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An explicit arithmetic formula for the Fourier coefficients of Siegel-Eisenstein series of degree two with square free odd level

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1 Introduction

We give an explicit arithmetic formula for the Fourier coefficients of the Siegel-Eisenstein series $E_{k,\overline{\chi}}^{(2)}$ of degree two on the congruence subgroup $\Gamma_0^{(2)}(N)$ with a square free odd level N, where k is the weight and χ is a primitive Dirichlet character mod N. If the level N exceeds one, then any explicit formula for the Fourier coefficients of $E_{k,\overline{\chi}}^{(2)}$ was not available as far as the author knows.

We state the main result precisely. Let H_2 be the Siegel upper half-space of degree two and Z be the variable on H_2. Then for any integer $k>3$, the Siegel-Eisenstein series $E_{k,\overline{\chi}}^{(2)}$ with level N is defined by

$$E_{k,\overline{\chi}}^{(2)}(Z)=\sum_{\{C,D\}}\overline{\chi}(\det D)\det(CZ+D)^{-k},$$

where the summation is taken over all representatives $\gamma=\begin{pmatrix} A & B \\ C & D \end{pmatrix}\in\Gamma_{\infty}^{(2)}\backslash\Gamma_0^{(2)}(N)$ with

$$\Gamma_0^{(2)}(N)=\{\gamma\in Sp_2(\mathbf{Z}); C\equiv O_2 \pmod N\},\quad \Gamma_{\infty}^{(2)}=\{\gamma\in Sp_2(\mathbf{Z}); C=O_2\},$$

and χ is a Dirichlet character mod N such that $\chi(-1)=(-1)^k$.

Theorem 1. *Let $k>3$ be an integer, N be a square free odd natural number exceeds one and χ be a primitive Dirichlet character mod N satisfying $\chi(-1)=(-1)^k$.*

Then for any positive definite half integral symmetric matrix T of size two, the T-th Fourier coefficient $A(T,E_{k,\overline{\chi}}^{(2)})$ of the Siegel-Eisenstein series $E_{k,\overline{\chi}}^{(2)}$ with level N is given by

$$A(T,E_{k,\overline{\chi}}^{(2)})=\frac{(-2\pi i)^k\tau_N(\overline{\chi})}{N^k\Gamma(k)L(k,\overline{\chi})}\sum_{d|e(T)}\chi(d)d^{k-1}e_{\overline{\chi}}^{\infty}\left(\frac{-\det 2T}{d^2}\right),$$

where $\tau_N(\overline{\chi})$ *is the Gauss sum* $\tau_N(\overline{\chi}) = \sum_{r=1}^{N} \overline{\chi}(r)e^{2\pi i r/N}$, $\Gamma(s)$ *is the Gamma function,* $L(s,\overline{\chi})$ *is the Dirichlet L-function of* $\overline{\chi}$, $e(T) = (n,r,m)$ *is the greatest common divisor of* n,r,m *for* $T = \begin{pmatrix} n & r/2 \\ r/2 & m \end{pmatrix}$, *and* $e^{\infty}_{\overline{\chi}}(D)$ *has the form*

$$e^{\infty}_{\overline{\chi}}(D) = \frac{\pi^{k-1/2}\overline{\chi}(-4)}{i^k 2^{k-2}\Gamma(k-1/2)}|D|^{k-3/2}\frac{L(k-1,\chi_K\overline{\chi})}{L(2k-2,\overline{\chi}^2)}$$
$$\times \prod_{\text{prime } p|N}\left\{\sum_{e=1}^{1+ord_p D}\frac{\overline{\chi^*_p}(p^e)}{p^{(k-1/2)e}}\epsilon^3_{p^e}C^{\infty}_{\overline{\chi},p}(D,p^e)\right\}$$
$$\times \sum_{d|f}\mu(d)\chi_K(d)\overline{\chi}(d)d^{1-k}\sigma_{3-2k,\overline{\chi}^2}(f/d).$$

