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Subrepresentation Theorem for p-adic Symmetric Spaces

Shin-ichi Kato (Dept. of Math., Kyoto University)
加藤 信一 (京都大・理学研究科)
Keiji Takano (Akashi College of Technology)
高野 啓児 (明石工業高専)

Abstract

The notion of relative cuspidality for distinguished representations attached to p-adic symmetric spaces is introduced. A generalization of Jacquet's subrepresentation theorem to the relative case (symmetric space case) is given, under a reasonable assumption on the relative Cartan decomposition.

1 Introduction

Let $\underline{G}$ be a connected reductive group over a non-archimedean local field F and $\underline{Z}$ be the F-split component (i.e., the maximal F-split central torus) of $\underline{G}$. The group $\underline{G}(F)$ of F-points of $\underline{G}$ is denoted by G, and similarly $\underline{Z}(F)$ by Z.

First we shall recall some of fundamental theory (due to Jacquet and Harish-Chandra) on admissible representations of reductive p-adic groups.

(a) **Definition.** An admissible representation (π, V) of G is said to be *cuspidal* if the support of every matrix coefficient of π is compact modulo Z.

For a parabolic F-subgroup $\underline{P}$ of $\underline{G}$, let (π_P, V_P) denote the (normalized) Jacquet module of (π, V) along $P = \underline{P}(F)$. The following criterion for cuspidality in terms of Jacquet modules is well-known.

(b) <u>Theorem.</u> (Jacquet, Harish-Chandra) *An admissible representation (π, V) of G is cuspidal if and only if $V_P = 0$ for any proper parabolic F-subgroup $\underline{P}$ of $\underline{G}$.*

Let $\underline{P} = \underline{M} \ltimes \underline{U}$ be a Levi decomposition of $\underline{P}$. For an admissible representation ρ of $M = \underline{M}(F)$ let $\mathrm{Ind}_P^G(\rho)$ be the normalized induction. The Frobenius reciprocity asserts that the (functorial) isomorphism

$$\mathrm{Hom}_G\left(\pi, \mathrm{Ind}_P^G(\rho)\right) \simeq \mathrm{Hom}_M(\pi_P, \rho)$$

holds. After **(b)** Jacquet's *subrepresentation theorem* directly follows from the above isomorphism (and the induction on the semi-simple rank etc).

(c) <u>Theorem.</u> (Jacquet) *For any irreducible admissible representation (π, V) of G, there exists a parabolic F-subgroup $\underline{P} = \underline{M} \ltimes \underline{U}$ of $\underline{G}$ and an irreducible cuspidal representation ρ of $M = \underline{M}(F)$ such that π is embedded in* $\mathrm{Ind}_P^G(\rho)$.

In this work we shall give the *relative version* (denoted by **(A)**, **(B)**, **(C)** below) of the above **(a)**, **(b)**, **(c)** respectively, assuming a version of the *relative Cartan decomposition* ($\sharp$) which is a description of orbits in G/H under a maximal compact subgroup of G. Details of ($\sharp$) will be explained in section 4. There are many concrete examples for which ($\sharp$) is valid.

From now on we assume that the residual characteristic of F is not equal to 2. Let σ be an F-involution on $\underline{G}$ and let H be the σ-fixator, that is, the subgroup consisting of σ-fixed points in G. An admissible representation (π, V) of G is said to be *H-distinguished* if the space $(V^*)^H$ of H-invariant linear forms on V is non-zero. For each $\lambda \in (V^*)^H$ and $v \in V$ let $\phi_{\lambda,v}$ be the corresponding *generalized* matrix coefficient given by

$$\phi_{\lambda,v}(g) = \langle \lambda, \pi(g^{-1})v \rangle$$

for $g \in G$. These are right H-invariant smooth functions on G. We call such functions *H-matrix coefficients* of π. We put the following definition.

(A) <u>Definition.</u> An H-distinguished representation (π, V) of G is said to be *H-relatively cuspidal* if the support of every H-matrix coefficient of π is compact modulo ZH.

