Let $\mathcal{V}_k$ be the categories of projective schemes, smooth over a field k. For a scheme X, we denote by $\mathcal{Z}^i(X)$, the group generated by irreducible codimension i cycles on X. For a morphism $\phi : X \to Y$ in $\mathcal{V}_k$ with irreducible Y, we denote $\Gamma_\phi \in \mathcal{Z}^{\dim X}(X \times Y)$, the *graph of* Y. (If Y is not irreducible, then consider it componentwise.) Let us define $\mathrm{CH}^i(X) := \mathcal{Z}^i(X)/\sim_{rat}$. Here, for $Z_1, Z_2 \in \mathcal{Z}^i(X)$, $Z_1 \sim_{rat} Z_2$ if and only if there is a rational function $f \in k(X)$, such that $\mathrm{div}(f) = [Z_1] - [Z_2]$. On the group $\mathrm{CH}^*(X)$, we can define the product structure by intersection theory, and pull-backs and push-forwards by maps in $\mathcal{V}_k$. Then, for pure d-dimensional X, we define the group of r-th algebraic correspondences, $\mathrm{Corr}^r(X, Y) := \mathrm{CH}^{r+d}(X \times Y)$. *The category of Chow motives* $\mathcal{M}_k$, is defined to be a pseudo-abelian category (i.e. exact category, which is closed under taking images and kernels of projectors) with tensor structures, as follows. Objects are the triplets (X, p, m) for $X \in \mathcal{V}_k, p = p^2 \in \mathrm{Corr}^0(X, X), m \in \mathbb{Z}$. Morphisms are defined by

$$\mathrm{Hom}_{\mathcal{M}_k}((X,p,m),(Y,q,n)) = q \cdot \mathrm{Corr}^{n-m}(X, Y) \cdot p.$$

We also denote $h^i(X)(m) := (X, p_i, m)$, for p_i is the Künneth projector for i-th cohomology. The *Tate object* $\mathbb{Q}(r)$ is defined to be $(\mathrm{Spec}\, k, id, r)$. This definition is compatible with the tensor structure. We use the term *E-motif*, we consider these motives by extending correspondences from $\mathbb{Q}$ to E.

Remark 1.2. If we do not assume the standard conjecture of Grothendieck, we can not prove the existence of projectors p_i, satisfying $(X, p_i, 0) = h^i(X)$, which gives i-th cohomologies of X with pure dimension d, for Weil cohomologies via realization functors, for $i \neq 0, 1, 2d-1$. But we can define without any conjecture, $h^1(X)$ for any curve X over k.

We will define realizations only for pure motives. Readers can also find realization functors from the Voevodsky's category $\mathrm{DM}_{gm}(k)$, for any subfield k of $\mathbb{C}$ in [Hu].

Definition 1.3 (Realizations). Let $M = h^i(X)(j)$ be a pure motif over $\mathbb{Q}$, with coefficients in E. We define the *Betti realization* M_B, *de Rham realization* M_{dR}, and *ℓ-adic realization* M_ℓ *of* M, to be the cohomology groups

$$H^i_{sing}(X(\mathbb{C}), \mathbb{Q}(j)) \otimes_{\mathbb{Q}} E, H^i_{dR}(X/\mathbb{Q}) \otimes_{\mathbb{Q}} E, H^i_{\acute{e}t}(X \times_{\mathbb{Q}} \bar{\mathbb{Q}}, \mathbb{Q}_\ell(j)) \otimes_{\mathbb{Q}} E.$$

These are E-vector apace, E-vector space, E_ℓ-module respectively, which are given by additional structures; the action of complex conjugation, the Hodge filtration, the Galois action of $G_{\mathbb{Q}}$. And they are compared by comparison maps.

Example 1.4 (Realizations of Dirichlet Motives). For the case of a Dirichlet motif, we can define them as follows. The readers who do not like motivic treatment can consider the following system of realizations plus the motivic cohomology, as the definition of a Dirichlet motif.

- p-adic étale realization :
  $M_p(\chi)(r) := p_{\chi^{-1}}[H^0_{\acute{e}t}(\operatorname{Spec}\mathbb{Q}(\zeta_N)\otimes_{\mathbb{Q}}\bar{\mathbb{Q}}, \mathbb{Q}_p(r))\otimes_{\mathbb{Q}} E]$.
- Betti realization :
  $M_B(\chi)(r) := p_{\chi^{-1}}[H^0_B(\operatorname{Spec}\mathbb{Q}(\zeta_N)\otimes_{\mathbb{Q}}\mathbb{C}, \mathbb{Q}(r))\otimes_{\mathbb{Q}} E]$, where $E(r) = E\cdot(2\pi i)^r$.
- de Rham realization :
  $M_{dR}(\chi)(r) := p_{\chi^{-1}}H^0_{dR}(\operatorname{Spec}\mathbb{Q}(\zeta_N)/\mathbb{Q})\otimes_{\mathbb{Q}} E$, with Hodge filtlation shifted by r.

Definition 1.5 (Motivic Cohomologies). Let us consider a pure $\mathbb{Q}$-motif $M = h^i(X)(j)$ over k. For this motif, we define the rational *motivic cohomology*, by the following

$$H^0_{\mathcal{M}}(X,\mathbb{Q}(j)) = \begin{cases}\operatorname{Hom}_{\mathcal{M}_k}(M,\mathbb{Q}) & j=0,\\ 0 & \text{else,}\end{cases} \quad H^1_{\mathcal{M}}(X,\mathbb{Q}(j)) = \begin{cases} K_{2j-i-1}(X)^{(j)}_{\mathbb{Q}} & 2j-i\neq 1,\\ CH_0(X)_{\mathrm{hom}=0} & 2j-i=1.\end{cases}$$

We also abbreviate $H^0_{\mathcal{M}}(M) = H^0_{\mathcal{M}}(X,\mathbb{Q}(j))$, and $H^1_{\mathcal{M}}(M) = H^1_{\mathcal{M}}(X,\mathbb{Q}(j))$. These groups are known to be extentions in the motivic derived category $DM_{gm}(k)$ of Hanamura, Levine, and Voevodsky. We further need the *finite motivic cohomology.* If there is a regular model $\mathfrak{X}$ of X, which is proper over $\mathbf{Z}$, let us define

$$H^1_f(M) = H^1_f(X,\mathbb{Q}(j)) = \operatorname{Image}(K_{2j-i-1}(\mathfrak{X})_{\mathbb{Q}}{}^{(j)} \longrightarrow K_{2j-i-1}(X)_{\mathbb{Q}}).$$

Here, we denote by $K_n(\mathfrak{X})^{(j)}_{\mathbb{Q}}$ the eigenspace for Adams operations ψ_k, for any $k\geq 1$. This group is not yet interpreted as the extentions in the motivic category. For the definition without taking the model, see [Sch2] using alterration. These groups are conjecturally finite dimensional.

Example 1.6 (The case of Dirichlet Motives). Let $F = \mathbb{Q}(\zeta_N)^{\operatorname{Ker}\chi}$ and consider the E-motif $M(\chi)(r)$ over $\mathbb{Q}$. We assume that $\mathbb{Q}(\chi)$ in contained in E. Then, we have

$$H^0_f(M(\chi)(r)) := \begin{cases} E & r=0,\chi=1,\\ 0 & \text{else,}\end{cases} \quad H^1_f(M(\chi)(r)) := \begin{cases} p_{\chi^{-1}}[K_{2r-1}(\mathcal{O}_F)\otimes_{\mathbb{Q}} E], & r\geq 1,\\ 0, & \text{else.}\end{cases}$$

The case for which the finite dimensionality is known is only for $\operatorname{Spec}\mathcal{O}_F$, that is the miraculous result of A. Borel ([Bu]). Note that for this case, we have $K_{2j-1}(\operatorname{Spec}\mathcal{O}_F)^{(j)} = K_{2j-1}(\operatorname{Spec}\mathcal{O}_F)$. See the proof in [W][Thoerem 47], where the proof reduces to the computation in the étale cohomology, done by Soulé.