Here we use the following notations. Let ord_pD *be the integer such that* p^{ord_pD} *is the exact power of* p *dividing* D, $\mu(d)$ *be the Mobius function,* $\sigma_{s,\overline{\chi}^2}(f)$ *is defined by* $\sigma_{s,\overline{\chi}^2}(f) = \sum_{d|f}\overline{\chi}^2(d)d^s$, *the natural number* f *is defined by* $D = D_K f^2$ *with the discriminant* D_K *of* $K = \mathbf{Q}(\sqrt{D})$ *and* $\chi_K(*) = (\frac{D_K}{*})$ *is the Kronecker symbol of* K. *Let* χ_p *be the primitive characters mod* p *so that* $\chi = \prod_{\text{prime } p|N}\chi_p$. *Then* χ^*_p *is defined by* $\chi^*_p = \prod_{\text{prime } q|\frac{N}{p}}\chi_q$. *We put* $\epsilon_d = 1$ *or* i *according to* $d \equiv 1 \pmod 4$ *or* $3 \pmod 4$, *and* $C^{\infty}_{\overline{\chi},p}(D,p^e)$ *are explicitly given as follows.*

Let $\tau_p(\chi) = \sum_{r=1}^{p}\chi(r)e^{2\pi i r/p}$ *be the Gauss sum,* $\left(\frac{*}{p}\right)$ *be the Legendre symbol and put* $m = ord_pD$. *Then we have*

(a)*for* $e \le m$,

$$C^{\infty}_{\overline{\chi},p}(D,p^e) = \begin{cases} p^{e-1}(p-1), & \chi_p = \left(\frac{*}{p}\right) \text{ and } e \text{ is odd,} \\ 0, & \text{otherwise.} \end{cases}$$

(b)*for* $e = m+1$,

$$C^{\infty}_{\overline{\chi},p}(D,p^e) = \chi_p(D/p^m)\left(\frac{D/p^m}{p}\right)^{m+1}p^m\tau_p(\overline{\chi_p}\left(\frac{*}{p}\right)^{m+1}).$$

(c)*for* $e \ge m+2$, $C^{\infty}_{\overline{\chi},p}(D,p^e) = 0$.

For the Siegel-Eisenstein series on the full Siegel modular group $Sp_2(\mathbf{Z})$ of degree two, equivalently the case level $N = 1$, an explicit formula for its Fourier coefficients was obtained by Maass [12], [13]. His starting point is Siegel's formula which expresses the Fourier coefficients as an infinite product of the local densities of quadratic forms over all primes. Then he calculated the local densities explicitly to get his formula. For the case level $N > 1$ we cannot proceed by the same way as Maass, since Siegel type formula does not hold for the Fourier coefficients of the Siegel-Eisenstein series $E^{(2)}_{k,\overline{\chi}}$ on the congruence subgroup $\Gamma^{(2)}_0(N)$, especially for the Euler p-factors with primes p dividing the level N.

There is another proof for Maass' formula due to Eichler and Zagier (see corollary 2 [6] p.80). They showed that the Maass lift of the Jacobi Eisenstein series on $SL_2(\mathbf{Z}) \ltimes \mathbf{Z}^2$ equals the Siegel-Eisenstein series on $Sp_2(\mathbf{Z})$. Our formula for the case of the congruence subgroup $\Gamma_0^{(2)}(N)$ given in Theorem 1 follows from an analogous result to this fact. In fact, we will show that the Maass lift $\mathcal{M}E_{k,1,\overline{\chi}}^{\infty}$ of the Jacobi Eisenstein series $E_{k,1,\overline{\chi}}^{\infty}$ on $\Gamma_0(N) \ltimes \mathbf{Z}^2$ is equal to the Siegel-Eisenstein series $E_{k,\overline{\chi}}^{(2)}$ with level N up to a constant. Eichler and Zagier use the characterization of the Siegel-Eisenstein series on $Sp_2(\mathbf{Z})$ as the unique eigenform of all Hecke operators whose zero-th Fourier coefficient is one. Our method is completely different from Eichler-Zagier's argument. Our main tools are Koecher-Maass series $D^*(f, \mathcal{U}, s)$ of a Siegel modular form f with a Grossencharacter $\mathcal{U}$ and the Roelcke-Selberg spectral decomposition, which are used to formulate the converse theorem for Siegel modular forms [8], [5], [2], [3], [7].