A parabolic F-subgroup $\underline{P}$ is said to be *σ-split* if $\underline{P}$ and $\sigma(\underline{P})$ are opposite. For such a $\underline{P}$ we shall always take $\underline{M} = \underline{P} \cap \sigma(\underline{P})$ as a (σ-stable) Levi subgroup of $\underline{P}$. In section 3 we shall construct a linear mapping

$$r_P : (V^*)^H \to ((V_P)^*)^{M \cap H}$$

between the spaces of invariant linear forms by using Casselman's *canonical lifts*. We shall give an asymptotic relation between H-matrix coefficients of π defined by $\lambda \in (V^*)^H$ and $(M \cap H)$-matrix coefficients of the Jacquet module π_P defined by $r_P(\lambda)$. Using this relation, our criterion for relative cuspidality is given, in terms of Jacquet modules along σ-split parabolics, as follows.

> **(B) Theorem.** *Assume ($\sharp$). An H-distinguished representation (π, V) of G is H-relatively cuspidal if and only if $r_P\left((V^*)^H\right) = 0$ for any proper σ-split parabolic F-subgroup $\underline{P}$ of $\underline{G}$.*

After this characterization of relative cuspidality, our *relative subrepresentation theorem* is given (by the Frobenius reciprocity etc) as follows.

> **(C) Theorem.** *Assume ($\sharp$). For any irreducible H-distinguished representation (π, V) of G, there exists a σ-split parabolic F-subgroup $\underline{P} = \underline{M} \ltimes \underline{U}$ of $\underline{G}$ and an irreducible $(M \cap H)$-relatively cuspidal representation ρ of $M = \underline{M}(F)$ such that π is embedded in* $\mathrm{Ind}_P^G(\rho)$.

We hope that this theorem **(C)** provides a new foundation for the classification of distinguished representations attached to symmetric spaces, as **(c)** did for the classification of admissible representations.

Our statements **(A)**, **(B)**, **(C)** are generalizations of **(a)**, **(b)**, **(c)** respectively in the following sense: take a connected reductive F-group $\underline{G}_0$ and let $\underline{G}$ be the direct product $\underline{G} = \underline{G}_0 \times \underline{G}_0$. Consider the involution σ on $\underline{G}$ which permutes the factors. Then the corresponding symmetric space $\underline{G}/\underline{H}$ is isomorphic to the underlying space of $\underline{G}_0$. Such a situation is refered to as *the group case*. The assumption ($\sharp$) is true for the group case by the ordinary Cartan decomposition for G_0. The statements **(A)**, **(B)**, **(C)** applied to the group case will recover **(a)**, **(b)**, **(c)** for the group G_0 respectively. See section 6 for details.

Complete proofs of the statements in this article will be given in our forthcoming paper.

2 Notation for subgroups associated to σ

Let $\underline{G}$, σ and H be as in the introduction. For any F-subgroup $\underline{R}$ of $\underline{G}$, the group $\underline{R}(F)$ of F-points of $\underline{R}$ is denoted by R (by deleting the underbar).

An F-split subtorus $\underline{S}$ of $\underline{G}$ is said to be (σ, F)-*split* if $\sigma(s) = s^{-1}$ for all $s \in \underline{S}$. Fix a maximal (σ, F)-split torus $\underline{S}_0$ of $\underline{G}$. Take a maximal F-split torus $\underline{A}_\emptyset$ of $\underline{G}$ containing $\underline{S}_0$ and let Φ be the root system of $(\underline{G}, \underline{A}_\emptyset)$. Since $\underline{A}_\emptyset$ turns out to be σ-stable ([HW]), σ naturally acts on Φ. As in [HH] choose a σ-*basis* Δ of Φ satisfying

$$\alpha > 0,\ \sigma(\alpha) \neq \alpha \implies \sigma(\alpha) < 0$$

under the corresponding order.

Let $\underline{P}_\emptyset$ be the minimal parabolic F-subgroup of $\underline{G}$ containing $\underline{A}_\emptyset$, corresponding to the choice of Δ as above. Parabolic F-subgroups containing $\underline{P}_\emptyset$ are called standard parabolics. They correspond to subsets of Δ. For a subset $I \subset \Delta$ let $\underline{P}_I$ be the corresponding standard parabolic subgroup. Let $\underline{A}_I$ be the identity component of the intersection of all $\ker(\alpha)$, $\alpha \in I$, and set $\underline{M}_I = Z_{\underline{G}}(\underline{A}_I)$, the centralizer of $\underline{A}_I$ in $\underline{G}$. Then $\underline{A}_I$ is the F-split component of $\underline{M}_I$. One has a Levi decomposition $\underline{P}_I = \underline{M}_I \ltimes \underline{U}_I$ where $\underline{U}_I$ denotes the unipotent radical of $\underline{P}_I$. Let $\underline{P}_I^-$ be the unique parabolic subgroup such that $\underline{P}_I \cap \underline{P}_I^- = \underline{M}_I$ and $\underline{U}_I^-$ be its unipotent radical.

