CAP forms on $U(2,2)$ II. Cusp forms *

Takuya KONNO † Kazuko KONNO ‡

Abstract

This is a report of our work on non-tempered automorphic representations of $U_{E/F}(2,2)$. Few years ago, we obtained a complete description of the local components of such automorphic forms. This time, we construct all the expected automorphic forms with these components.

Contents

1   Introduction to CAP forms

The term CAP is a short hand for the phrase "Cuspidal but Associated to Parabolic subgroups". This is the name given by Piatetski-Shapiro [PS83] to those cuspidal automorphic representations which apparently contradict the generalized Ramanujan conjecture. An up-to-date definition of CAP forms might be given as follows.

Let G be a connected reductive group defined over a number field F. We write $\mathbb{A} = \mathbb{A}_F$ for the adele ring of F. By an *automorphic representation* of $G(\mathbb{A})$, we mean an *irreducible subquotient* of the right regular representation

$$R(g)\phi(x) = \phi(xg), \quad g \in G(\mathbb{A})$$

of $G(\mathbb{A})$ on the Hilbert space

$$L^2(G(F)\mathfrak{A}_G\backslash G(\mathbb{A})) := \left\{ \begin{array}{l|l} \phi : G(\mathbb{A}) \to \mathbb{C} \\ \text{measurable} \end{array} \begin{array}{l} \text{(i) } \phi(\gamma a g) = \phi(g), \\ \quad \gamma \in G(F),\ a \in \mathfrak{A}_G,\ g \in G(\mathbb{A}) \\ \text{(ii) } \int_{G(F)\mathfrak{A}_G\backslash G(\mathbb{A})} |\phi(g)|^2\, dg < \infty \end{array} \right\}.$$

Here, $\mathfrak{A}_G$ is the maximal $\mathbb{R}$-vector subgroup in the center $Z(G)(\mathbb{A})$ of $G(\mathbb{A})$ and the measure is taken to be $G(\mathbb{A})$-invariant. The *discrete spectrum* $L^2_{\mathrm{disc}}(G(F)\mathfrak{A}_G\backslash G(\mathbb{A}))$ is the maximum subspace of $L^2(G(F)\mathfrak{A}_G\backslash G(\mathbb{A}))$ which is a direct sum of irreducible subrepresentations. Further this decomposes as

$$L^2_{\mathrm{disc}}(G(F)\mathfrak{A}_G\backslash G(\mathbb{A})) = L^2_0(G(F)\mathfrak{A}_G\backslash G(\mathbb{A})) \oplus L^2_{\mathrm{res}}(G(F)\mathfrak{A}_G\backslash G(\mathbb{A})).$$

Here $L^2_0(G(F)\mathfrak{A}_G\backslash G(\mathbb{A}))$ is the completion of the space of cusp forms with respect to the Petersson (*i.e.*, L^2-) norm and called the *cuspidal spectrum*. On the other hand, $L^2_{\mathrm{res}}(G(F)\mathfrak{A}_G\backslash G(\mathbb{A}))$ is spanned by certain iterated residues of Eisenstein series

$$\mathrm{Res}_{\lambda=s}\, E^G_P(\phi), \quad \phi \in \mathrm{Ind}^G_P(\tau_\lambda), \tau \subset L^2_0(M(F)\mathfrak{A}_M\backslash M(\mathbb{A})),$$

where $P = MU \subset G$ is a proper parabolic subgroup. We observe that

- Let us write $t(\tau_v)$ for the *Hecke* (formerly called *Satake*) *matrix* of τ at any unramified place v for M and τ. Then the Hecke matrix for the residue $\mathrm{Res}_{\lambda=s}\, E^G_P(\tau_\lambda)$ is $q_v^{-s}t(\tau_v)$. Here q_v is the cardinality of the residue field of F_v.

- According to Langlands' criterion for square integrability, we must have $\Re\varpi^\vee(s) > 0$ for any "fundamental coweight" ϖ for P.

In particular, even if τ satisfies the Ramanujan conjecture for M (*i.e.*, $t(\tau_v)^{\mathbb{Z}}$ is bounded), any residue $\mathrm{Res}_{\lambda=s}\, E^G_P(\tau_\lambda)$ in the discrete spectrum cannot satisfy the same conjecture for G.

Now let G^* be the quasisplit inner form of G. At almost all places v of F, $G_v := G \otimes_F F_v$ is isomorphic to G^*_v.

Definition 1.1. *An irreducible cuspidal representation* $\pi = \bigotimes_v \pi_v \subset L^2_0(G(F)\mathfrak{A}_G\backslash G(\mathbb{A}))$ *of* $G(\mathbb{A})$ *is a* CAP *form if there exists an irreducible residual automorphic representation* $\pi^* = \bigotimes_v \pi^*_v \subset L^2_{\mathrm{res}}(G^*(F)\mathfrak{A}_G\backslash G^*(\mathbb{A}))$ *of* $G^*(\mathbb{A})$ *such that the absolute values of the eigenvalues of the Hecke matrices* $t(\pi_v)$ *and* $t(\pi^*_v)$ *coincide at almost all* v.

Example 1.2. *(i) Combining the results of Jacquet-Shalika [JS81b], [JS81a] and Moeglin-Waldspurger [MW89], one finds that there are no CAP forms on* $G = GL(n)$.

(ii) If $G = D^\times$, *the unit group of a central division algebra over* F, *the trivial representation* $\mathbb{1}_{G(\mathbb{A})}$ *is a CAP form.*

(iii) The CAP forms on $U_{E/F}(3)$ (any unitary group in 3 variables) are the θ-liftings of automorphic characters on $U_{E/F}(1, \mathbb{A})$ [GR90], [GR91].