Definition 1.7 (L-function of motives). Let M be an E-motif over $\mathbb{Q}$. We consider the function

$$L(M,s) = \prod_v P_v(M,s)^{-1}.$$

Here, v runs over rational primes, and we put $P_v(M,s) = \det_{\mathbb{Q}_\ell}[1-\operatorname{Fr}_v v^{-s}|M_\ell^{I_v}]$, where I_v is an inertia subgroup at $v\neq\ell$. This is conjecturally independent of the choice of ℓ, which is proved at the good reduction prime v. We call this function *L-function of M*, or *Hasse-Weil L-function of M.*

2 Statements of TNC and the Main Theorem.

2.1 Motivating Examples - Special Values Side -

The Tamagawa number conjecture of Bloch-Kato is a vast generalization of the class number formula of Dirichlet, the Birch-Swinnerton-Dyer conjecture, and astonishingly, the Iwasawa theory. But without difficult definitions, the idea and philosophy of the conjecture can be understood already in these formulas. (And recall that Iwasawa Main Conjecture is also reduced to the class number formula.) So, let us see the motivating cases first, before stating the general TNC. The difficulties for $p = 2$ can also be seen below.

Example 2.1 (The Class Number Formula). The simplest case of TNC is the case of Dedekind zeta function, that is, $E = \mathbb{Q}, M = h^0(\operatorname{Spec} F)$. By definition, we have

$$\begin{aligned} L(M,s) &= \prod_p \det_{\mathbb{Q}_\ell}[1 - \mathrm{Fr}_p p^{-s} | H^0(\operatorname{Spec} F \otimes_{\mathbb{Q}} \bar{\mathbb{Q}}, \mathbb{Q}_\ell)]^{-1} \\ &= \prod_p \det_{\mathbb{Q}_\ell}[1 - \mathrm{Fr}_p p^{-s} | H^0(\operatorname{Spec} \mathcal{O}_F[1/p] \otimes_{\mathbb{Z}} \bar{\mathbb{Q}}, \mathbb{Q}_\ell)]^{-1}. \end{aligned}$$

The Euler factor is interpreted via Shapiro's Lemma,

$$\begin{aligned} &\det_{\mathbb{Q}_\ell}[1 - \mathrm{Fr}_p p^{-s} | H^0(\operatorname{Spec} \mathcal{O}_F[1/p] \otimes_{\mathbb{Z}} \bar{\mathbb{Q}}, \mathbb{Q}_\ell)] \\ &= \prod_{v|p} \det_{\mathbb{Q}_\ell}[1 - \mathrm{Fr}_v N(v)^{-s} | H^0(\operatorname{Spec} \mathbb{Z}[1/p] \otimes_{\mathbb{Z}} \bar{\mathbb{Z}}, \mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Q}_\ell)] = \prod_{v|p}(1 - N(v)^{-s}). \end{aligned}$$

Here, N_v is the cardinarity of the residue field of v. So, we have $\zeta_F(s) = L(M,s)$. For $r \in \mathbb{Z}$, we define

$$\zeta_F^*(r) = \lim_{s \to 0} s^{-r_M} \zeta_F(s), \ r_M := \text{ orders of zeroes of } \zeta_F \text{ at s = r.}$$

In the case $r = 0$ or 1, we have the classical class number formula :

$$\zeta_F^*(0) = \lim_{s \to 0} s^{r_1+r_2-1} \zeta_F(s) = -\frac{h_F R_F}{w_F}, \ \ \zeta_F^*(1) = \lim_{s \to 1}(s-1)\zeta_F(s-1) = \frac{2^{r_1}(2\pi)^{r_2} h_F R_F}{w_F \sqrt{d_F}}.$$

Here, we adopt the conventions h_F, R_F, w_F, d_F respectively to be the class number, the Dirichlet regulator, the number of the roots of unity, and the absolute value of the discriminant of F. For simplicity, consider the case of $s = 0$ and the p-part of this formula, modulo the irrational part. Then, we have by [Mi][Chapter 2, Proposition 2.1] and using the Kummer sequence, we have

$$H^0_{\acute{e}t}(\operatorname{Spec} \mathcal{O}_F[1/p], \mathbb{Z}_p(1)) = 0, H^1_{\acute{e}t}(\operatorname{Spec} \mathcal{O}_F[1/p], \mathbb{Z}_p(1))_{tors} = |w_F|_p, H^2_{\acute{e}t}(\operatorname{Spec} \mathcal{O}_F[1/p], \mathbb{Z}_\ell(1)) = |h|_p.$$

So, we are able to see that the value $\zeta_F^*(0)/R_F$ has the p-adic interpretation via p-adic Euler characteristic up to sign. For the case $p = 2$, it is easy to imagine that 2^{r_1}-power makes complecated in this formula to see exactly the effect of the 2-adic part of cohomology. This consideration above is highly generalized to the Cohomological Lichtenbaum Conjucture. See Theorem 2.16.

Example 2.2 (BSD). Let A be an abelian variety over $\mathbb{Q}$. In this case, we consider $M = h^1(A)(1)$. Then, we have the conjectural formula for the special value of $L(M,s) = L(A, s+1)$, by

$$L^*(M,0) = 2^r \frac{\Omega_A^+ R_A |A^\vee(\mathbb{Q})_{tors}| \cdot |A(\mathbb{Q})_{tors}|}{\text{Ш}(A/\mathbb{Q})} \prod_\ell c_\ell(A).$$

Here, $r = \operatorname{rank} A(\mathbb{Q}), R_A =$ regulator of $A(\mathbb{Q})/A(\mathbb{Q})_{tors}$, and Ω_A^+ is the Néron period, and $c_\ell(M)$ is Tamagawa factor. The Tate-Shafarevich group Ш$(A/\mathbb{Q})$ is conjectured to be a finite group. In Appendix, we will see these values are interpreted via motivic cohomology groups, i.e. motivic meaning of these values and prove that this formula and TNC for the motif M is equivalent. Note that also in this conjectural formula, the power of the prime 2 appears, and 2 is also distinguished in this case.

If we take the irrational parts above two examples (i.e. regulators and periods) into account , it seems natural to expect the following exact sequence.

Conjecture 2.3 (The Period-Regulator sequence). *For a $\mathbb{Q}$-motif M over $\mathbb{Q}$, let α_M be the map, which is induced by taking the c-fixed part of the Hodge's comparison morphism $M_B \otimes_{\mathbb{Q}} \mathbb{C} \simeq M_{dR} \otimes_{\mathbb{Q}} \mathbb{C}$,*

$$\alpha : M_B^+ \otimes_{\mathbb{Q}} \mathbb{R} \longrightarrow M_{dR} \otimes_{\mathbb{Q}} \mathbb{R} / \operatorname{Fil}^0 M_{dR} \otimes_{\mathbb{Q}} \mathbb{R}.$$

Then, we have the following exact sequence of finite dimensional $\mathbb{R}$*-vector spaces,*

$$0 \longrightarrow H^0_f(M)_{\mathbb{R}} \xrightarrow{cl} \ker(\alpha_M) \xrightarrow{r^*} (H^1_f(M^*(1)))_{\mathbb{R}})^*$$
$$\xrightarrow{h} H^1_f(M)_{\mathbb{R}} \xrightarrow{r} \mathrm{coker}(\alpha_M) \xrightarrow{c^*} (H^0_f(M^*(1))_{\mathbb{R}})^* \longrightarrow 0.$$

Here, $r = r_M, cl_M = cl, h = h_M$ *is called (Beilinson) regulator map, cycle map, and height pairing.* $(-)^*$ *is the dual of those maps.*

2.2 Preliminaries for TNC.

In the followings, we assume the motives are defined over $\mathbb{Q}$, with coefficients in E. We need more preliminaries for our result. These are important objects in the cohomological side. Let V be an E_p-linear continuous $G_{\mathbb{Q}}$-representation. We regard V as $G_{\mathbb{Q}_p}$-module via $G_{\mathbb{Q}_p} \hookrightarrow G_{\mathbb{Q}}$.