Recall that a parabolic F-subgroup $\underline{P}$ of $\underline{G}$ is said to be σ-split if $\underline{P}$ and $\sigma(\underline{P})$ are opposite. Let Δ_σ be the set of all σ-fixed roots in Δ. The condition for a standard parabolic subgroup $\underline{P}_I$ to be σ-split is given as follows ([HH]).

> $\underline{P}_I$ is σ-split if and only if $\Delta_\sigma \subset I$ and the subsystem Φ_I generated by I is σ-stable.

Note that every σ-split parabolic F-subgroup of $\underline{G}$ arises as $\underline{P}_I$ in this way, for a suitable choice of $\underline{S}_0$, $\underline{A}_\emptyset$ and Δ.

For a standard σ-split parabolic F-subgroup $\underline{P}_I$ of $\underline{G}$, let $\underline{S}_I$ be the identity component of $\underline{A}_I \cap \underline{S}_0$. We call $\underline{S}_I$ *the* (σ, F)-*split component of* $\underline{P}_I$. Given a positive real number $\varepsilon > 0$, set

$$S_I^-(\varepsilon) = \{ s \in S_I \mid |s^\alpha|_F \leqq \varepsilon \ (\forall \alpha \in \Delta \setminus I) \}.$$

We shall often drop the subscript I if there is no fear of confusion. We shall say briefly that P is a σ-split parabolic subgroup of G if it is the group of F-points of a σ-split parabolic F-subgroup $\underline{P}$ of $\underline{G}$, and also say that S is the

(σ, F)-split component of P if it is the group of F-points of the (σ, F)-split component $\underline{S} = \underline{S}_I$ of $\underline{P} = \underline{P}_I$, and so on.

Lemma 2.1. *Let $P = M \ltimes U$ be a σ-split parabolic subgroup with the (σ, F)-split component S. For any two open compact subgroups U_1, U_2 of U, there exists a positive real number $\varepsilon \leqq 1$ such that*

$$sU_1s^{-1} \subset U_2$$

for all $s \in S^-(\varepsilon)$.

For an open compact subgroup K of G and a parabolic subgroup $P = M \ltimes U$, set $U_K = U \cap K$, $M_K = M \cap K$ and $U_K^- = U^- \cap K$. If K is σ-stable and P is σ-split, it is obvious that $\sigma(U_K) = U_K^-$ and $\sigma(U_K^-) = U_K$. We say that K has *the Iwahori factorization with respect to P* if the product map

$$U_K^- \times M_K \times U_K \to K$$

is bijective.

To study Jacquet modules along σ-split parabolics we use a particular fundamental system $\{K_n\}$ of open neighborhoods of the identity in G: it consists of *σ-stable* open compact subgroups of G, having the Iwahori factorization with respect to all *standard σ-split* parabolic subgroups. (We just replace each K_n in [C, 1.4.4] by $K_n \cap \sigma(K_n)$.) We say that such a family $\{K_n\}$ is *adapted to* $(\underline{S}_0, \underline{A}_\emptyset, \Delta)$.

The following lemma is important for the investigation of invariant linear forms on Jacquet modules.

Lemma 2.2. *Let $K = K_n$ be an open compact subgroup in the family adapted to $(\underline{S}_0, \underline{A}_\emptyset, \Delta)$. Then for any corresponding standard σ-split parabolic subgroup P of G one has*

$$U_K \subset HM_KU_K^-.$$

3 Invariant linear forms on Jacquet modules

In this section we shall explain how to construct the mapping

$$r_P : (V^*)^H \to ((V_P)^*)^H$$

between the spaces of invariant linear forms mentioned in the introduction, and give the result on the asymptotic behaviour of H-matrix coefficients.