(iv) The CAP forms on $Sp(2)$ are either the Saito-Kurokawa liftings (θ-liftings of automorphic representations of the metaplectic cover $\widetilde{SL(2, \mathbb{A})}$) or the θ_{10}-type representations constructed by Howe-Piatetski-Shapiro [PS83] (θ-liftings of automorphic representations of various orthogonal groups in 2-variables). It is expected but I do not know if these two families are disjoint.

(v) Some CAP forms on the split exceptional group of type G_2 are studied by Gan-Gurevich-Jiang [GGJ02].

(vi) The Ikeda lift on $Sp(2n)$ and the Miyawaki lift on $Sp(3)$ [Ike01] are CAP forms.

Besides its importance as counter examples to the Ramanujan conjecture, we propose the following three motivation of studying CAP forms.

- Construct and explicitly describe certain mixed motives associated to Shimura varieties. This point of view is discussed in detail in [Har93].

- Capture some periods of automorphic forms. This is related to the Ikeda-Ichino conjecture.

- Construct unipotent and other singular supercuspidal representations of p-adic groups.

In 2003, we have described the expected local components of the CAP forms of the quasisplit unitary group $U_{E/F}(2, 2)$ in 4-variables [KKa]. In this talk, we construct the cusp forms with those local components.

2 A-parameters

In order to put non-tempered automorphic forms into the framework of Langlands' conjecture, J. Arthur proposed a series of conjectures [Art89]. The conjectural description is given through the A-parameters. On the other hand, these parameters are not well related to the definition1.1 of CAP forms, because the Ramanujan conjecture is not yet established for any non-abelian reductive group G. In order to obtain a nice framework to study CAP forms, it is best to introduce the following *ad hoc* notion of A-parameters for unitary groups.

Let E/F be a quadratic extension of number fields, and write σ for the generator of $\mathrm{Gal}(E/F)$. We fix an algebraic closure $\bar{F}$ of E (or F) and write W_F (resp. W_E) for the Weil group of $\bar{F}/F$ (resp. $\bar{F}/E$). Recall the (non-split) extension $1 \to W_E \to W_F \to \mathrm{Gal}(E/F) \to 1$. We fix an inverse image $w_\sigma \in W_F$ of σ.

First we consider the group $H_n := \mathrm{Res}_{E/F}GL(n)$. Its L-group is given by $^L H_n = \widehat{H}_n \rtimes_{\rho_{H_n}} W_F$ with $\widehat{H}_n = GL(n, \mathbb{C})^2$ and

$$\rho_{H_n}(w)(h_1, h_2) = \begin{cases} (h_1, h_2) & \text{if } w \in W_E, \\ (h_2, h_1) & \text{otherwise.} \end{cases}$$

We write $\Phi_0(H_n)$ for the set of (isomorphism classes of) irreducible unitary cuspidal representations of $H_n(\mathbb{A})$. Conjecturally, this should be in 1-1 correspondence with the set of isomorphism classes of irreducible n-dimensional representations with bounded image of the hypothetical

Langlands group $\mathcal{L}_E$ of E. We adopt this latter point of view, since it is convenient for some observations. There should be a natural morphism $p_{W_F} : \mathcal{L}_F \to W_F$. As in the Weil group case, $\mathcal{L}_F$ should be an extension $1 \to \mathcal{L}_E \to \mathcal{L}_F \to \mathrm{Gal}(E/F) \to 1$. Again we take an inverse image $w_\sigma \in \mathcal{L}_F$ of the above fixed $w_\sigma \in W_F$. By [Bor79, Prop.8.4], each $\varphi_E \in \Phi_0(H_n)$ is identified with the homomorphism $\varphi : \mathcal{L}_F \to {}^L H_n$ given by

$$\varphi(w) := \begin{cases} (\varphi_E(w), \varphi_E(w_\sigma w w_\sigma^{-1})) \times p_{W_F}(w) & \text{if } w \in \mathcal{L}_E, \\ (\varphi_E(w w_\sigma^{-1}), \varphi_E(w_\sigma w)) \rtimes p_{W_F}(w) & \text{otherwise.} \end{cases} \tag{2.1}$$

Definition 2.1. *An A-parameter for H_n is a homomorphism $\phi : \mathcal{L}_F \times SL(2,\mathbb{C}) \to {}^L H_n$ such that*

(i) $\phi|_{SL(2,\mathbb{C})} : SL(2,\mathbb{C}) \to \widehat{H_n}$ is analytic.

(ii) $\mathcal{L}_F \overset{\phi}{\to} {}^L H_n \overset{\mathrm{proj}}{\to} W_F$ coincides with $p_{W_F} : \mathcal{L}_F \to W_F$. Thus ϕ is determined by the representation $\phi_E : \mathcal{L}_E \times SL(2,\mathbb{C}) \overset{\phi}{\longrightarrow} {}^L H_n \overset{1st. proj.}{\longrightarrow} GL(n,\mathbb{C})$ (under (2.1)).

(iii) ϕ_E is semisimple, so that we have an irreducible decomposition $\phi_E \simeq \bigoplus_{i=1}^r \varphi_{i,E} \otimes \rho_{d_i}$. Here, $\varphi_{i,E}$ is an m_i-dimensional irreducible representation of $\mathcal{L}_E$ and ρ_d denotes the d-dimensional irreducible representation of $SL(2,\mathbb{C})$. Note $\sum_{i=1}^r d_i m_i = n$.

(iv) $\varphi_{i,E} \in \Phi_0(H_{m_i})$.