More precisely, we proceed as follows. For the Siegel-Eisenstein series $E_{k,\overline{\chi}}^{(2)}$ with level N and the Maass lift $\mathcal{M}E_{k,1,\overline{\chi}}^{\infty}$ of the Jacobi Eisenstein series $E_{k,1,\overline{\chi}}^{\infty}$ on $\Gamma_0(N) \ltimes \mathbf{Z}^2$, we will show that their Koecher-Maass series with any Grossencharacter are equal up to a constant,

$$D^*(E_{k,\overline{\chi}}^{(2)}, \mathcal{U}, s) = \frac{(-2\pi i)^k \tau_N(\overline{\chi})}{N^k \Gamma(k) L(k, \overline{\chi})} D^*(\mathcal{M}E_{k,1,\overline{\chi}}^{\infty}, \mathcal{U}, s),$$

where $\tau_N(\overline{\chi})$ is the Gauss sum, $\Gamma(s)$ is the Gamma function and $L(s, \overline{\chi})$ is the Dirichlet L-function. Consider

$$F = E_{k,\overline{\chi}}^{(2)} - \frac{(-2\pi i)^k \tau_N(\overline{\chi})}{N^k \Gamma(k) L(k, \overline{\chi})} \mathcal{M}E_{k,1,\overline{\chi}}^{\infty}.$$

We can show that the image ΦF of the Siegel operator Φ is zero. This says that the Fourier expansion of F has only the terms indexed by positive definite half integral symmetric matrices. Let the variable on the Siegel upper half-space be $Z = it^{1/2}W$, where $t > 0$ and W is a positive definite real symmetric matix of size two whose determinant is one. We identify W with the variable τ on the upper half-plane. Then we have the Roelcke-Selberg spectral decomposition of $F_t(W) = F(it^{1/2}W)$. As shown in [8], [3], each spectral coefficient with respect to a Grossencharacter $\mathcal{U}(\tau)$ is the inverse Mellin transform of the Koecher-Maass series $D^*(F, \overline{\mathcal{U}}, s)$. Since the Koecher-Maass series $D^*(F, \overline{\mathcal{U}}, s)$ is zero as we can see from above identity, we conclude that F is zero i.e.

$$E_{k,\overline{\chi}}^{(2)} = \frac{(-2\pi i)^k \tau_N(\overline{\chi})}{N^k \Gamma(k) L(k, \overline{\chi})} \mathcal{M}E_{k,1,\overline{\chi}}^{\infty}.$$

Since the Fourier coefficients of images of the Maass lift can be described easily in terms of the Fourier coefficients of Jacobi form, our formula for the Fourier coefficients of the Siegel-Eisenstein series $E_{k,\overline{\chi}}^{(2)}$ with level N follows from an explicit calculation of the Fourier coefficients of the Jacobi Eisenstein series $E_{k,1,\overline{\chi}}^{\infty}$.

To show the coincidence of two Koecher-Maass series, we calculate each Koecher-Maass series. It is easy for that of the Maass lift. To calculate $D^*(f,\mathcal{U},f)$, we usually need a formula for the Fourier coefficients of f. Since any formula of the Fourier coefficients of the Siegel-Eisenstein series $E_{k,\overline{\chi}}^{(2)}$ does not available, we first calculate the Koecher-Maass series $D^*(F_{k,\overline{\chi}}^{(2)},\mathcal{U},s)$ for the twisted Siegel-Eisenstein series $F_{k,\overline{\chi}}^{(2)}$ defined by

$$F_{k,\overline{\chi}}^{(2)}(Z)=N^{-k}\det Z^{-k}E_{k,\overline{\chi}}^{(2)}(-(NZ)^{-1}).$$

This is possible, since a Siegel type formula holds for the Fourier coefficients of $F_{k,\overline{\chi}}^{(2)}$ and so an explicit formula for the Fourier coefficients of $F_{k,\overline{\chi}}^{(2)}$ is available by the explicit form of the Siegel series due to Katsurada [10]. The resulting formula of $D^*(F_{k,\overline{\chi}}^{(2)},\mathcal{U},s)$ can be seen as the Rankin-Selberg transform of certain automorphic forms on $\Gamma_0(N)$ by the explicit calculation of the Fourier coefficients of the Jacobi Eisenstein series $E_{k,1,\chi}^{0}$ associated with the cusp 0 and the Shimura correspondence for Maass wave forms due to Katok-Sarnak [9] and Duke-Imamoglu [5]. Since we can prove the identity

$$D^*(f,\mathcal{U},k-s)=(-1)^kD^*(f|_k\omega_N^{(2)},\mathcal{U},s),$$

where $f|_k\omega_N^{(2)}(Z)=N^{-k}\det Z^{-k}f(-(NZ)^{-1})$ for any Siegel modular form of weight k on $\Gamma_0^{(2)}(N)$, we get

$$D^*(E_{k,\overline{\chi}}^{(2)},\mathcal{U},k-s)=(-1)^kD^*(F_{k,\overline{\chi}}^{(2)},\mathcal{U},s).$$

Hence we can compute $D^*(E_{k,\overline{\chi}}^{(2)},\mathcal{U},s)$ from the explicit formula of $D^*(F_{k,\overline{\chi}}^{(2)},\mathcal{U},s)$ by the Rankin-Selberg method. We remark that, since involved automorphic forms are not always cuspidal according with Maass wave forms $\mathcal{U}(\tau)$, we cannot use the usual Rankin-Selberg method and we must use the method given in our previous work [14].