Let (π, V) be an admissible representation of G and $P = M \ltimes U$ be a σ-split parabolic subgroup of G, with the (σ, F)-split component S. The Jacquet module (π_P, V_P) of (π, V) along P is defined as follows: the space V_P is the quotient $V/V(U)$, where $V(U)$ denotes the subspace of V generated by all the elements of the form $\pi(u)v - v$, $u \in U$, $v \in V$. Let $j_P : V \to V_P$ be the canonical projection. The action π_P of M is normalized so that

$$\pi_P(m)j_P(v) = \delta_P^{-1/2}(m)j_P(\pi(m)v)$$

for $m \in M$.

Now we recall Casselman's *canonical lifting* ([C, §4]). For a compact subgroup K of G let V^K be the subspace of V of all K-fixed vectors and let $\mathcal{P}_K : V \to V^K$ be the projection operator given by

$$\mathcal{P}_K(v) = \frac{1}{\mathrm{vol(K)}} \int_K \pi(k)v dk.$$

For a compact subgroup U_1 of U set

$$V(U_1) = \{\, v \in V \mid \int_{U_1} \pi(u)vdu = 0\,\}.$$

It is known ([C, 3.2.1]) that $V(U)$ is the union of all $V(U_1)$ where U_1 ranges over all compact subgroups of U. Now, given $\overline{v} \in V_P$, take an open compact subgroup $K = K_n$ from the family $\{K_n\}$ adapted to $(\underline{S}_0, \underline{A}_0, \Delta)$ so that $\overline{v} \in (V_P)^{M_K}$. Next let us choose an open compact subgroup U_1 of U so that $V^K \cap V(U) \subset V(U_1)$. Finally, by **2.1** we can choose a positive real number $\varepsilon \leqq 1$ so that for all $s \in S^-(\varepsilon)$, we have $sU_1s^{-1} \subset U_K$. Then, for all $s \in S^-(\varepsilon)$ the spaces $\mathcal{P}_K(\pi(s)V^K)$ are identical ([C, 4.1.6]) and by the restriction of $j_P : V \to V_P$ we have an isomorphism

$$\mathcal{P}_K\left(\pi(s)V^K\right) \xrightarrow{\simeq} (V_P)^{M_K}$$

for any $s \in S^-(\varepsilon)$ ([C, 4.1.4]). The element $v \in \mathcal{P}_K\left(\pi(s)V^K\right)$ such that $j_P(v) = \overline{v}$ is called the canonical lift of $\overline{v} \in V_P$ with respect to K. It depends on the choice of K, but not on U_1 and ε. If v' is another canonical lift of $\overline{v}$, say, with respect to K', then assuming that K' is contained in K we have ([C, 4.1.8])

$$v' \in V^{M_K U_K^-}, \quad v = \mathcal{P}_K(v') = \mathcal{P}_{U_K}(v').$$

By this relation and the lemma **2.2** we have:

Lemma. *Let λ be an H-invariant linear form on an H-distinguished representation (π, V) of G. Let $P = M \ltimes U$ be a σ-split parabolic subgroup of G and v, $v' \in V$ be canonical lifts of the same element $\overline{v} \in V_P$. Then*

$$\langle \lambda, v \rangle = \langle \lambda, v' \rangle.$$

After this lemma we may define a linear form $r_P(\lambda)$ on the Jacquet module V_P along a σ-split parabolic subgroup P as follows:

Definition. Let $\lambda \in (V^*)^H$ be an H-invariant linear form on an H-distinguished representation (π, V) of G and P be a σ-split parabolic subgroup of G. The linear form $r_P(\lambda)$ on the Jacquet module V_P is defined by

$$\langle r_P(\lambda), \overline{v} \rangle = \langle \lambda, v \rangle$$

for each $\overline{v} \in V_P$ if $v \in V$ is a canonical lift of $\overline{v}$.

This construction of $r_P(\lambda)$ is a relative version of Casselman's *canonical pairing* of Jacquet modules ([C, 4.2.2]). See section 6.

Next we give the following proposition which describes the asymptotic behaviour of H-matrix coefficients.