A-parameters ϕ, ϕ' for H_n are equivalent *if they are $\widehat{H_n}$-conjugate, or equivalently, if ϕ_E and ϕ'_E are isomorphic. An A-parameter ϕ contributes to the discrete spectrum if and only if it is* elliptic, *i.e., ϕ_E is irreducible.*

Now we turn to the quasisplit unitary group $G = G_n$ in n-variables for E/F. This can be realized in such a way that

$$G_n(R) := \{ g \in \mathbb{M}_n(R \otimes_F E)^\times \mid \theta_n(g) = \sigma(g) \},$$

for any abelian F-algebra R. Here $\theta_n(g) := \mathrm{Ad}(I_n){}^t g^{-1}$ with

$$I_n := \begin{pmatrix} & & & 1 \\ & & -1 & \\ & \cdot^{\cdot^{\cdot}} & & \\ (-1)^{n-1} & & & \end{pmatrix}.$$

The L-group ${}^L G_n = \widehat{G_n} \rtimes_{\rho_{G_n}} W_F$ is given by $\widehat{G_n} = GL(n,\mathbb{C})$ and

$$\rho_{G_n}(w) = \begin{cases} \mathrm{id} & \text{if } w \in W_E, \\ \theta_n & \text{otherwise.} \end{cases}$$

Definition 2.2. *An A-parameter for G is a homomorphism $\phi : \mathcal{L}_F \times SL(2,\mathbb{C}) \to {}^L G$ such that*

(BC) $\phi_E : \mathcal{L}_E \times SL(2,\mathbb{C}) \overset{\phi}{\to} {}^L G_n \overset{1st. proj.}{\to} GL(n,\mathbb{C})$ coincides with ϕ_E^H for some A-parameter ϕ^H for H_n.

Two A-parameters are equivalent *if they are $\widehat{G}$-conjugate. Let* $\Psi(G)$ *be the set of equivalence classes of A-parameters for G. For an A-parameter ϕ, we write $S_\phi(G)$ for the centralizer of* $\phi(\mathcal{L}_F \times SL(2,\mathbb{C}))$ *in $\widehat{G}$, and $\mathcal{S}_\phi(G)$ for the group of connected components of* $S_\phi(G)/Z(\widehat{G})^{\mathrm{Gal}(\bar{F}/F)}$. $\phi \in \Psi(G)$ *is called* elliptic *if the identity component $S_\phi(G)^0$ of $S_\phi(G)$ is contained in* $Z(\widehat{G})^{\mathrm{Gal}(\bar{F}/F)}$. *We write $\Psi_0(G)$ for the subset elliptic classes in $\Psi(G)$. An elliptic ϕ is of* CAP-type *if $\phi|_{SL(2,\mathbb{C})}$ is non-trivial. We write $\Psi_{\mathrm{CAP}}(G)$ for the set of classes of CAP-type in $\Psi_0(G)$.*

An elementary exercise in representation theory shows that each $\phi \in \Phi_0(G_n)$ can be written as

$$\phi_E \simeq \bigoplus_{i=1}^{r} \xi_i \cdot \varphi_{i,E} \otimes \rho_{d_i} \qquad (2.2)$$

where,

- $\varphi_i \in \Psi(G_{m_i})$ is such that $\varphi_{i,E}|_{\mathcal{L}_E}$ is irreducible;

- ξ_i is an idele class character of E such that $\xi_i|_{\mathbb{A}^\times} = \omega_{E/F}^{n-d_i-m_i+1}$. Here $\omega_{E/F}$ is the quadratic character of $\mathbb{A}^\times/F^\times$ associated to E/F by the classfield theory.

- $\xi_i \cdot \varphi_{i,E} \not\simeq \xi_j \cdot \varphi_{j,E}$, $(1 \le i \ne j \le r)$.

Thus it suffices to describe the set

$$\Phi_{\mathrm{st}}(G_m) := \{\varphi \in \Psi_0(G_m) \mid \varphi_E|_{\mathcal{L}_E} \text{ is irreducible}\}.$$

For $\varphi \in \Phi_{\mathrm{st}}(G_m)$, φ_E viewed as a parameter for H_m corresponds to a cuspidal automorphic representation π_E of $H_m(\mathbb{A})$. According to Langlands' functoriality conjecture, the map $\varphi \mapsto \varphi_E$ corresponds to the *standard base change* lifting from $G_m(\mathbb{A})$ to $H_m(\mathbb{A})$ [Rog90]. Hence the description of $\Phi_0(G_m)$ amounts to that of the image of the standard base change. As for this question, the following expectation is well-known.

Conjecture 2.3. *Let π_E be an irreducible cuspidal representation of $H_m(\mathbb{A})$ and $\varphi^H : \mathcal{L}_F \to {}^L H_m$ be its Langlands parameter. Take an idele class character μ of E such that $\mu|_{\mathbb{A}^\times} = \omega_{E/F}$. Then $\varphi_E^H = \varphi_E$ for some $\varphi \in \Phi_{\mathrm{st}}(G_m)$ (i.e., π_E is the standard base change lift of some stable L-packet of $G_m(\mathbb{A})$) if and only if*

(i) $\sigma(\pi_E) := \pi_E \circ \sigma \simeq \pi_E^\vee$ (the contragredient);

(ii) the twisted tensor L-function $L_{\mathrm{Asai}}(s, \mu^{n+1}(\det)\pi_E)$ [Gol94] has a pole at $s = 1$.

Using the base change for $GU_{E/F}(2)$, we deduced the case $m = 2$ of the conjecture from [HLR86, Th.3.12] ([KKa, Cor.3.3]). This avails us to deduce the following description of $\Psi_{\mathrm{CAP}}(G_4)$ from (2.2). Note that this does not involve the hypothetical Langlands group $\mathcal{L}_F$ anymore.