Definition 2.4 (Local Finite Cohomologies). We define the *finite cohomology of Bloch-Kato* by

$$H^1_f(\mathbb{Q}_p, V) := \ker(H^1(\mathbb{Q}_p, V) \to H^1(\mathbb{Q}_p, V \otimes \mathbb{B}_{cris})).$$

Here, we used the p-adic period ring of Fontaine, which is the p-adic period ring of good reduction varieties (see [Co][p512]). Also, we define the subcomplex $\mathbb{R}\Gamma_f(\mathbb{Q}_p, V)$ of $\mathbb{R}\Gamma(\mathbb{Q}_p, V)$, which sits in degree 0 and 1 (cf. Section 3.2), defined by

$$\mathbb{R}\Gamma_f(\mathbb{Q}_p, V) := [D_{cris}(V) \overset{(1-\phi, \mathrm{pr})}{\longrightarrow} D_{cris}(V) \oplus D_{dR}(V)].$$

This complex has the following cohomologies (cf. Section 3.2)

$$H^0(\mathbb{R}\Gamma_f(\mathbb{Q}_P, V)) = H^0(\mathbb{Q}_p, V), H^1(\mathbb{R}\Gamma_f(\mathbb{Q}_p, V)) = H^1_f(\mathbb{Q}_p, V).$$

For $\ell \neq p$, we define $\mathbb{R}\Gamma_f(\mathbb{Q}_\ell, V)$ by the complex $\mathbb{R}\Gamma_f(\mathbb{Q}_\ell, V) = [V^{I_\ell} \overset{1-\mathrm{Fr}_\ell}{\longrightarrow} V^{I_\ell}]$. We put $\mathbb{R}\Gamma_{/f}(\mathbb{Q}_v, V) :=$ Cone[$\mathbb{R}\Gamma_f(\mathbb{Q}_v, V) \to \mathbb{R}\Gamma(\mathbb{Q}_v, V)$] for all v. These are objects in the derived category of $\mathbb{Q}_p$-vector spaces.

Definition 2.5 (Global Finite Cohomologies, cf. [FP][CHAPITRE II, p643). Let V be an E_p-linear continuous $G_{\mathbb{Q}}$-representation. We define a complex $\mathbb{R}\Gamma_f(\mathbb{Z}[1/Sp], V)$ by the mapping fiber

$$\mathbb{R}\Gamma_f(\mathbb{Z}[1/Sp], V) := \mathrm{Cone}[\ \mathbb{R}\Gamma(\mathrm{Spec}\,\mathbb{Z}[1/Sp], V) \to \bigoplus_{v \in Sp\infty} \mathbb{R}\Gamma_{/f}(\mathrm{Spec}\,\mathbb{Q}_v, V)\ ][-1].$$

Using the octahedral axiom in the derived category (see [H][p21, (TR4)]) to the destinguished triangle $\mathbb{R}\Gamma_c(\mathbb{Z}[1/Sp], V) \to \mathbb{R}\Gamma(\mathbb{Z}[1/Sp], V) \to \bigoplus_{v \in Sp\infty} \mathbb{R}\Gamma(\mathbb{Q}_v, V)$, and to the defining tiriangle above, we have the following distinguished triangle,

$$\mathbb{R}\Gamma_c(\mathbb{Z}[1/Sp], V_p(r)) \to \mathbb{R}\Gamma_f(\mathbb{Z}[1/Sp], V_p(r)) \to \bigoplus_{v \in Sp} \mathbb{R}\Gamma_f(\mathbb{Q}_v, V_p(r)) \oplus \mathbb{R}\Gamma(\mathbb{R}, V_p(r)). \qquad (\spadesuit)$$

This cohomological complex $\mathbb{R}\Gamma_f(\mathbb{Z}[1/Sp], V)$ is conjecturally closely related to the integral motivic cohomology in the previous section, as follows.

Conjecture 2.6 ("Finite Cohomologies have Motivic Origin"). *In the terminology above, we should have the isomorphisms (cycle map and p-adic regulator)*

$$H^0_f(M) \otimes_{\mathbb{Q}} \mathbb{Q}_p \longrightarrow H^0_f(\mathbb{Z}[1/Sp], M_p),\ \ H^1_f(M) \otimes_{\mathbb{Q}} \mathbb{Q}_p \longrightarrow H^1_f(\mathbb{Z}[1/Sp], M_p).$$

Proposition 2.7. Assume Conjecture 2.6 holds. Then, these maps induce the isomorphism

$$\det_{E_p}\mathbb{R}\Gamma_f(\mathbb{Z}[1/Sp], M_p) \simeq L_f(M) \otimes L_f(M^*(1)) \otimes \mathbb{Q}_p.$$

Here, we defined $L_f(M) = \det_E H_f^0(M) \otimes \det_E^{-1} H_f^1(M)$.

The proposition above tells us that $H_f^q(M)$'s should behave like some kind of Euler chatacteristics of M. The only example, for which Conjecture 2.6 is known until now, is the case $M = h^0(\operatorname{Spec} \mathcal{O}_F)(r)$ by the miraculous result of A. Borel.

In the followings, we always assume the finite dimensionality of $H_f^q(M), q = 0, 1, 2, 3$. Upon this conjecture, we can define the following E-vector space, which plays the key role to formulate TNC.

Definition 2.8 (Fundamental Line). For an E-motif M over $\mathbb{Q}$, let us define an E-vector space,

$$\Delta_f(M) := L_f(M) \otimes L_f(M^*(1)) \otimes \det_E M_{dR}/Fil^0 M_{dR} \otimes \det_E^{-1} M_B^+.$$

Proposition 2.9 (θ_∞). For a $\mathbb{Q}$-motif M, we have an identification θ_∞: $\Delta_f(M) \otimes_\mathbb{Q} \mathbb{R} \simeq \mathbb{R}$, by taking the $\mathbb{R}$-determinant of the exact sequence in Conjecture 2.3. For the case of E-motif, tensor E over $\mathbb{Q}$.

For the space $\Delta_f(M)$, the following E_p-module is associated, under Conjecture 2.6 and finite dimensionality of $H_f^q(M)$'s.

Definition 2.10 (Euler-Poncaré line). Define $\Delta_{EP}(M_p) := \det_{E_p}^{-1}\mathbb{R}\Gamma_c(\mathbb{Z}[1/Sp], M_p)$, for p-adic realization M_p. Also for any $G_\mathbb{Q}$-equivariant $\mathcal{O}_p$-lattice T_p of M_p, we put the *Euler Poincaré line*, which is the $\mathcal{O}_p$-lattice of $\Delta_{EP}(T_p)$, by $\Delta_{EP}(T_p) := \det_{\mathcal{O}_p}\mathbb{R}\Gamma_c(\mathbb{Z}[1/Sp], T_p)$.

Claim. This is independent of the choice of T_p, i.e. well-defined.