2 Jacobi Eisenstein series of index 1 with level N

Let N be a square free odd natural number exceeds one and k be an integer. Let χ be a primitive Dirichlet character mod N such that $\chi(-1)=(-1)^k$. For $G\subset SL_2(\mathbf{R})\ltimes\mathbf{R}^2$, we define

$$G_\infty=\{g\in G;1|_{k,1}g=1\}.$$

For any cusp κ of $\Gamma_0(N)$, we take $g\in SL_2(\mathbf{Z})$ such that

$$g(i\infty)=\kappa. \tag{1}$$

Then we define the Jacobi Eisenstein series of weight k and index 1 associated with cusp κ by

$$E^{\kappa}_{k,1,\chi}(\tau,z) = \sum_{\gamma\in(g\Gamma^J g^{-1})_\infty\backslash g\Gamma^J} \chi(g^{-1}\gamma)1|_{k,1}\gamma, \tag{2}$$

where $\chi(\gamma)$ is defined by

$$\chi(\gamma)=\chi(d), \quad \gamma=\left(\left(\begin{array}{cc} a & b \\ c & d \end{array}\right),(\lambda,\mu)\right)\in\Gamma^J.$$

The author learned this definition of the Jacobi Eisenstein series from Professor Boecherer. This satisfies

$$E^{\kappa}_{k,1,\chi}|_{k,1}\gamma = \overline{\chi}(\gamma)E^{\kappa}_{k,1,\chi}, \tag{3}$$

for all $\gamma\in\Gamma^J$.

For the cusp 0, we take the above g in (1) by $g=\left(\begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array}\right)$ and for the cusp $i\infty$, we take $g=I_2$.

Let

$$E^0_{k,1,\chi}(\tau,z) = \sum_{\substack{D<0,r\in\mathbf{Z}\\ D\equiv r^2 \pmod 4}} e^0_\chi(D)q^{\frac{r^2-D}{4}}\zeta^r \tag{4}$$

be the Fourier development of $E^0_{k,1,\chi}$ and

$$E^{\infty}_{k,1,\overline{\chi}}(\tau,z) = \sum_{r\equiv 0 \pmod 2} q^{\frac{r^2}{4}}\zeta^r + \sum_{\substack{D<0,r\in\mathbf{Z}\\ D\equiv r^2 \pmod 4}} e^{\infty}_{\overline{\chi}}(D)q^{\frac{r^2-D}{4}}\zeta^r \tag{5}$$

be the Fourier development of $E^{\infty}_{k,1,\overline{\chi}}$.

To prove Theorem 1, we need to know the behavior of $E^0_{k,1,\chi}$ at each cusp of $\Gamma_0(N)$. As the set of representatives of non equivalent cusps of $\Gamma_0(N)$, we can take

$$\{i\infty,0\}\bigcup\{1/\mu;1<\mu<N,\mu|N\}, \tag{6}$$

since we assume that N is square free. As the elements of $SL_2(\mathbf{Z})$ which transforms $i\infty$ to the cusp of $\Gamma_0(N)$, we can take

$$\sigma_\infty = I_2, \sigma_0=\left(\begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array}\right), \sigma_\mu=\left(\begin{array}{cc} 1 & \alpha \\ \mu & N\beta/\mu \end{array}\right), \tag{7}$$

where integers α and β are chosen so that $N\beta/\mu-\alpha\mu=1$. For the cusp κ, we will also denote σ_κ instead of the above notations (7) by a trivial identification.

Let

$$E^0_{k,1,\chi}|_{k,1}\sigma_0(\tau,z) = \sum_{r\equiv 0 \pmod 2} q^{\frac{r^2}{4}}\zeta^r + \sum_{\substack{D<0, r\in\mathbf{Z} \\ D\equiv Nr^2 \pmod 4}} a^0_\chi(ND) q^{\frac{Nr^2-D}{4N}}\zeta^r$$

be the Fourier development of $E^0_{k,1,\chi}|_{k,1}\sigma_0$.