Proposition 2.4. *The set $\Psi_{\mathrm{CAP}}(G_4)$ consists of the following classes. We write η, μ for typical idele class characters of E such that $\eta|_{\mathbb{A}^\times} = 1$, $\mu|_{\mathbb{A}^\times} = \omega_{E/F}$, respectively.*

Name	ϕ_E	$\{(d_i, m_i)\}$	$\mathcal{S}_\phi(G)$
(1.a) ϕ_η	$\eta \otimes \rho_4$	$\{(4,1)\}$	$\{1\}$
(1.b) $\phi_{\pi_E, \mu}$	$\mu \varphi_{\pi_E} \otimes \rho_2$	$\{(2,2)\}$	$\{1\}$
(2.a) ϕ_μ	$(\mu \otimes \rho_3) \oplus \mu'$	$\{(3,1), (1,1)\}$	$\mathbb{Z}/2\mathbb{Z}$
(2.b) $\phi_{\pi_E, \eta}$	$(\eta \otimes \rho_2) \oplus \varphi_{\pi_E}$	$\{(2,1), (1,2)\}$	$\mathbb{Z}/2\mathbb{Z}$
(2.c) $\phi_{\underline{\eta}}$	$(\eta \otimes \rho_2) \oplus (\eta' \otimes \rho_2)$	$\{(2,1), (2,1)\}$	$\mathbb{Z}/2\mathbb{Z}$
(2.d) $\phi_{\underline{\eta}, \mu}$	$(\eta \otimes \rho_2) \oplus \mu \oplus \mu'$	$\{(2,1), (1,1), (1,1)\}$	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$

Here, in (1.b), (2.b), π_E runs over the set of irreducible cuspidal automorphic representation of $H_2(\mathbb{A})$ such that $\sigma(\pi_E) \simeq \pi_E^\vee$ and $L_{\text{Asai}}(s, \pi_E)$ is holomorphic at $s = 1$. In (2.a) $\underline{\mu} = (\mu, \mu')$ where μ' can be μ. In (2.c) $\underline{\eta} = (\eta, \eta')$ modulo permutation, with $\eta \neq \eta'$. Finally, in (2.d) $\underline{\mu} = (\mu, \mu')$ modulo permutation and $\mu \neq \mu'$.

3 Review of the local theory

Let ϕ be an A-parameter for $G = G_4$. By restriction, we obtain the local component

$$\phi_v : \mathcal{L}_{F_v} \times SL(2, \mathbb{C}) \to {}^L G_v$$

of ϕ at each place v of F. Here the local Langlands group $\mathcal{L}_{F_v}$ is given by $W_{F_v} \times SU(2, \mathbb{R})$ if v is non-archimedean and W_{F_v} otherwise [Kot84, §12]. ${}^L G_v$ is the L-group of the scalar extension $G_v = G \otimes_F F_v$. Arthur's local conjecture, among other things, associates to each ϕ_v a finite set $\Pi_{\phi_v}(G_v)$ of isomorphism classes of irreducible unitarizable representations of $G(F_v)$, called an A-packet. At all but finite number of v, $\Pi_{\phi_v}(G_v)$ is expected to contain a unique unramified element π_v^1. Using such elements, we can form the global A-packet associated to ϕ:

$$\Pi_\phi(G) := \left\{ \bigotimes_v \pi_v \ \middle| \ \begin{array}{ll} \text{(i)} & \pi_v \in \Pi_{\phi_v}(G_v), \ \forall v; \\ \text{(ii)} & \pi_v = \pi_v^1, \ \forall' v \end{array} \right\}.$$

It is conjectured that any CAP-form on G is contained in $\Pi_\phi(G)$ for some $\phi \in \Psi_{\text{CAP}}(G)$. Thus our problem can be stated as follows.

Problem 3.1. *(i) Describe $\Pi_\phi(G)$ (or equivalently, its local components $\Pi_{\phi_v}(G_v)$).*
(ii) Describe the multiplicity of each $\pi \in \Pi_\phi(G)$ in $L^2_{\text{disc}}(G(F)\backslash G(\mathbb{A}))$. (Note $\mathfrak{A}_G = \{1\}$ for the unitary group G.)

Example 3.2. *The A-packets associated to some of the parameters listed in Prop.2.4 can be easily described.*

(1.a) For ϕ_η, we have $\Pi_\phi(G) = \{\eta_G := \eta_u(\det)\}$, where $\eta_u : U_{E/F}(1, \mathbb{A}) \ni z/\sigma(z) \mapsto \eta(z) \in \mathbb{C}^\times$.

(1.b) For $\phi_{\pi_E, \mu}$, $\Pi_\phi(G)$ consists of the unique irreducible quotient $J_P^G(\mu(\det)\pi_E | \det |_{\mathbb{A}_E}^{1/2})$, of the global parabolically induced representation from the Siegel parabolic subgroup $P = MU$.

(2.a) For ϕ_μ, $\Pi_\phi(G)$ consists of the θ-lifting $\theta_\mu((\mu/\mu')_u, W)$ of the automorphic character $(\mu/\mu')_u$ of $U_{E/F}(1, \mathbb{A})$.

In particular, no CAP forms occur in these cases. All of these representations are known to occur in the residual discrete spectrum [Kon98]. Hence from now on, we concentrate on the rest cases (2.b–d).