Proof. Let T_p, T_p' be two choices. By cosidering intersection of these two lattices, the claim is reduced to showing : For a finite p-primary $G_\mathbb{Q}$-module T, we have the equality $\prod_q |H_c^q(\mathbb{Z}[1/Sp], T)|^{(-1)^q} = 1$. To prove this claim, it suffices to compute

$$\frac{\prod_q |H^q(\mathbb{Z}[1/Sp], T)|^{(-1)^q}}{\prod_{q, v \in Sp\infty} |H^q(\mathbb{Q}_v, T)|^{(-1)^q}}.$$

But from the local and global Tate's Euler characteristic formula ([Mi][Theorem 2.8, Theorem 5.1]), we can compute the numerator $|T^+|/|T|$, and the inverse of the denominator, $|T|\cdot|\hat{H}^1(\mathbb{R}, T)|/|T^+|\cdot|\hat{H}^2(\mathbb{R}, T)|$. Here, $\hat{H}^*(\mathbb{R}, T)$ is Tate's modified cohomology. Because T is finite, $|\hat{H}^1(\mathbb{R}, T)| = |\hat{H}^2(\mathbb{R}, T)|$. So everything is canceled and we have the claim. □

Proposition 2.11. There is an isomorphism $\theta_p : \Delta_f(M) \otimes_\mathbb{Q} \mathbb{Q}_p \simeq \Delta_{EP}(M_p)$.

Proof. Use Proposition 2.7, and the distinguished triangle (♠). □

Finally, we can state our conjecture. The conjecture is stated by the behavior of the zeta element.

Definition 2.12 (Zeta elements of Motives). For an E-motif M, define $\delta(M) \in \Delta_f(M)$ which goes $L^*(M)^{-1}$ via the map θ_∞. We call it the *zeta element of* M.

Conjecture 2.13 (Bloch-Kato, Tamagawa Number Conjecture (=TNC)). *Let M be an E-motif over $\mathbb{Q}$, fix a $G_\mathbb{Q}$-equivariant $\mathcal{O}_p$-lattice T_p of M_p. Then, we have the followings.*

(1) (*Beilinson-Deligne conjecture*) $\delta(M)$ *is in* $\Delta_f(M)$, *which is apriori only in* $\Delta_f(M) \otimes_{\mathbb{Q}} \mathbb{R}$.

(2) (*Bloch-Kato conjecture*) $\theta_p(\delta(M) \otimes 1_{\mathbb{Q}_p}) = \Delta_{EP}(T_p)$.

Theorem 2.14 (Main Theorem, Burns-Flach, Flach, Itakura). *Let M be a Dirichlet motif with Tate twists over* $\mathbb{Q}$. *Then,* TNC_M *holds also for* $p = 2$.

Remark 2.15. If $p \neq 2$, this is deduced from the results of Burns-Greither, and Huber-Kings for $M(\chi)(r)$, which is the refined version of TNC (called ETNC). For $p = 2$, this is also proved recently by Flach [Fl] and Burns-Flach [BF], independently by the author [I] with slightly different method. The author needs to remark that their result is even stronger than Theorem 2.14. The difficulty for the prime 2 is due to the fact : Prime number 2 is the king of prime numbers, as is said by Prof. H. Hida with his humour.

We have a striking consequence for the special values of the Dedekind zeta functions for an abelian extention of $\mathbb{Q}$. This is my original motivation for the problem.

Corollary 2.16 (Cohomological Lichtenbaum Conjecture). Let the case $E = \mathbb{Q}$, and F is an abelian extension of $\mathbb{Q}$. Put $M = h^0(F)(1-k), k \geq 2$. Then, TNC_M implies the following formula :

$$\zeta_F^*(1-k) = \pm \begin{cases} \prod_p \dfrac{|H^2_{\acute{e}t}(\mathcal{O}_F[1/p], \mathbb{Z}_p(k))|}{|H^1_{\acute{e}t}(\mathcal{O}_F[1/p], \mathbb{Z}_p(k))|}, & \text{for } k \text{ even}, F \text{ any field}, \\ \prod_p \dfrac{|H^2_{\acute{e}t}(\mathcal{O}_F[1/p], \mathbb{Z}_p(k))|}{|H^1_{\acute{e}t}(\mathcal{O}_F[1/p], \mathbb{Z}_p(k))_{tors}|} \times R_k(F), & \text{for } k \text{ odd}, F \text{ totally imaginary field.} \end{cases}$$

Remark 2.17. For the case k is even and $p = 2$, this is the result of Wiles via Main Conjecture, and totally real F is the one of Kolster, via Bloch-Kato-Milnor conjecture. Other cases are new. In the survey of Flach, this is announced for all abelian F. But it seems to be false, because it relies on the argument of Huber-Kings, which fails for $p = 2$.

3 Key Ingredients

Proof goes on along the "bootstrapping process using functional equation" by Huber-Kings. We will introduce the key ingredient to go on the process, which is named "compatibility of the conjecture with functional equation". Assume in this section, $M = M(\chi)(r)$ with $r \geq 2$ for simplicity. But concerning the matter of this section, we do not need any conjecture for the finite dimensionality of the cohomologies.

3.1 Definitions.

Definition 3.1 (ε-line). Define the 1-dimensional E-vector space $\Delta_{loc}(M) = \det_E M_{dR} \otimes_E \det_E^{-1} M_B$. We will call this space *ε-line of M*. Easily to guess, $\Delta_{loc}(M)$ and $\Delta_f(M), \Delta_f(M^*(1))$ are related by the following Poincare duality θ^{PD}, which is defined by

$$\begin{aligned} \theta^{PD}\colon \Delta_f(M) \otimes \Delta_f^*(M^*(1)) &\simeq \det_E M_{dR}/\operatorname{Fil}^0 M_{dR} \otimes \det_E^{-1} M^*_{dR}/\operatorname{Fil}^{-1} M^*_{dR} \otimes \det_E^{-1} M_B^+ \otimes \det M_B^*(1)^+ \\ &\simeq \det_E^{-1} M_B \otimes \det_E M_{dR} \simeq \Delta_{loc}(M). \end{aligned}$$

For all rational places v, let us introduce an identification θ_v^{loc}, which are analogues of those for θ_v. We define the identification $\theta_\infty^{loc}\colon \Delta_{loc}(M) \otimes_{\mathbb{Q}} \mathbb{R} \simeq E_\infty$, induced by the $+$-part of Hodge's comparison map $M_B \otimes_{\mathbb{Q}} \mathbb{C} \simeq M_{dR} \otimes_{\mathbb{Q}} \mathbb{C}$. For the p-adic realizations, we define

$$\theta_p^{loc}\colon \Delta_{loc}(M) \otimes_{\mathbb{Q}} \mathbb{Q}_p \simeq \det_{E_p} M_{dR} \otimes_{\mathbb{Q}} \mathbb{Q}_p \otimes \det_{E_p}^{-1} M_p \xrightarrow{\eta_p \otimes \mathrm{id}} \det_{E_p}^{-1} \mathbb{R}\Gamma(\mathbb{Q}_p, M_p) \otimes \det_{E_p}^{-1} M_p,$$

to be the composite map of base change of determinants and the map η_p explained below. We call the last E_p-module $\Delta_{EP}(M_p) = \det_{E_p}^{-1} \mathbb{R}\Gamma(\mathbb{Q}_p, M_p) \otimes \det_{E_p}^{-1} M_p$, the *functional equation line of M at p*. (This is only my terminology, so maybe readers should not use this term as if everyone knew it.)

Definition 3.2 (ε-element). Let us put the element in $\Delta_{loc}(M)\otimes_{\mathbb{Q}}\mathbb{R}$, which satisfies $\theta_{\infty}^{loc}(\varepsilon)=\dfrac{L^*(M^*,1)}{L^*(M,0)}$. We call it *$\varepsilon$-element of M*. In other words, $\varepsilon=\theta^{PD}(\delta(M)\otimes\delta^*(M^*(1))$.

Now, we introduce the compatibility with functional equation of TNC, which is the key ingredient to prove the whole case of TNC via bootstrapping process.