Proposition. *Let (π, V) be an H-distinguished representation of G and λ be an H-invariant linear form on V. Let $P = M \ltimes U$ be a σ-split parabolic subgroup of G with the (σ, F)-split component S.*

(i) *For each $v \in V$, there exists a positive real number $\varepsilon \leqq 1$ such that for any $s \in S^-(\varepsilon)$ one has*

$$\langle \lambda, \pi(s)v \rangle = \delta_P^{1/2}(s) \langle r_P(\lambda), \pi_P(s) j_P(v) \rangle.$$

(ii) *Assume that $\overline{\lambda}$ is a linear form on V_P having the following property: for each $v \in V$, there exists a positive real number $\varepsilon \leqq 1$ such that for any $s \in S^-(\varepsilon)$ one has*

$$\langle \lambda, \pi(s)v \rangle = \delta_P(s)^{1/2} \langle \overline{\lambda}, \pi_P(s) j_P(v) \rangle.$$

Then $\overline{\lambda}$ coincides with $r_P(\lambda)$.

This is a relative version of [C, 4.2.3].

The $(M \cap H)$-invariance of the linear form $r_P(\lambda)$ is shown after **(ii)** of the above proposition.

Corollary(1). *The linear form $r_P(\lambda)$ on V_P is $M \cap H$-invariant and the mapping $r_P : (V^*)^H \to ((V_P)^*)^{M\cap H}$ is linear.*

This is seen as follows: for any $m \in M \cap H$ put $\overline{\lambda} = r_P(\lambda) \circ \pi_P(m)$. Then $\overline{\lambda}$ has the property that $r_P(\lambda)$ must have in **(ii)**. As a consequence we have $r_P(\lambda) = r_P(\lambda) \circ \pi_P(m)$.

(ii) of the above proposition has one more important corollary on the transitivity with respect to the inclusion of σ-split parabolics: let P, Q be σ-split parabolic subgroups of G with $P \supset Q$. Let M, L be the σ-stable Levi subgroup of P, Q respectively. In such a case, $M \cap Q$ is a σ-split parabolic subgroup of M. As is well-known, $(V_P)_{M\cap Q}$ is naturally isomorphic to V_Q as an L-module. There are induced mappings

$$r_P : (V^*)^H \to ((V_P)^*)^{M\cap H}, \quad r_{M\cap Q} : ((V_P)^*)^{M\cap H} \to (((V_P)_{M\cap Q})^*)^{L\cap H}$$

and

$$r_Q : (V^*)^H \to ((V_Q)^*)^{L\cap H} \left(\simeq (((V_P)_{M\cap Q})^*)^{L\cap H}\right)$$

of invariant linear forms.

Corollary(2). *For P, Q as above, one has*

$$r_{M\cap Q} \circ r_P = r_Q.$$

That is, the diagram

$$\begin{tikzcd}
(V^*)^H \arrow[r, "r_P"] \arrow[d, "r_Q"'] & (V_P^*)^{M\cap H} \arrow[d, "r_{M\cap Q}"] \\
(V_Q^*)^{L\cap H} \arrow[r, "\simeq"] & (((V_P)_{M\cap Q})^*)^{L\cap H}
\end{tikzcd}$$

is commutative.

Indeed, $\overline{\lambda} := r_{M\cap Q} \circ r_P(\lambda)$ has the property that $r_Q(\lambda)$ must have in **(ii)** of the proposition.

4 Characterization of relative cuspidality

In this section we shall explain our assumption on the orbit decomposition of G/H, which we call *relative Cartan decomposition*. Then we shall give a rough sketch of the way to obtain our theorem **(B)** from the assumption ($\sharp$).

Choose a maximal (σ, F)-split torus $\underline{S}_0$, a maximal F-split torus $\underline{A}_\emptyset$ containing $\underline{S}_0$, and a σ-basis Δ. Let $\underline{M}_0$ be the centralizer of $\underline{S}_0$ in $\underline{G}$ (which coincides with $\underline{M}_{\Delta_\sigma}$) and set

$$S_0^+ = \{s \in S_0 \mid |s^\alpha|_F \geqq 1\,(\forall \alpha \in \Delta)\} = \{s^{-1} \mid s \in S_0^-(1)\}.$$