Local A-packets Now let E/F be a quadratic extension of non-archimedean local fields of characteristic zero. We also have corresponding results in the archimedean case, but we need some extra notation to state them. Let ϕ be (local analogue of) an A-parameter of type (2.b–2.d). In [KKa], we have constructed $\Pi_\phi(G)$ by the local θ-correspondence. Let us briefly recall the construction. First note that ϕ can be written in the form

$$\phi_E = \varphi_{\pi_E} \oplus (\eta \otimes \rho_2). \tag{3.1}$$

Here $\varphi_{\pi_E} : \mathcal{L}_E \to GL(2, \mathbb{C})$ corresponds to an irreducible admissible representation π_E of $H_2(F) = GL(2, E)$ under the local Langlands correspondence [HT01], [Kut80]. Also notice that $\mathcal{S}_\phi(G) = \mathbb{Z}/2\mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

For a 2-dimensional hermitian space $(V, (\cdot, \cdot))$, we write G_V for its unitary group. $(W, \langle \cdot, \cdot \rangle) = (W_n, \langle \cdot, \cdot \rangle_n)$ denotes the hyperbolic skew-hermitian space of dimension $2n$, so that $G = G_4$ is the unitary group G_{W_2} of W_2. Fix a character pair $\underline{\xi} = (\mathbb{1}, \eta)$ of $E^\times$ such that $\eta|_{F^\times} = \mathbb{1}$, and a non-trivial character $\psi_F : F \to \mathbb{C}^\times$. These specify the Weil representation $\omega_{V,W,\underline{\xi}} = \omega_{W,\mathbb{1}} \times \omega_{V,\eta}$ of $G_V(F) \times G_W(F)$. As usual, this determines the *local θ-correspondence*

$$\mathscr{R}(G_V, \omega_{W,\mathbb{1}}) \ni \begin{array}{ccc} \pi_V & \longmapsto & \theta_{\underline{\xi}}(\pi_V, W) \\ \theta_{\underline{\xi}}(\pi_W, V) & \longleftarrow & \pi_W \end{array} \in \mathscr{R}(G_W, \omega_{V,\eta})$$

between certain subsets $\mathscr{R}(G_V, \omega_{W,\mathbb{1}}) \subset \Pi(G_V(F))$, $\mathscr{R}(G_W, \omega_{V,\eta}) \subset \Pi(G_W(F))$. Here $\Pi(G_V(F))$ denotes the set of isomorphism classes of irreducible admissible representations of $G_V(F)$.

Definition 3.3. *In the notation of 3.1, let $\Pi_{\eta\pi_E^\vee}(G_V)$ be the L-packet of $G_V(F)$ whose standard base change to $H_2(F)$ is $\eta(\det)\pi_E^\vee$. (Empty if V is anisotropic and π_E is in the principal series.) We define*

$$\Pi_\phi(G) := \coprod_V \theta_{\underline{\xi}}(\Pi_{\eta\pi_E^\vee}(G_V), W),$$

where V runs over the set of isometry classes of 2-dimensional hermitian space over E.

4 Presentation of the problem

We now go back to the global setting. Let ϕ be an A-packet of type (2.b–d) in Prop.2.4. Having defined the local A-packets, we have the global packet $\Pi_\phi(G) = \bigotimes_v \Pi_{\phi_v}(G_v)$. In the present case, the multiplicity formula in Arthur's conjecture is stated as follows.

Conjecture 4.1. *There exists a pairing $\langle \cdot, \cdot \rangle : \mathcal{S}_{\phi_v}(G_v) \times \Pi_{\phi_v}(G_v) \to \{\pm 1\}$ such that the multiplicity of $\pi = \bigotimes_v \pi_v \in \Pi_\phi(G)$ in $L^2_{\mathrm{disc}}(G(F)\backslash G(\mathbb{A}))$ is given by*

$$m(\pi) := \frac{1}{|\mathcal{S}_\phi(G)|} \sum_{s \in \mathcal{S}_\phi(G)} \epsilon_\phi(s) \prod_v \langle s, \pi_v \rangle.$$

Here, ϵ_ϕ is the sign character of the first $\mathbb{Z}/2\mathbb{Z}$ of $\mathcal{S}_\phi(G)$ if $\varepsilon(1/2, \pi_E \times \eta^{-1}) = -1$, and is the trivial character otherwise.

Our main result states that $m(\pi)$ is equal to or larger than the right hand side of the conjectural formula. But this makes sense only after the pairing $\langle \cdot, \cdot \rangle : \mathcal{S}_{\phi_v}(G_v) \times \Pi_{\phi_v}(G_v) \to \{\pm 1\}$ is described.

Pairing in the stable case The pairing $\langle \cdot, \cdot \rangle : \mathcal{S}_{\phi_v}(G_v) \times \Pi_{\phi_v}(G_v) \to \{\pm 1\}$ is given locally as the notation indicates. Thus we may go back to the local non-archimedean situation of §3. First we recall some basic requirements on $\Pi_\phi(G)$ from [Art89].

(i) For $\phi \in \Psi(G)$, we have a Langlands' parameter

$$\varphi_\phi : \mathcal{L}_F \ni w \longmapsto \phi\left(w, \begin{pmatrix} |w|_F^{1/2} & \\ & |w|_F^{-1/2} \end{pmatrix}\right) \rtimes p_{W_F}(w) \in {}^L G,$$

where $| \cdot |_F$ is the transport of the module of F by the reciprocity isomorphism $F^\times \xrightarrow{\sim} W_{F,\mathrm{ab}}$ (or its composite with $\mathcal{L}_F \xrightarrow{p_{W_F}} W_F \twoheadrightarrow W_{F,\mathrm{ab}}$). Then the associated L-packet $\Pi_{\varphi_\phi}(G)$ should be contained in $\Pi_\phi(G)$.