Theorem 3.3 (Huber-Kings [HK], Burns-Flach [BF], Itakura [I]). *Let $M=M(\chi)(r)$ be a Dirhchlet motif, and fix a $G_{\mathbb{Q}}$-equivariant $\mathcal{O}_p$-lattice T_p of M_p. Then, we have the followings.*

(1) *ε is in $\Delta_{loc}(M)$, which is apriori only in $\Delta_{loc}(M)\otimes_{\mathbb{Q}}\mathbb{R}$.*

(2) $\theta_p^{loc}(\varepsilon\otimes 1_{\mathbb{Q}_p})=2^{\chi(-1)}\Delta_{FE}(T_p)$.

(3) *The right hand side of* (2) $=\Delta_{EP}(T_p)\otimes\Delta_{EP}(T_p^{\vee}(1))$.

Corollary 3.4. Suppose Theorem 3.3 holds for Dirhchlet motif $M=M(\chi)(r)$. Then it is equivalent to the both of TNC_M and $\mathrm{TNC}_{M^*(1)}$.

Proof. Consider the following diagram,

$$\begin{array}{ccc} \delta(M)\otimes\delta(M^*(1))\in\Delta_f(M)\otimes_E\Delta_f^*(M^*(1))\otimes\mathbb{Q}_p & \xrightarrow{\theta^{PD}\otimes 1_{\mathbb{Q}_p}} & \Delta_{loc}(M)\otimes\mathbb{Q}_p\ni\varepsilon \\ \Big\downarrow{\scriptstyle \theta_p(M)\otimes\,\theta_p{}^*(M^*(1))} & & \Big\downarrow{\scriptstyle \theta_p{}^{loc}} \\ \Delta_{EP}(M_p)\otimes_{\mathcal{O}_p}\Delta_{EP}^*(M_p^*(1))\otimes_{\mathbb{Z}_p}\mathbb{Q}_p & \xrightarrow{\theta_p{}^{AV}} & \Delta_{FE}(T_p(r))\otimes_{\mathbb{Z}_p}\mathbb{Q}_p. \end{array}$$

First, we have by Theorem 3.3 (2), $\varepsilon\in\Delta_{loc}(M(\chi)(r))$ goes to $2^{\chi(-1)}\Delta_{EP}(T_p(\chi)(r))$. On the other hand in the line below, we have the lattices $\Delta_{EP}(T_p(\chi)(r))\otimes\Delta_{EP}(T_p(\chi^{-1})(r-1))$ and $2^{\chi(-1)}\Delta_{FE}(T_p(\chi)(r))$. And Theorem 3.3 (3) shows these are equal. So, we have $\delta(M)$ goes to a generator of $\Delta_{EP}(T_p)$ whenever $\delta(M^*(1))$ goes to a generator of $\Delta_{EP}(T_p^*(1))$. ☐

3.2 On the map η_p.

We need to remark that, not only for the case of Dirichlet motives, we have the identification (*), for all pure motives of proper smooth varieties, via the great results of G. Faltings and T. Tsuji. In the p-adic world, there is an exact sequence of p-adic period rings

$$0\to\mathbb{Q}_p\longrightarrow\mathbb{B}_{cris}\overset{(1-\phi,\mathrm{pr})}{\longrightarrow}\mathbb{B}_{cris}\oplus\mathbb{B}_{dR}/\operatorname{Fil}^0\mathbb{B}_{dR}\to 0.$$

Here, ϕ is the arithmetic Frobenius on $\mathbb{B}_{cris}$, and pr is the composition of the natural maps $\mathbb{B}_{cris}\hookrightarrow\mathbb{B}_{dR}\to\mathbb{B}_{dR}/\operatorname{Fil}^0\mathbb{B}_{dR}$. For the definition of these p-adic period rings and exactness of this sequence, see [Co][III Proposition 3.1]. The author wants to remark that this sequence is the p-adic analogue of the exponential sequence in the classical topology. It is reasonable to call the boundary map of this sequence,

$$\exp_p: D_{dR}(M_p)/\mathrm{Fil}^0 D_{dR}(M_p)\longrightarrow H_f^1(\mathbb{Q}_p,M_p).$$

If M_p is a de Rham representation, this is an isomorphism. So, if we consider the derived functor of $(-\otimes_{\mathbb{Q}_p}M_p)^{G_{\mathbb{Q}_p}}$, we have an identification of $\det_{E_p}^{-1}\mathbb{R}\Gamma_f(\mathbb{Q}_p,M_p)$ to the determinant

$$\det_{E_p}[0\to H^0(\mathbb{Q}_p,M_p)\to D_{cris}(M_p)\to D_{cris}(M_p)\oplus D_{dR}(M_p)/D_{dR}^0(M_p)\xrightarrow{\exp_p}H_f^1(\mathbb{Q}_p,M_p)\to 0].$$

For the case $r\geq 2$, we have $H_f^1(\mathbb{Q}_p,M_p)=H^1(\mathbb{Q}_p,M_p)$ (every extension of $\mathbb{Q}_p$ by $\mathbb{Q}_p(r)$ is cristalline for $r\geq 2$) and $\mathrm{Fil}^0M_{dR,p}=0$. So $\mathbb{R}\Gamma_f(\mathbb{Q}_p,M_p)\simeq\mathbb{R}\Gamma(\mathbb{Q}_p,M_p)$. Therefore, we have the identification

$$\eta_p\colon \det_{E_p}^{-1}\mathbb{R}\Gamma(\mathbb{Q}_p,M_p)\simeq\det_{E_p}^{-1}\mathbb{R}\Gamma_f(\mathbb{Q}_p,M_p)\simeq\det_{E_p}^{-1}M_{dR,p}.$$

4 Outline of the Proof.

Because we need a lot of pages, we will only see in this section, how Theorem 3.3 (3) is proved, and give some comments on the whole proof of TNC for Dirichlet motives.

Definition 4.1 (Basis' of realizations). Suppose we are given a Dirichlet character χ with conductor N. Let us fix an embbeding τ_0: $\mathbb{Q}(\zeta_N) \hookrightarrow \mathbb{C}$, which maps to $\zeta_N \mapsto \exp(2\pi/N)$. Let us denote a basis δ_{τ_0} of $T_B(h^0(\mathbb{Q}(\zeta_N)) = \mathcal{O}^{\mathrm{Hom}(\mathbb{Q}(\zeta),\mathbb{C})}$, the "delta function at τ_0". We define $t_B(\chi) = p_{\chi^{-1}}\delta_{\tau_0}$, which is a basis of $T_B(\chi)$. We define a basis $t_{dR}(\chi) = p_{\chi^{-1}}\zeta_N$ of $T_{dR}(\chi) = p_{\chi^{-1}}[\mathcal{O} \otimes_{\mathbb{Z}} \mathbb{Z}[\zeta_N]]$, by taking $\zeta_N \otimes 1_{\mathbb{Z}[\zeta_N]}$ as a basis of $T_{dR}(h^0(\mathbb{Q}(\zeta_N))) = \mathcal{O} \otimes \mathbb{Z}[\zeta_N]$.

Proposition 4.2 (Explicit description of ε-elemant). Let $N, r \geq 1$ be a natural number, χ is a Dirhchlet character with conductor N. We put $\delta = \delta_\chi = 0$ if χ satisfies $\chi(-1) = (-1)^r$ and χ is non-trivial, and put 1 if $\chi(-1) = (-1)^{r-1}$. We put $\delta = 0$ for the case χ is trivial. We denote $\tau(\chi) = \sum_{\sigma \in G} \chi(\sigma)\cdot\zeta_N^{\sigma}$, the Gauss sum of χ. Then we have the following.

(1) From the functional equation of L-function of χ, we have $\dfrac{L^*(\chi^{-1}, 1-r)}{L^*(\chi, r)} = 2^{\chi(-1)}\cdot\dfrac{N^r(r-1)!}{\tau(\chi)\cdot(2\pi i)^{r-\delta}}$.