The Fourier coefficients $e^{\infty}_{\overline{\chi}}(D)$ of $E^{\infty}_{k,1,\overline{\chi}}$ and $a^0_\chi(ND)$ of $E^0_{k,1,\chi}|_{k,1}\sigma_0$ have the following relation, which is important to prove Theorem 1.

Proposition 1. *One has*

$$e^{\infty}_{\overline{\chi}}(D) = a^0_\chi(N^2D).$$

3 Siegel modular forms with level N and their Koecher-Maass series

Denote by $M_k(\Gamma_0^{(2)}(N),\chi)$ the space of all holomorphic functions f on H_2 which satisfy

$$f|_k M = \chi(\det D) f, \quad M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_0^{(2)}(N).$$

Each $f \in M_k(\Gamma_0^{(2)}(N),\chi)$ has a Fourier expansion of the form

$$f(Z) = \sum_{T\in L_2, T\geq O} A(T)\exp(2\pi i tr(TZ)), \tag{8}$$

where the summation extends over all semi-positive definite half integral symmetric matrices T of size two.

Let $\mathcal{P}_2$ be the set of all positive definite real symmetric matrices of size two and $\mathcal{SP}_2$ be the determinant one surface of $\mathcal{P}_2$. We identify $\mathcal{SP}_2$ with the upper half-plane H_1 by

$$\begin{pmatrix} v^{-1} & -uv^{-1} \\ -uv^{-1} & v^{-1}(u^2+v^2) \end{pmatrix} \to \tau = u + iv. \tag{9}$$

We mean by a Grossencharacter any function $\mathcal{U}$ on H_1 satisfying the following three conditions.

(G-i)$\mathcal{U}(\gamma\tau) = \mathcal{U}(\tau)$ for all $\gamma \in SL_2(\mathbf{Z})$.

(G-ii)$\mathcal{U}(\tau)$ is a C^∞-function on H_1 with respect to $u = \Re\tau, v = \Im\tau$ which verifies a differential equation $\Delta\mathcal{U} = -\lambda\mathcal{U}$ with some $\lambda \in \mathbf{C}$, where $\Delta = v^2(\frac{\partial^2}{\partial v^2} + \frac{\partial^2}{\partial u^2})$ is the Laplacian on H_1.

(G-iii)$\mathcal{U}$ is a moderate growth as $v = \Im\tau$ tends to ∞.

A Grossencharacter is also called a Maass wave form.

We extend a Grossencharacter $\mathcal{U}$ to a function on $\mathcal{P}_2$ by setting

$$\mathcal{U}(T) = \mathcal{U}(\tau_T),$$

where τ_T corresponds to $\det T^{-1/2}T$, in other words $T \in \mathcal{P}_2$ is identified with $\tau_T \in H_1$ by

$$T = \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} \to \tau_T = \frac{-b + i\sqrt{\det 2T}}{2a}.$$

Now for $f \in M_k(\Gamma_0^{(2)}(N), \chi)$ which has a Fourier expansion (8), we define the Koecher-Maass series with a Grossencharacter $\mathcal{U}$ by

$$D(f, \mathcal{U}, s) = \sum_{T \in L_2^+/SL_2(\mathbf{Z})} \frac{A(T)\mathcal{U}(T)}{\epsilon(T) \det T^s}, \tag{10}$$

where L_2^+ is the set of all positive definite half integral symmetric matrices of size two and the summation extends over all $T \in L_2^+$ modulo the usual action $T \to T[U] = {}^tUTU$ of the group $SL_2(\mathbf{Z})$ and $\epsilon(T) = \sharp\{U \in SL_2(\mathbf{Z}); T[U] = T\}$ is the order of the unit group of T.