(ii) More precisely, there exists a parabolic subgroup $P_\phi = M_\phi U_\phi$ such that $\phi(\mathcal{L}_F) \subset {}^L M_\phi$ and

$$\mu_\phi : W_F \ni w \longmapsto \phi\left(1, \begin{pmatrix} |w|_F^{1/2} & \\ & |w|_F^{-1/2} \end{pmatrix}\right) \in {}^L G$$

is a P_ϕ-dominant element of $\mathfrak{a}_{M_\phi}^* = (\mathrm{Lie}\,\mathfrak{A}_{M_\phi})^*$. Then $\Pi_{\varphi_\phi}(G) = \{J_{P_\phi}^G(\pi \otimes e^{\mu_\phi}) \mid \pi \in \Pi_{\phi|_{\mathcal{L}_F}}(M_\phi)\}$, where $J_{P_\phi}^G(\pi \otimes e^{\mu_\phi})$ is the "Langlands' quotient[1]" of the standard parabolically induced representation $I_{P_\phi}^G(\pi \otimes e^{\mu_\phi})$. Now let us fix a Borel subgroup $B = TU$ and a non-degenerate character ψ_U of $U(F)$. According to the *generic packet conjecture*, $\Pi_{\phi|_{\mathcal{L}_F}}(M_\phi)$ contains a unique generic representation π_1 with respect to $\psi_U|_{(U \cap M_\phi)(F)}$. Then, the pairing between $\Pi_\phi(G)$ and $\Pi(\mathcal{S}_\phi(G))$ should be chosen in such a way that $\langle J_{P_\phi}^G(\pi_1 \otimes e^{\mu_\phi}), \cdot \rangle$ is the trivial character of $\mathcal{S}_\phi(G)$.

(iii) The following diagram should commute.

$$
\begin{array}{ccc}
\Pi_{\varphi_\phi}(G) \ni J_{P_\phi}^G(\tau \otimes e^{\mu_\phi}) & \longrightarrow & \langle \cdot, \tau \rangle \in \Pi(\mathcal{S}_{\phi|_{\mathcal{L}_F}}(M_\phi)) \\
\text{inclusion} \downarrow & & \downarrow \text{inclusion} \\
\Pi_\phi(G) \ni \pi & \longrightarrow & \langle \cdot, \pi \rangle \in \Pi(\mathcal{S}_\phi(G)).
\end{array}
$$

Going back to ϕ of type (2.b–d), the construction of the local packet $\Pi_\phi(G)$ involved the following, so-called ε-*dichotomy* property of the local θ-correspondence. Recall that there are only two isometry classes of 2-dimensional hermitian space V over E. They are classified by the signature $\omega_{E/F}(- \det V)$.
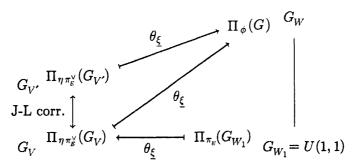
Theorem 4.2 ([KKa] Th.6.4). *We adopt the notation of Def.3.3. The local θ-correspondent* $\theta_{\underline{\xi}}(\Pi_{\pi_E}(G_2), V)$ *of the L-packet* $\Pi_{\pi_E}(G_2)$ *to* $G_V(F)$ *is the L-packet* $\Pi_{\eta \pi_E^\vee}(G_V)$ *if*

$$\varepsilon(1/2, \pi_E \times \eta^{-1}, \psi_E)\omega_{\Pi_{\pi_E}(G_2)}(-1) = \omega_{E/F}(- \det V),$$

---

[1]Again not precisely, because π is not always tempered in our definition of A-parameters.

and is zero otherwise. Here $\psi_E := \psi_F \circ \mathrm{Tr}_{E/F}$ *and* $\varepsilon(s, \pi_E \times \eta^{-1}, \psi_E)$ *is the Jacquet-Langlands local constant of* $\pi_E \times \eta^{-1}$. *Also* $\omega_{\Pi_{\pi_E}(G_2)}$ *denotes the common central character of the members of* $\Pi_{\pi_E}(G_2)$.

If we write V for the (isometry class of the) 2-dimensional hermitian space over E satisfying the condition of Th.4.2 and V' for the other one, the construction of $\Pi_\phi(G)$ is summerized in the following diagram.

Moreover, the *induction principle* of the local θ-correspondence [Kud86], [MVW87, Ch.3] shows that $\Pi_{\varphi_\phi}(G) = \theta_{\underline{\xi}}(\Pi_{\eta\pi_E^\vee}(G_V), W_2)$. This together with the requirement (iii) above yield the following.

Theorem 4.3. *Suppose* $\Pi_{\pi_E}(G_2)$ *is stable, i.e., consists of a single element, so that* $S_\phi(G) \simeq \mathbb{Z}/2\mathbb{Z}$. *Then we have*

$$\langle \theta_{\underline{\xi}}(\Pi_{\eta\pi_E^\vee}(G_V), W), \cdot \rangle = \mathrm{sgn}, \quad \langle \theta_{\underline{\xi}}(\Pi_{\eta\pi_E^\vee}(G_{V'}), W), \cdot \rangle = 1,$$

where V *and* V' *are labeled as above.*

5 Endoscopy for $U_{E/F}(2)$

It remains to consider the case where $\Pi_{\pi_E}(G_2)$ is *endoscopic*. This is the case (2.d) in Prop.2.4 (see [KKb, 4.3]):

$$\varphi_E = \mu \oplus \mu', \quad \pi_E = I(\mu \otimes \mu').$$

We write $\Pi_{\underline{\mu}}(G_V) := \Pi_{\pi_E}(G_V) = \{\pi_V(\underline{\mu})^{\pm}\}$ with $\underline{\mu} = (\mu, \mu')$.