(2) $\varepsilon \in \Delta_{loc}(M(\chi)(r))$ is given by $\varepsilon = 2^{\chi(-1)}\cdot N^{r-1}(r-1)! t_{dR}(\chi) \otimes t_B(r-\delta)^{-1}$ in $\Delta_{loc}(M(\chi)(r))$.

Proof. (1) is easy computation. Note that $2^{\chi(-1)}$ is from the differenciation by s of $\sin(\pi(s-\delta)/2)$. This formula is also valid for the case χ is trivial. (2) is from explicit computation of θ_∞^{loc} via t_{dR}, t_B. □

From this proposition, Conjecture 3.3 (3) is reduced to showing ([HK][Proposition 1.2.5]),

$$\exp_2(t_{dR}(\chi) \otimes 1_{\mathbb{Z}_2}) = \frac{(1-\chi(2)2^{-r})(1-\chi^{-1}(2)2^{r-1})^{-1}}{(r-1)!N^{r-1}} \det{}^{-1}_{\mathcal{O}_2}\mathbb{R}\Gamma(\mathbb{Q}_2, T_2(\chi)(r)). \qquad (\Diamond)$$

Proposition 4.3. (cf. [HK][Corollary B. 2.7] for $p \neq 2$.) Let the Galois group of $\mathbb{Z}_2^\times$-extension $\Gamma = \mathrm{Gal}(\mathbb{Q}(\mu_{2^\infty})/\mathbb{Q})$, and put $\Gamma_n = \mathrm{Gal}(\mathbb{Q}(\mu_{2^\infty})/\mathbb{Q}(\mu_{2^n})), G_n = \Gamma/\Gamma_n$. The Iwasawa algebra $\tilde{\Lambda} = \varprojlim_n \mathcal{O}_2[G_n]$ is not regular. Put intermidiate fields $k_n = \mathbb{Q}_2(\mu_{N'2^n}), K_n = \mathbb{Q}(\mu_{N'2^n}) \otimes_{\mathbb{Q}} \mathbb{Q}_2 \simeq \prod k_n$, Galois groups $\Delta = \mathrm{Gal}(\mathbb{Q}(\mu_{N'})/\mathbb{Q}), H = \mathrm{Gal}(k_0/\mathbb{Q}_2)$. We identify $\mathrm{Gal}(K_\infty/\mathbb{Q}) \simeq \Delta \times \Gamma, \mathrm{Gal}(k_\infty/\mathbb{Q}_2) \simeq H \times \Gamma$. Then, we have the flollowing isomorphism

$$\det{}^{-1}_{\mathcal{O}_2}\mathbb{R}\Gamma(\mathbb{Q}_2(\mu_{2^n}), T_2(\chi)(r)) \simeq \det{}^{-1}_{\mathcal{O}_2}\mathrm{Hom}_{\Gamma_n\times\Delta}(\mathcal{O}_{K_0}[[\Gamma]], T_2(\chi)(r)).$$

This isomorphism is rationally induced by an isomorphism

$$s_{\chi^{-1}} : H^1(\mathbb{Q}_2(\mu_{2^n}), V_2(\chi)(r)) \to \mathrm{Hom}_{\Gamma_n\times\Delta}(K_0[[\Gamma]], V_2(\chi)(r)).$$

By Proposition 4.3, seeing the image of $t_{dR}(\chi)$ by $\exp_2$ is reduced to doing the image of $t_{dR}(\chi)$ by $s_\chi \circ \exp_2$ in $\mathrm{Hom}_{\Gamma_n\times\Delta}(\mathcal{O}_{K_0}[[\Gamma]], V_2(\chi)(r))$. Let us choose a $\mathbb{Z}_2[\Delta]$-generator $\tilde{\zeta}_{N'}$ of $\mathcal{O}_{K_0}$, and we fix an isomorphism evaluation at $\tilde{\zeta}_{N'}$, noted $\mathrm{ev}(\tilde{\zeta}_{N'})$:

$$\mathrm{Hom}_{\Gamma_n\times\Delta}(\mathcal{O}_{K_0}[[\Gamma]], V_2(\chi)(r)) \simeq V_2(\chi)(r).$$

Lemma 4.4. (cf. [HK, Lemma B 3.1]) There is an equality in $V_2(\chi)(r)$:

$$ev[\tilde{\zeta}_{N'}]((s_\chi \circ \exp_2)(t_{dR}(\chi)) = \frac{(N')^r(1-\chi(2)2^{-r})}{N^{r-1}(r-1)!(1-\chi^{-1}(2)2^{r-1})} t_2(\chi)(r).$$

Proof. From the choice of $\tilde{\zeta}_{N'}$, we have $\mathbb{Z}_2[\Delta]\tilde{\zeta}_{N'} \simeq \mathcal{O}_{K_0}$. Furthermore, this choice induces

$$\begin{aligned}\mathrm{Hom}_{\Gamma_m\times\Delta}(\mathcal{O}_{K_0}[[\Gamma]], E_2(r)) &\simeq \mathrm{Hom}(\mathcal{O}_{K_0}[G_m], E_2(r))\\ &\simeq \mathrm{Hom}(\mathbb{Z}_2[G_m\times\Delta], E_2(r)).\end{aligned}$$

Then, the following diagram commutes :

$$\begin{tikzcd} V_{dR}(\chi)\otimes_{\mathbf{Q}}\mathbf{Q}_2 \arrow[r, "s_\chi\circ\exp_2"] \arrow[d] & \mathrm{Hom}_{\Gamma\times\Delta}(\mathcal{O}_{K_0}[[\Gamma]], V_2(\chi)(r)) \arrow[r, "\mathrm{ev}(\tilde{\zeta}_{N'})"] \arrow[d, "\iota"] & V_2(\chi)(r) \arrow[d, "\iota'"] \\ K_m \arrow[r, "s_{K_m}\circ\exp_2"] & \mathrm{Hom}_{\Gamma_m}(\mathcal{O}_{K_0}[[\Gamma]], \bigoplus_\chi V_2(\chi)(r)) \arrow[r, "\simeq"] & E_2(r). \end{tikzcd}$$

Here, the vertical maps are inclusions into the χ-part summand. So, if we put $e \in \Delta\times G_m$ the generator corresponding to $\mathrm{ev}(\tilde{\zeta}_{N'})$, we have $\iota'(t_2(\chi)(r))(g) = p_{\chi^{-1}}\delta(g)$. Here, δ is a standard generator, satisfying $\delta(g) = \begin{cases} t_2(r) & g = e \\ 0 & g \neq e. \end{cases}$ Then, the commutativity in the right square leads $\iota(t_2(\chi)(r))(p_\chi\tilde{\zeta}_{N'}) = \iota'(t_2(\chi)(r))(e) = \frac{1}{\varphi(N)}t_2(r)$. Hence if we see in the whole square, we have for $\alpha\in V_{dR}(\chi)$,

$$(s_\chi\circ\exp_2)(\alpha)(\tilde{\zeta}_{N'}) = (s_{K_m}\circ\exp_2)(\alpha)(p_\chi\tilde{\zeta}_{N'})\frac{t_2(\chi)(r)}{\frac{1}{\varphi(N)}t_2(r)}.$$

It suffices to compute $s_{K_m}\circ\exp_2$, and it is done in [HK, Lemma B.3.1], using the Kato's explicit reciprocity law unless $m=0$ (unramified case). They do not prove it in this case, because [BK, Claim 4.8] needs the Fontaine-Messing theory and it fails in the case $p=2$. By means of a slightly different argument from that in [HK, p460], it suffices to check that the target of the map $s_{K_0}\circ\exp_2 : K_{0,2}\to\mathrm{Hom}(K_0, E_2(r))$ is the same as the following map :

$$x\mapsto [y\mapsto \frac{1}{(r-1)!}\mathrm{Tr}_{K_0/\mathbf{Q}_2}(x\cdot(1-2^{-r}\,\mathrm{Fr}_2)(1-2^{r-1}\,\mathrm{Fr}_2^{-1})^{-1}(y))\otimes t_2(r)].$$