<u>**Assumption ($\sharp$).**</u> For a suitable choice of maximal compact subgroup $K_{\max}$ of G, there exists a finite subset Γ of $(\underline{M}_0 \cdot \underline{H})(F)$ such that

$$G = K_{\max} \cdot S_0^+ \cdot \Gamma \cdot H.$$

There are many examples of symmetric pairs (G, H) satisfying this assumption, such as

$$(G,H) = (\mathrm{GL}_n(F), \mathrm{O}_n(F)),\ (\mathrm{GL}_n(E), \mathrm{U}_n(E/F)),\ (\mathrm{GL}_{2n}(F), \mathrm{Sp}_n(F)),$$
$$(\mathrm{GL}_n(F), \mathrm{GL}_r(F) \times \mathrm{GL}_{n-r}(F)),\ (\mathrm{GL}_n(E), \mathrm{GL}_n(F)),\ \ldots$$

where E/F is a quadratic extension. See [H] for the first four and [T] for the last one. See also [U] for related matters.

We shall briefly explain how to derive theorem **(B)** under the assumption ($\sharp$). Let (π, V) be an H-distinguished representation of G and $\lambda \in (V^*)^H$ be an H-invariant linear form on V. For each $v \in V$ consider the H-matrix coefficient $\phi_{\lambda,v}$ defined by

$$\phi_{\lambda,v}(g) = \langle \lambda, \pi(g^{-1})v \rangle.$$

Let P be a σ-split parabolic subgroup of G and S be the (σ, F)-split component of P. Since $v \in V$ is $K_{\max}$-finite, we may choose a positive real number $\varepsilon \leqq 1$ in **(i)** of the proposition of section 4 so that the relation

$$\langle \lambda, \pi(s)\pi(k)v \rangle = \delta_P^{1/2}(s) \langle r_P(\lambda), \pi_P(s) j_P(\pi(k)v) \rangle$$

holds for all $s \in S^-(\varepsilon)$ and all $k \in K_{\max}$. Now assume that $r_P(\lambda) = 0$. Then for all $h \in H$, $s \in S^-(\varepsilon)$ and $k \in K_{\max}$ we must have

$$\phi_{\lambda,v}(k^{-1}s^{-1}h) = \langle \lambda, \pi(s)\pi(k)v \rangle = 0.$$

That is, $\phi_{\lambda,v}$ is zero on the double coset $K_{\max} \cdot s^{-1} \cdot H$ for all $s \in S^{-}(\varepsilon)$. The compactness of the support of $\phi_{\lambda,v}$ in the union

$$\bigcup_{s \in S_0^-(1)} K_{\max} \cdot s^{-1} \cdot H$$

modulo ZH readily follows by varying P in the proper standard σ-split parabolics. Unfortunately $\bigcup K_{\max} \cdot s^{-1} \cdot H$ does not cover all of G in any example we examined. We need a complementary finite set Γ to cover all of G as in ($\sharp$). Roughly speaking, it is possible to show the compactness of the support in

$$\bigcup_{s \in S_0^-(1)} K_{\max} \cdot s^{-1}\gamma^{-1} \cdot H$$

by a similar discussion at least if $\gamma \in (\underline{M}_0 \cdot \underline{H})(F)$. Thus, assuming that the complementary elements γ can be chosen from $(\underline{M}_0 \cdot \underline{H})(F)$, we have one direction of the theorem.

(B) **Theorem.** **(Characterization of Relative Cuspidality)** *Assume ($\sharp$) for (G, σ). An H-distinguished representation (π, V) of G is H-relatively cuspidal if and only if $r_P\left((V^*)^H\right) = 0$ for any proper σ-split parabolic F-subgroup $\underline{P}$ of $\underline{G}$.*

5 Relative subrepresentation theorem

We need the following lemma. It is non-trivial but the proof is elementary.

Lemma. *A finitely generated H-relatively cuspidal representation has an irreducible H-distinguished quotient.*

Now we shall give a rough sketch of the proof of our main theorem **(C)**. If (π, V) is H-relatively cuspidal there is nothing to prove. If not, then there is a proper σ-split parabolic subgroup P of G such that $r_P\left((V^*)^H\right) \neq 0$. Let $P = M \ltimes U$ be minimal one. If we assume ($\sharp$) for G, then by the corollary **(2)** of section 3 it is seen that the Jacquet module π_P is $(M \cap H)$-relatively cuspidal. Apply the above lemma to take an irreducible $M \cap H$-distinguished quotient ρ of π_P. By the Frobenius reciprocity

$$\mathrm{Hom}_G(\pi, \mathrm{Ind}_P^G(\rho)) \simeq \mathrm{Hom}_M(\pi_P, \rho) \ (\neq 0),$$

there is an embedding of π into $\mathrm{Ind}_P^G(\rho)$. If ρ is not relatively cuspidal apply the same procedure for ρ. Here we need to assume ($\sharp$) also for M. In this way we have