We briefly recall the endoscopic lifting for G_V from [KKb]. The unique non-trivial *elliptic endoscopic data* for G_V is $(H, {}^L H, s, \xi)$, where $H = U_{E/F}(1)^2$, $s = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $\xi : {}^L H \hookrightarrow {}^L G_2$ is the L-embedding given by

$$\widehat{H} \ni (z_1, z_2) \longmapsto \begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix} \times 1$$

$$\xi : \quad W_E \ni w \longmapsto \begin{pmatrix} \mu_0(w) & 0 \\ 0 & \mu_0'(w) \end{pmatrix} \times w \ \in {}^L G_2.$$

$$w_\sigma \longmapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \rtimes w_\sigma$$

Here $\underline{\mu}_0 = (\mu_0, \mu_0')$ are characters of $E^\times$ such that $\mu_0|_{F^\times} = \mu_0'|_{F^\times} = \omega_{E/F}$. The isomorphism class of the data is independent of $\underline{\mu}_0$.

We fix a generator δ of E over F such that $\mathrm{Tr}_{E/F}(\delta) = 0$, and take $\varepsilon \in F^\times \smallsetminus \mathrm{N}_{E/F}(E^\times)$. We may realize $(V, (\cdot, \cdot))$ as $V = E^2$ and

$$
(v, v') = \begin{cases}
{}^t\sigma(v) \begin{pmatrix} 0 & (2\delta)^{-1} \\ -(2\delta)^{-1} & 0 \end{pmatrix} v' & \text{if } V \text{ is hyperbolic,} \\[2em]
{}^t\sigma(v) \begin{pmatrix} -\varepsilon & 0 \\ 0 & 1 \end{pmatrix} v' & \text{if } V \text{ is anisotropic.}
\end{cases}
$$

Then we fix an embedding

$$
\eta_V : H \ni \gamma_H = (zz', z\sigma(z')) \longmapsto \begin{cases}
z \begin{pmatrix} x & -y\Delta \\ y & x \end{pmatrix} \in G_V & \text{if } V \text{ is hyperbolic,} \\[2em]
\begin{pmatrix} zz' & 0 \\ 0 & z\sigma(z') \end{pmatrix} \in G_V & \text{if } V \text{ is anisotropic.}
\end{cases}
$$

Here, each element $\gamma_H \in H$ is written as $(zz', z\sigma(z'))$ for some $z, z' \in \mathrm{Res}_{E/F}\mathbb{G}_m$ with $\mathrm{N}_{E/F}(z) = \mathrm{N}_{E/F}(z')^{-1}$ and $\Delta := -\delta^2$. These data together with the non-trivial character ψ_F in §3 determines the *Langlands-Shelstad transfer factor* $\Delta_V : H(F)_{G\text{-reg}} \times G_V(F)_{\text{reg}} \to \mathbb{C}$. This is characterized by the formula

$$
\Delta_V(\gamma_H, \eta_V(\gamma_H)) = \lambda(E/F, \psi_F)\omega_{E/F}\left(\frac{z' - \sigma(z')}{-2\delta}\right)\mu_0(x_1)\mu_0'(x_2)\frac{|z' - \sigma(z')|_E^{1/2}}{|z'|_E^{1/2}}. \tag{5.1}
$$

Here $\lambda(E/F, \psi_F)$ is *Langlands' λ-factor* for E/F with respect to ψ_F, and we have written $zz' = x_1/\sigma(x_1)$, $z\sigma(z') = x_2/\sigma(x_2)$ for some $x_1, x_2 \in E^\times$.

Fact 5.1 (Labesse-Langlands, [KKb] Ch.3). *For any $f \in C_c^\infty(G_V(F))$,*

$$
f^H : H(F)_{G\text{-reg}} \ni \gamma_H \longmapsto \sum_{\substack{\gamma \in \mathrm{Ad}(G_V(\bar{F}))\eta_V(\gamma_H) \cap G_V(F) \\ \text{mod. } G_V(F)\text{-conj.}}} \Delta_V(\gamma_H, \gamma)O_\gamma(f) \in \mathbb{C}
$$

extends to an element of $C_c^\infty(H(F))$. Here $O_\gamma(f)$ denotes the orbital integral of f at γ.

The *endoscopic lifting* which we need is the adjoint map of $f \mapsto f^H$ from the space of invariant distributions on $G(F)$ to that on $H(F)$. In particular, the L-packet $\Pi_{\underline{\mu}}(G_V) = \{\pi_V(\underline{\mu})^\pm\}$ is labeled in such a way that

$$
\mathrm{tr}\,\pi_V(\underline{\mu})^+(f) - \mathrm{tr}\,\pi_V(\underline{\mu})^-(f) = \left((\mu/\mu_0)_u \otimes (\mu'/\mu_0')_u\right)(f^H)
$$

holds. If V is hyperbolic in the realization (5.1), then $\pi_V(\underline{\mu})^+$ is the unique generic member in $\Pi_{\underline{\mu}}(G_V)$ with respect to the character [KKb, Prop.4.8]

$$
\psi_{U_2} : U_2(F) \ni \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \longmapsto \psi_F(b) \in \mathbb{C}^\times.
$$

(This is a consequence of the *Whittaker normalization* of the transfer factor (5.1).) Combining these with the *seesaw duality* [Kud84]

$$
\begin{array}{ccc}
G_V & & U_{E/F}(1) \times U_{E/F}(1) \\
\Big| \eta_V & \times & \Big| \\
U_{E/F}(1) \times U_{E/F}(1) & & U_{E/F}(1)
\end{array}
$$

we obtain a *Saito-Tunnell type* character formula for $\pi_V(\underline{\mu})^{\pm}$.