Here, K_0 is the product of k_0. The deduction of the lemma from this claim, is as follows. Let us put $x = p_{\chi^{-1}}\zeta_N, y = \zeta_{N'}$ in this formula. Then, we have

$$\begin{aligned}&\frac{1}{(r-1)!}Tr_{K_0/\mathbf{Q}_2}(p_{\chi^{-1}}(\zeta_N)(1-\chi(2)2^{-r})(1-\chi^{-1}(2)2^{r-1})^{-1}p_{\chi^{-1}}(\zeta_N))t_2(r)\\ &=\frac{(1-\chi(2)2^{-r})N}{(r-1)!(1-\chi^{-1}(2)2^{r-1})\varphi(N)}t_2(r).\end{aligned}$$

The $\frac{1}{\varphi(N)}t_2(r)$ cancels out in the above formula, and we can prove the lemma. So, it is reduced to proving the claim. But we need to omit it for the shortage of pages. □

Proposition 4.5. The equality ($\Diamond$) holds. Hence, Theorem 3.3 (3) holds.

Proof. What we have to see is that

$$\varepsilon' := \frac{(r-1)!N^{r-1}(1-\chi^{-1}(2)2^{r-1})}{(1-\chi(2)2^{-r})}\exp_2 t_2(\chi)\cdot\mathcal{O}_2 = \det{}^{-1}_{\mathcal{O}_2}\mathbb{R}\Gamma(\mathbf{Q}_2, T_2(\chi)(r)).$$

So, it suffices to show $ev(\tilde{\zeta}_{N'})(s_\chi\varepsilon') = (N')^r t_2(\chi)(r)$ is a generator of $T_2(\chi)(r)$. If we compare $t_2(\chi)(r)$ with the standard generator δ in the last lemma, $s_\chi\varepsilon'$ differs by $(N')^r$ times a generator. Because $(2, N') = 1$, we have the claim. □

The proof of TNC for Dirichlet motives goes on using Theorem 3.3 and Iwasawa Main Conjecture. But there is not enough pages to give a whole proof, so we introduce its summary as follows.

$\mathrm{TNC}_{M(\chi)(r)}$	$r<0$	$r=0,\chi(2)\neq 1$	$=1$	$r=1,\chi(2)\neq 1$	$=1$	$r>1$
$\chi(-1)=(-1)^r$	③	③	①'	②	②	②
$\chi(-1)=(-1)^{r-1}$	②'	④"	②'	④"	①	③'
$\mathrm{TNC}_{M(F)(r)}$	—	①'	①'	①	①	—

Here, ⓘ' is deduced from ⓘ via Corollary 3.4, for all $i=1,2,3,4$.

Remark 4.6. ② is deduced from non-critical case of Main Conjecture via Euler system argument. And ④" is deduced from critical case of Main Conjecture.

5 Appendix.

In this section, we will see the BSD conjecture for an abelian variety A over $\mathbb{Q}$ and the TNC for $M = h^1(A^\vee)(1)$ is equivalent following [V]. ($A^\vee$ is the dual abelian variety of A.) For simplicity, assume p is an odd prime. And we always assume that $\text{Ш}(A/\mathbb{Q})$ is finite. T_p is the Tate module $H_1(A^\vee{}_{\bar{\mathbb{Q}}}, \mathbb{Z}_p)$, which is a $G_\mathbb{Q}$-stable lattice of M_p.

Lemma 5.1. For $M = h^1(A^\vee)(1)$ and any ℓ, we have the following cohomology groups.

(0) (Motivic) $H^0_f(M) = H^1_f(M^*(1)) = 0, H^1_f(M) = A^\vee(\mathbb{Q}), H^0_f(M^*(1)) = \mathrm{Hom}_\mathbb{Z}(A(\mathbb{Q}),\mathbb{Q})$.

(1) (Local) $H^0_f(\mathbb{Q}_\ell, T_p) = 0, H^1_f(\mathbb{Q}_\ell, T_p) \simeq A^\vee(\mathbb{Q}_\ell)^{\wedge p}, H^i_f(\mathbb{Q}_\ell, T_p) = 0, i \neq 0, 1$.

(2) (Global) For $i \neq \{0,1,2,3\}$, $H^i_f(\mathbb{Z}[1/Sp], T_p) = 0$. For remaining i's, $H^0_f(\mathbb{Z}[1/Sp], T_p) = 0$, $H^1_f(\mathbb{Z}[1/Sp], T_p) \simeq A^\vee(\mathbb{Q})_{\mathbb{Z}_p}$, $H^3_f(\mathbb{Z}[1/Sp], T_p) \simeq \mathrm{Hom}_\mathbb{Z}(A(\mathbb{Q})_{tors}, \mathbb{Q}_p/\mathbb{Z}_p)$. $H^2_f(\mathbb{Z}[1/Sp], T_p)$ is described by the following exact sequcnce,

$$0 \to \text{Ш}(A/\mathbb{Q})[p] \to H^2_f(\mathbb{Z}[1/Sp], T_p) \to \mathrm{Hom}_\mathbb{Z}(A(\mathbb{Q}), \mathbb{Z}_p) \to 0.$$

Proof. (0) is by definition. (1) is the result of Fontaine. (2) is implied from (1). □

By Lemma 5.1 (0) and by definition, we have the fundamendal line of M as follows;

$$\Delta_f(M) = \det{}^{-1} A^\vee(\mathbb{Q})_\mathbb{Q} \otimes \det \mathrm{Hom}_\mathbb{Z}(A(\mathbb{Q}),\mathbb{Q}) \otimes \det{}^{-1} H_1(A^\vee(\mathbb{C}),\mathbb{Q})^+ \otimes \det \mathrm{Lie}\, A^\vee.$$

For further argument, we need to fix a $\mathbb{Z}$-basis of $A^\vee(\mathbb{Q})$, $\{P_1^\vee, \dots, P_d^\vee\}$. If we take a standard choice of the dual basis, we have the $\mathbb{Z}$-basis of $\mathrm{Hom}_\mathbb{Z}(A(\mathbb{Q}),\mathbb{Z}), \{P_1, \dots, P_d\}$. Similarly, choose a $\mathbb{Z}$-basis of $T_B^+ := H_1(A^\vee(\mathbb{C}),\mathbb{Z})^+$ and $\mathrm{Lie}_\mathbb{Z} A^\vee := \mathrm{Hom}_\mathbb{Z}(\Omega^1_{\mathfrak{B}/\mathbb{Z}}, \mathbb{Z})$ by $\{\gamma_1, \dots, \gamma_{d^+}\}$, and $\{\omega_1, \dots, \omega_{d^+}\}$ respectively. Here, $\mathfrak{B}/\mathbb{Z}$ is the Néron model of $A/\mathbb{Q}$. Then, we define a lattice of $\Delta_f(M)$, generated by

$$\delta_0(M) := \det{}_\mathbb{Z}^{-1} T_{A^\vee} \otimes_\mathbb{Z} \det{}_\mathbb{Z} T_A^d \otimes_\mathbb{Z} \det{}_\mathbb{Z}^{-1} T_\mathfrak{B}^+ \otimes_\mathbb{Z} \det{}_\mathbb{Z} \mathrm{Lie}_\mathbb{Z} A^\vee.$$