(C) Theorem. (Relative Subrepresentation Theorem)
Assume ($\sharp$) for all σ-stable Levi subgroups of σ-split parabolic subgroups of G. For any irreducible H-distinguished representation (π, V) of G, there exists a σ-split parabolic F-subgroup $P = M \ltimes U$ of G and an irreducible $(M \cap H)$-relatively cuspidal representation ρ of M such that π is embedded in $\mathrm{Ind}_P^G(\rho)$.

6 The group case

Take a connected reductive F-group $\underline{G}_0$ and let $\underline{G}$ be the direct product $\underline{G} = \underline{G}_0 \times \underline{G}_0$. Let σ be the involution on $\underline{G}$ which permutes the factors. Then the σ-fixator H in G is the diagonal subgroup

$$H = \Delta(G_0) = \{(g, g) \in G_0 \times G_0 \mid g \in G_0\}.$$

The map $(g_1, g_2) \mapsto g_1 g_2^{-1}$ induces an identification $G/H = (G_0 \times G_0)/\Delta(G_0) \simeq G_0$. We shall apply our theory to this situation.

- Distinguishedness.

  Any *irreducible* admissible representation π of $G = G_0 \times G_0$ is of the form $\pi_0 \otimes \pi_0'$ where π_0, π_0' are irreducible admissible representations of G_0. It is $H = \Delta(G_0)$-distinguished if and only if $\pi_0' \simeq \widetilde{\pi}_0$, that is, π is of the form $\pi_0 \otimes \widetilde{\pi}_0$ for an irreducible admissible representation π_0 of G_0.

- (A) for the group case means (a).

  The natural pairing between π_0 and $\widetilde{\pi}_0$ gives a non-zero $\Delta(G_0)$-invariant linear form $\lambda \in ((\pi_0 \otimes \widetilde{\pi}_0)^*)^{\Delta(G_0)}$ (which is unique up to constant) by

  $$\langle \lambda, v_0 \otimes \widetilde{v}_0 \rangle = \langle \widetilde{v}_0, v_0 \rangle_{\widetilde{\pi}_0 \times \pi_0}.$$

  The H-matrix coefficients defined by λ are identified with the usual matrix coefficients of π_0 through the map $(g_1, g_2) \mapsto g_1 g_2^{-1}$ as follows:

  $$\langle \lambda, (\pi_0 \otimes \widetilde{\pi}_0)(g_1, g_2)^{-1}(v_0 \otimes \widetilde{v}_0) \rangle = \langle \lambda, \pi_0(g_1^{-1}) v_0 \otimes \widetilde{\pi}_0(g_2^{-1}) \widetilde{v}_0 \rangle$$

$$= \langle \widetilde{\pi}_0(g_2^{-1})\widetilde{v}_0, \pi_0(g_1^{-1})v_0 \rangle_{\widetilde{\pi}_0 \times \pi_0} = \langle \widetilde{v}_0, \pi_0\left((g_1 g_2^{-1})^{-1}\right) v_0 \rangle_{\widetilde{\pi}_0 \times \pi_0}.$$

Thus it is obvious that $\pi = \pi_0 \otimes \widetilde{\pi}_0$ is H-relatively cuspidal if and only if π_0 is cuspidal as a representation of G_0.

- ($\sharp$) is true for the group case.

In the group case the decomposition in ($\sharp$) follows from the *ordinary Cartan decomposition* for the group G_0: take a maximal F-split torus $\underline{A_0}$ of $\underline{G_0}$ and let K_0 be an $\underline{A_0}$-good maximal compact subgroup of G_0. The ordinary Cartan decomposition asserts that

$$G_0 = K_0 \cdot A_0^+ \cdot \Gamma_0 \cdot K_0$$

for a suitable finite subset Γ_0 of $M_0 = Z_{G_0}(A_0)$ [S, §0.6]. Now the map $(g_1, g_2) \mapsto g_1 g_2^{-1}$ induces an identification

$$(K_0 \times K_0)\backslash(G_0 \times G_0)/\Delta(G_0) \simeq K_0\backslash G_0/K_0,$$

which implies that ($\sharp$) is true by taking $K_{\max} = K_0 \times K_0$.