Let

$$D^*(f, \mathcal{U}, s) = \int_{SL_2(\mathbf{Z})\backslash \mathcal{P}_2} \det Y^s \mathcal{U}(Y) f^{(2)}(i\frac{Y}{\sqrt{N}}) \frac{dY}{\det Y^{3/2}}, \tag{11}$$

where $f^{(2)}$ is defined from (8) by

$$f^{(2)}(Z) = \sum_{T \in L_2^+} A(T) \exp(2\pi i tr(TZ)).$$

If a Grossencharacter $\mathcal{U}$ corresponds to the eigenvalue $-\left(\frac{1}{4} + r^2\right)$ of Δ, then it is known that (see [11])

$$\begin{aligned} & D^*(f, \mathcal{U}, s) && (12) \\ = \; & 2\pi^{1/2} N^s (2\pi)^{-2s} \Gamma(s - 1/4 + ir/2) \Gamma(s - 1/4 - ir/2) D(f, \mathcal{U}, s). \end{aligned}$$

Put

$$\omega_N^{(2)} = \begin{pmatrix} O_2 & -I_2 \\ NI_2 & O_2 \end{pmatrix}. \tag{13}$$

Then $f|_k\omega_N^{(2)}$ belongs to $M_k(\Gamma_0^{(2)}(N), \overline{\chi})$ for $f \in M_k(\Gamma_0^{(2)}(N), \chi)$.

By the similar way as in [1] and Theorem 10 of [3] p.209, we can show

Proposition 2. *For $f \in M_k(\Gamma_0^{(2)}(N), \chi)$, we have*

$$D^*(f, \mathcal{U}, k - s) = (-1)^k D^*(f|_k\omega_N^{(2)}, \mathcal{U}, s).$$

If the Fourier coefficients of $f \in M_k(\Gamma_0^{(2)}(N), \chi)$ satisfy a Maass type relation, then $D(f, \mathcal{U}, s)$ is a convolution product of two Dirichlet series as follows. This result is due to Boecherer (see Satz 3 of [4] p.20).

Proposition 3. *Let $f \in M_k(\Gamma_0^{(2)}(N), \chi)$ has a Fourier expansion* (8). *Suppose that there exists a function c on the set of all negative integers such that*

$$\dot{A}(T) = \sum_{d \mid e(T)} \chi(d) d^{k-1} c\left(\frac{-\det 2T}{d^2}\right), \tag{14}$$

where $e(T) = (n, r, m)$ for $T = \begin{pmatrix} n & r/2 \\ r/2 & m \end{pmatrix}$. Then we have

$$D(f, \mathcal{U}, s) = 2^{2s} L(2s - k + 1, \chi) \sum_{n=1}^{\infty} \frac{c(-n) b(-n) n^{3/4}}{n^s}, \tag{15}$$

where

$$b(-n) = n^{-3/4} \sum_{\substack{T \in L_2^+ / SL_2(\mathbf{Z}) \\ \det 2T = n}} \frac{\mathcal{U}(T)}{\epsilon(T)} \tag{16}$$

and $\epsilon(T)$ is the same as in (10).

The final task in this section is to define the Maass lift $\mathcal{M}$ from the space $J_{k,1}(\Gamma_0(N), \chi)$ of Jacobi forms of weight k and index one to the space $M_k(\Gamma_0^{(2)}(N), \chi)$ of Siegel modular forms of weight k.

For $\phi \in J_{k,1}(\Gamma_0(N), \chi)$ and natural number m, we define the operator V_m by

$$\phi|_{k,1} V_m(\tau, z) = m^{k-1} \sum_{M \in \Gamma_0(N) \backslash M_2^*(m)} \chi(a)(c\tau + d)^{-k} e\left(-\frac{cmz^2}{c\tau + d}\right) \phi\left(M\tau, \frac{mz}{c\tau + d}\right),$$

where the summation is taken over all representatives of

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N) \setminus M_2^*(m)$$

with

$$M_2^*(m) = \{M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbf{Z}); \det M = m, c \equiv 0 \pmod{N}, (a, N) = 1\}.$$

It is known that

$$\phi|_{k,1} V_m \in J_{k,m}(\Gamma_0(N), \chi)$$

and if

$$\phi(\tau,z)=\sum_{\substack{n,r\in\mathbf{Z}\\4n-r^2\geq 0}} c(n,r)q^n\zeta^r,$$

then

$$\phi|_{k,1}V_m(\tau,z)=\sum_{\substack{n,r\in\mathbf{Z}\\4mn-r^2\geq 0}}\left(\sum_{d|(n,r,m)}\chi(d)d^{k-1}c\left(\frac{mn}{d^2},\frac{r}{d}\right)\right)q^n\zeta^r. \tag{17}$$