By definition, Ω_A^+, R_A is the determinant of the maps α_M, h respectively. So, if we assume the full Birch-Swinnerton-Dyer conjecture, we have

$$\begin{aligned} L^*(M,0)^{-1} &= 2^{-r} \frac{|\text{Ш}(A/\mathbb{Q})|}{\Omega_A^+ \cdot R_A \cdot |A^\vee(\mathbb{Q})_{tors}| \cdot |A(\mathbb{Q})_{tors}|} \prod_\ell c_\ell(M)^{-1} \\ &= 2^{-r} \frac{|\text{Ш}(A/\mathbb{Q})|}{|A^\vee(\mathbb{Q})_{tors}| \cdot |A(\mathbb{Q})_{tors}|} \prod_\ell c_\ell(M)^{-1} \theta_\infty(\delta_0(M) \otimes 1_\mathbb{R}). \end{aligned}$$

So, let us put $\delta := 2^{-r}|\text{Ш}(A/\mathbb{Q})||A^\vee(\mathbb{Q})_{tors}|^{-1}\cdot|A(\mathbb{Q})_{tors}|^{-1}\prod_\ell c_\ell(M)^{-1}\delta_0$. This is the zeta element up to sign and modulo the BSD conjecture. For the second claim part of TNC, let us see the image of $\delta(M)$ by θ_p. Recall that the map θ_p is the composition of the following maps

$$\begin{aligned}\Delta_f(M)\otimes_{\mathbb{Q}}\mathbb{Q}_p &\simeq \det_{\mathbb{Z}}^{-1}\mathbb{R}\Gamma_f(\mathbb{Q},M_p)\otimes_{\mathbb{Q}_p}\det_{\mathbb{Z}}M_p^+\otimes_{\mathbb{Q}_p}\det_{\mathbb{Z}}^{-1}\mathrm{Lie}_{\mathbb{Q}_p}A^\vee\\ &\simeq \Delta_{EP}(M_p).\end{aligned}$$

If we use Lemma 5.1, we have

$$\begin{aligned}\det_{\mathbb{Z}_p}^{-1}T_{A^\vee}\otimes_{\mathbb{Z}}\mathbb{Z}_p &\simeq \det_{\mathbb{Z}_p}^{-1}A^\vee(\mathbb{Q})_{\mathbb{Z}_p}\otimes_{\mathbb{Z}_p}\det_{\mathbb{Z}_p}A^\vee(\mathbb{Q})_{p-tors}\\ &\simeq |A^\vee(\mathbb{Q})|_p^{-1}\cdot\det_{\mathbb{Z}_p}^{-1}H_f^1(\mathbb{Z}[1/Sp],T_p),\end{aligned}$$

$$\begin{aligned}\det_{\mathbb{Z}_p}T_A^d\otimes_{\mathbb{Z}}\mathbb{Z}_p &\simeq \det_{\mathbb{Z}_p}\mathrm{Hom}(A(\mathbb{Q}),\mathbb{Z}_p)\\ &\simeq \det_{\mathbb{Z}_p}H_f^2(\mathbb{Z}[1/Sp],T_p)\otimes_{\mathbb{Z}_p}\det_{\mathbb{Z}_p}^{-1}\text{Ш}(A/\mathbb{Q})\\ &\simeq |\text{Ш}(A/\mathbb{Q})|_p\cdot H_f^2(\mathbb{Z}[1/Sp],T_p).\end{aligned}$$

By Lemma 5.1 again, $H_f^3(\mathbb{Z}[1/Sp],T_p)\simeq \mathrm{Hom}_{\mathbb{Z}}(A(\mathbb{Q})_{tors},\mathbb{Q}_p/\mathbb{Z}_p)$. So, it follows

$$\det_{\mathbb{Z}_p}^{-1}T_{A^\vee}\otimes_{\mathbb{Z}}\mathbb{Z}_p\otimes_{\mathbb{Z}_p}\det_{\mathbb{Z}_p}T_A^d\otimes_{\mathbb{Z}}\mathbb{Z}_p\simeq\frac{|\text{Ш}(A/\mathbb{Q})|_p}{|A^\vee(\mathbb{Q})_{tors}|_p\cdot|A(\mathbb{Q})_{tors}|_p}\cdot\det_{\mathbb{Z}_p}\mathbb{R}\Gamma_f(\mathbb{Z}[1/Sp],T_p).$$

Next, we see the last two terms $\det_{\mathbb{Z}_p}^{-1}T_p^+,\det_{\mathbb{Z}_p}\mathrm{Lie}_{\mathbb{Z}_p}A^\vee$. Also if we use Lemma 5.1 (3), then we have $\det_{\mathbb{Z}_p}\mathbb{R}\Gamma_f(\mathbb{Q}_p,T_p)\simeq\det_{\mathbb{Z}_p}^{-1}H_f^1(\mathbb{Q}_p,T_p)$. For $p\neq 2$, we have $\det_{\mathbb{Z}_p}^{-1}\mathbb{R}\Gamma(\mathbb{R},T_p)\simeq T_p^+$. Now, we obtain

$$\theta_p(\delta\otimes 1_{\mathbb{Q}_p})\simeq 2^{-r}\prod_v|c_v(M)|_p^{-1}\cdot\det_{\mathbb{Z}_p}\mathbb{R}\Gamma_f(\mathbb{Z}[1/Sp],T_p)\otimes_{\mathbb{Z}_p}\bigotimes_{v\in p\infty}\det_{\mathbb{Z}_p}^{-1}\mathbb{R}\Gamma_f(\mathbb{Q}_v,T_p).$$

Assume now $\ell\neq p$. Denoting $H_f^1(\mathbb{Q}_\ell,T_p)=H^1(\mathbb{Q}_\ell^{ur},T_p)$, we have

$$\det_{\mathbb{Z}_p}\mathbb{R}\Gamma_f(\mathbb{Q}_\ell,T_p)\simeq\det_{\mathbb{Z}_p}[0\to T_p^{I_\ell}\xrightarrow{1-\phi_\ell}T_p^{I_\ell}\longrightarrow H_f^1(\mathbb{Q}_\ell,T_p)\longrightarrow H^1(I_\ell,T_p)_{tors}^{G_{\mathbb{Q}_\ell}}\to 0]\simeq\mathbb{Z}_p.$$

Put $c_\ell(M_p)=|H^1(I_\ell,T_p)_{tors}^{G_{\mathbb{Q}_\ell}}|_p$, which is trivial for good ℓ. For the case $\ell=p$, $c_p(M_p):=\eta_p\cdot(\psi_p)_{\mathbb{Q}_\ell}$, Here, we used the identification η_p: $\det_{\mathbb{Z}_p}^{-1}\mathbb{R}\Gamma_f(\mathbb{Q}_p,T_p)\otimes\det_{\mathbb{Z}_p}^{-1}\mathrm{Lie}_{\mathbb{Z}_p}A^\vee\simeq\mathbb{Z}_p$. Therefore, we have

$$\begin{aligned}\theta_p(\delta\otimes 1_{\mathbb{Q}_p}) &\simeq 2^{-r}\prod_v\frac{|c_v(M_p)|}{|c_v(M)|_p}\cdot\det_{\mathbb{Z}_p}\mathbb{R}\Gamma_f(\mathbb{Z}[1/Sp],T_p)\otimes_{\mathbb{Z}_p}\bigotimes_{v\in Sp\infty}\det_{\mathbb{Z}_p}^{-1}\mathbb{R}\Gamma_f(\mathbb{Q}_v,T_p)\\ &\simeq 2^{-r}\det_{\mathbb{Z}_p}\mathbb{R}\Gamma_c(\mathbb{Z}[1/Sp],T_p)=2^{-r}\Delta_{EP}(T_p).\end{aligned}$$

In [V][p14, 15], Venjakob proved $|c_v(M_p)|=|c_v(M)|_p$, i.e. $c_v(M_p)$ equals to the p-primary part of the usual definition by the Néron model $\mathfrak{B}$. Finally, we have the desired equality, which is the claim of TNC (2) for M : $\theta_p(\delta\otimes 1_{\mathbb{Z}_p})=\Delta_{EP}(T_p)\bmod\mathbb{Z}_p^\times$.

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