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On the Hecke eigenvalues of Siegel cusp forms of genus 2

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Denote by $S_k(\Gamma_1)$ be the space of cusp forms of integral weight k on the full modular group $\Gamma_1 := SL_2(\mathbf{Z})$ and let $f \in S_k(\Gamma_1)$ be a normalized Hecke eigenform. Denote by $\lambda(n)\,(n \in \mathbf{N})$ the Hecke eigenvalues of f. Then using a classical theorem of Landau together with the analytic properties of the Hecke L-function $L(f,s)$ and the Rankin-Selberg zeta function attached to f it is not difficult to see that the sequence $(\lambda(n))_{n\in\mathbf{N}}$ changes sign infinitely many often, i.e. there are infinitely many n such that $\lambda(n) > 0$ and there are infinitely many n such that $\lambda(n) < 0$. Indeed, this is true for the Fourier coefficients of any non-zero cusp form of any level (supposing that these coefficients are real).

A very natural question to ask is to what extent this result generalizes to Siegel modular forms. Here we consider the simplest case, namely the case of genus 2.

Let $S_k(\Gamma_2)$ be the space of Siegel cusp forms of integral weight k on $\Gamma_2 := Sp_2(\mathbf{Z}) \subset GL_4(\mathbf{Z})$ and let $F \in S_k(\Gamma_2)$ be a non-zero Hecke eigenform. Denote by $\lambda(n)\,(n \in \mathbf{N})$ the eigenvalues of F under the usual Hecke operators $T(n)\,(n \in \mathbf{N})$.

Note that the $\lambda(n)$ are no longer "proportional" (in any reasonable sense) to the Fourier coefficients of F.

One has

(1) $$\sum_{n\geq 1} \lambda(n) n^{-s} = \zeta(2s-2k+4)^{-1} Z_F(s) \qquad (\Re(s) >> 0)$$

where

$$Z_F(s) = \prod_p Z_{F,p}(p^{-s})^{-1} \qquad (\Re(s) >> 0)$$

is the spinor zeta function of F. Here

$$Z_{F,p}(X) = (1-\alpha_{0,p}X)(1-\alpha_{0,p}\alpha_{1,p}X)(1-\alpha_{0,p}\alpha_{2,p}X)(1-\alpha_{0,p}\alpha_{1,p}\alpha_{2,p}X)$$

and $\alpha_{0,p}, \alpha_{1,p}$ and $\alpha_{2,p}$ are "the" Satake p-parameters of F (cf. [1]).

If k is even let $S_k^*(\Gamma_2) \subset S_k(\Gamma_2)$ be the Maass subspace, in other words the subspace spanned by the images of the Saito-Kurokawa lifts of Hecke eigenforms in $S_{2k-2}(\Gamma_1)$. Recall that $S_k^*(\Gamma_2)$ is Hecke-invariant and for a non-zero Hecke eigenform $F \in S_k^*(\Gamma_2)$ there exist a unique normalized Hecke eigenform $f \in S_{2k-2}(\Gamma_1)$ such that

(2) $$Z_F(s) = \zeta(s-k+1)\zeta(s-k+2)L(f,s).$$

Theorem 1 [2]. *Let k be even and let $F \in S_k^*(\Gamma_2)$ be a non-zero Hecke eigenform. Then $\lambda(n) > 0$ for all n.*

The proof follows from explicitly exploiting the relations given by (2) between the $\lambda(n)$ and the eigenvalues of the form f and using Deligne's theorem (previously the Ramanujan-Petersson conjecture) for the latter.

Theorem 2 [4]. *Let $F \in S_k(\Gamma_2)$ be a non-zero Hecke eigenform and suppose that F is in the orthogonal complement of the space $S_k^*(\Gamma_2)$ if k is even. Then the sequence $(\lambda(n))_{n\in\mathbf{N}}$ has infinitely many sign changes.*

The proof uses (1) together with the analytic properties of the spinor zeta function $Z_F(s)$ coupled with the fact that the generalized Ramanujan-Petersson conjecture for F as considered is true (as proved by Weissauer), i.e. one has

$$|\alpha_{1,p}| = |\alpha_{2,p}| = 1 \qquad (\forall p).$$

For details we refer to [4].

Taking Theorem 2 for granted, a natural question is when the first negative eigenvalue occurs. Extending previous work in the case of elliptic modular forms [3], it seems possible that one can prove that there exists

$$n \ll_\epsilon k^{2+\epsilon}$$

such that $\lambda(n) < 0$ for F as in Theorem 2, where the constant implied in $\ll_\epsilon$ depends only on ϵ. For details we refer to [5].

References

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