- The mapping r_P for the group case.

The σ-split parabolic F-subgroups of $G = G_0 \times G_0$ are those of the form $P_0 \times P_0^-$ where P_0 and P_0^- are opposite parabolic F-subgroups of G_0. Set $M_0 = P_0 \cap P_0^-$. For an *irreducible* $\Delta(G_0)$-distinguished representation $\pi_0 \otimes \widetilde{\pi}_0$ of $G_0 \times G_0$, let $\lambda \in ((\pi_0 \otimes \widetilde{\pi}_0)^*)^{\Delta(G_0)}$ be as above. Then $r_P(\lambda) = r_{P_0 \times P_0^-}(\lambda)$ is a linear form on the Jacquet module

$$(\pi_0 \otimes \widetilde{\pi}_0)_{P_0 \times P_0^-} \simeq (\pi_0)_{P_0} \otimes (\widetilde{\pi}_0)_{P_0^-}$$

which is invariant under

$$(M_0 \times M_0) \cap \Delta(G_0) = \Delta(M_0).$$

So $r_P(\lambda)$ gives an M_0-invariant bilinear form on $(\pi_0)_{P_0} \times (\widetilde{\pi}_0)_{P_0^-}$. It coincides with the one constructed by Casselman in [C, §4].

- **(B)** for the group case implies **(b)**.

Now the linear form $r_{P_0 \times P_0^-}(\lambda)$ vanishes if and only if the Jacquet module $(\pi_0)_{P_0}$ vanishes, since Casselman's pairing was shown to be non-degenerate ([C, 4.2.4]). Thus our theorem **(B)** applied to the group case actually recovers **(b)**.

- (C) for the group case implies (c).

  Finally, apply our theorem (C) to the group case. We then assert that for any irreducible admissible ΔG_0-distinguished representation $\pi_0 \otimes \widetilde{\pi}_0$ of $G_0 \times G_0$, there exists a σ-split parabolic subgroup $P_0 \times P_0^-$, and an irreducible ΔM_0-relatively cuspidal representation $\rho_0 \otimes \widetilde{\rho}_0$ of $M_0 \times M_0$, such that $\pi_0 \otimes \widetilde{\pi}_0$ can be embedded in

$$\mathrm{Ind}_{P_0 \times P_0^-}^{G_0 \times G_0}(\rho_0 \otimes \widetilde{\rho}_0) \simeq \mathrm{Ind}_{P_0}^{G_0}(\rho_0) \otimes \mathrm{Ind}_{P_0^-}^{G_0}(\widetilde{\rho}_0).$$

  Now (c) is recovered at the first factor. At the second factor we also have an embedding $\widetilde{\pi}_0 \hookrightarrow \mathrm{Ind}_{P_0^-}^{G_0}(\widetilde{\rho}_0)$ as in [S, 3.3.1].

7 Concluding remarks

- By theorem (B) it turns out that *cuspidal distinguished representations are relatively cuspidal in our sense.* Examples of such representations were constructed by Hakim and Mao, for symmetric pairs $(G, H) = (\mathrm{GL}_n(F), \mathrm{O}_n(F))$, $(\mathrm{GL}_n(E), \mathrm{U}_n(E/F))$. See [HM1], [HM2].

- We have studied several examples of *non-cuspidal but relatively cuspidal* representations, for symmetric pairs $(G, H) = (\mathrm{GL}_{2n}(F), \mathrm{Sp}_n(F))$ and $(\mathrm{GL}_n(F), \mathrm{GL}_{n-1}(F) \times \mathrm{GL}_1(F))$. For these pairs it is known that there is no cuspidal distinguished representation (see [HR] and [P]). Details will be included in our forthcoming paper.

- In recent preprint ([BD]) Blanc and Delorme studied the distinguishedness of a class of induced representations. They used only σ-split parabolic subgroups as the inducing subgroups, and distinguished representations of σ-stable Levi subgroups as the inducing representations. Their work seems to include the *adjoint* operation (in some sense) to our construction of the mapping r_P.

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