Theorem 5.2. *For a character μ such that $\mu|_{F^\times} = \omega_{E/F}$, we introduce a sign $\varepsilon_{\psi_F}(\mu) := \varepsilon(1/2, \mu, \psi_E)\mu(-\delta)$.*
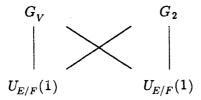*(i) If V is hyperbolic, the character (function) $\Theta_{\pi_V(\underline{\mu})^{\pm}}$ of $\pi_V(\underline{\mu})^{\pm}$ is given by (respecting signs)*

$$
\Theta_{\pi_V(\underline{\mu})^{\pm}} \circ \eta_V = \sum_{\eta|_{F^\times}=1} \frac{(1 \pm \varepsilon_{\psi_F}(\eta\mu^{-1}))(1 \pm \varepsilon_{\psi_F}(\eta\mu'^{-1}))}{4}(\mu\mu'\eta)_u \otimes \eta_u.
$$

(ii) If V is anisotropic, we have (respecting signs)

$$
\Theta_{\pi_V(\underline{\mu})^{\pm}} \circ \eta_V = \sum_{\eta|_{F^\times}=1} \frac{(1 \mp \varepsilon_{\psi_F}(\eta\mu^{-1}))(1 \pm \varepsilon_{\psi_F}(\eta\mu'^{-1}))}{4}(\mu\mu'\eta)_u \otimes \eta_u.
$$

Of course, these formulae indicates various interesting speculations. But this is not a place to discuss them. We only remark that the same formulae are also valid in the archimedean case. Now we combine the theorem with the seesaw duality

$$
\begin{array}{ccc}
G_V & & G_2 \\
\Big| & \times & \Big| \\
U_{E/F}(1) & & U_{E/F}(1)
\end{array}
$$

to obtain the following.

Theorem 5.3 (Howe duality for $\Pi_{\underline{\mu}}(G_V)$). *We write $\Pi_{\underline{\mu}}(G_2) = \{\pi(\underline{\mu})^{\pm}\}$ as above. Suppse $(V, (\cdot, \cdot))$ satisfies the condition of Th.4.2. Then we have $\theta_{\underline{\xi}}(\pi(\underline{\mu})^{\pm}, V) = \pi_V(\eta\underline{\mu}^{-1})^{\pm\varepsilon_{\psi_F}(\mu)}$, where $\eta\underline{\mu}^{-1} := (\eta\mu^{-1}, \eta\mu'^{-1})$.*

Pairing in the endoscopic case   We now define the pairing $\langle \cdot, \cdot \rangle : \Pi_\phi(G) \times \mathcal{S}_\phi(G) \to \{\pm 1\}$ for ϕ in Prop.2.4 (2.d). We retain the notation of the above discussion.

Definition 5.4. *Recall that $\mathcal{S}_\phi(G)$ for $\phi_E \simeq (\eta \otimes \rho_2) \oplus \mu \oplus \mu'$ is $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. The pairing is defined by*

$$
\langle \cdot, \theta_{\underline{\xi}}(\pi_V(\eta\underline{\mu}^{-1})^{\pm}, W_2) \rangle := \mathrm{sgn}^{(1-\varepsilon_{V,\eta}(\underline{\mu}))/2} \otimes \mathrm{sgn}^{(1 \mp \varepsilon_{\psi_F}(\mu))/2},
$$

where $\varepsilon_{V,\eta}(\underline{\mu}) := \varepsilon_{\psi_F}(\eta\mu^{-1})\varepsilon_{\psi_F}(\eta\mu'^{-1})\omega_{E/F}(-\det V)$.

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6 Automorphic forms

We now go back to the global situation, and consider the A-parameters ϕ of type (2.b)–(2.d) in Prop.2.4. As is announced in §4, our principal result is the following.

Theorem 6.1. *Each* $\pi = \otimes_v \pi_v \in \Pi_\phi(G)$ *occurs in the discrete spectrum* $L^2_{\mathrm{disc}}(G(F)\backslash G(\mathbb{A}))$ *with the multiplicity at least:*

$$\frac{1}{|\mathcal{S}_\phi(G)|} \sum_{s \in \mathcal{S}_\phi(G)} \epsilon_\phi(s) \prod_v \langle \pi_v, s \rangle. \tag{6.1}$$

Here, ϵ_ϕ *is the sign character of the first* $\mathbb{Z}/2\mathbb{Z}$ *of* $\mathcal{S}_\phi(G)$ *if* $\varepsilon(1/2, \pi_E \times \eta^{-1}) = -1$, *and is the trivial character otherwise.*

The proof involves the global θ-correspondence between $G_V(\mathbb{A})$ and $G(\mathbb{A})$ and the description of the discrete spectrum of $G_V(\mathbb{A})$ [KKb].

Remark 6.2. *Those* $\pi \in \Pi_\phi(G)$ *such that* $\varepsilon(1/2, \pi_E \times \eta^{-1}) = 1$ *and* $\langle \pi_v, \cdot \rangle$ *are trivial on the first* $\mathbb{Z}/2\mathbb{Z} \subset \mathcal{S}_{\phi_v}(G_v)$ *at all* v *are the residual discrete automorphic representations of* $G(\mathbb{A})$ *[Kon98]. All the other* π *with non-zero* (6.1) *are CAP autmorphic forms.*

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