By the same manner as in Theorem 2.2 of [2] p.173, we can prove

Proposition 4. *For $\phi\in J_{k,1}(\Gamma_0(N),\chi)$, we define the Maass lift $\mathcal{M}\phi$ by*

$$\mathcal{M}\phi(Z)=\phi_0(\tau,z)+\sum_{m\geq 1}\phi|_{k,1}V_m(\tau,z)e(m\tau') \tag{18}$$

with $Z=\begin{pmatrix}\tau & z\\ z & \tau'\end{pmatrix}\in H_2$, *where $\phi_0(\tau,z)$ is defined by*

$$\begin{aligned}\phi_0(\tau,z)&=\frac{N^k\Gamma(k)L(k,\overline{\chi})}{(-2\pi i)^k\tau_N(\overline{\chi})}\sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(N)}\overline{\chi}(d)(c\tau+d)^{-k}c(0,0) \qquad (19)\\ &=\left\{\frac{N^k\Gamma(k)L(k,\overline{\chi})}{(-2\pi i)^k\tau_N(\overline{\chi})}+\sum_{n\geq 1}(\sum_{d|n}\chi(d)d^{k-1})q^n\right\}c(0,0).\end{aligned}$$

Here $\tau_N(\overline{\chi})=\sum_{r=1}^N\overline{\chi}(r)e^{2\pi ir/N}$ is the Gauss sum, $\Gamma(s)$ is the Gamma function, $L(s,\chi)$ is the Dirichlet L function and

$$\Gamma_\infty=\left\{\pm\begin{pmatrix}1 & n\\ 0 & 1\end{pmatrix};n\in\mathbf{Z}\right\}.$$

Then we have

$$\mathcal{M}\phi\in M_k(\Gamma_0^{(2)}(N),\chi).$$

4 Proof of Theorem 1

In this section we explain that Theorem 1 follows from the coincidence of the Koecher-Maass series associated with the Siegel-Eisenstein series $E^{(2)}_{k,\overline{\chi}}$ with level N and the Maass lift $\mathcal{M}E^{\infty}_{k,1,\overline{\chi}}$ of the Jacobi Eisenstein series $E^{\infty}_{k,1,\overline{\chi}}$ on $\Gamma_0(N)\ltimes\mathbf{Z}^2$ for any with any Grossencharacter $\mathcal{U}$.

Define $\tilde{E} \in M_k(\Gamma_0^{(2)}(N), \chi)$ by

$$\tilde{E} = \frac{(-2\pi i)^k \tau_N(\overline{\chi})}{N^k \Gamma(k) L(k, \overline{\chi})} \mathcal{M} E_{k,1,\overline{\chi}}^{\infty}.$$

Then we will show the following result in the next section.

Proposition 5. *One has*

$$D^*(\tilde{E}, \mathcal{U}, s) = D^*(E_{k,\overline{\chi}}^{(2)}, \mathcal{U}, s).$$

Now let

$$F = E_{k,\overline{\chi}}^{(2)} - \tilde{E}.$$

Then it is easy to see that $\Phi F = 0$. This says that the Fourier expansion of F has only the terms indexed by positive definite half integral symmetric matrices. Let $Z = it^{1/2}W$ be the variable on H_2, where $t > 0$ and $W \in \mathcal{SP}_2$ is a positive definite real symmetric matrix of size two whose determinant is one. By identifying W with the variable on H_1 as in (9), we have the Roelcke-Selberg spectral decomposition of $F_t(W) = F(it^{1/2}W)$ as

$$F_t(W) = \sum_{j=0}^{\infty} < F_t, \mathcal{U}_j > \mathcal{U}_j(\tau) + \frac{1}{4\pi i} \int_{\Re u = 1/2} < F_t, E_u > E_u(\tau) du,$$

where $\mathcal{U}_0 = \sqrt{3/\pi}$, $\{\mathcal{U}_j\}_{j \geq 1}$ is an orthonormal basis consisting of cuspidal eigenfunctions for Δ,

$$E_u(\tau) = \sum_{\gamma \in \Gamma_\infty \backslash SL_2(\mathbf{Z})} (\Im \gamma \tau)^u$$

is the non holomorphic Eisenstein series and the inner product $< f, g >$ is defined by

$$< f, g > = \int_{SL_2(\mathbf{Z}) \backslash H_1} f(\tau) \overline{g(\tau)} \frac{du dv}{v^2}.$$

Then as in [8], (4. 8) of [3] p.219, we have

$$< F_t, \mathcal{U} > = \frac{1}{2\pi i} \int_{\Re s = s_0} (Nt)^{-s} D^*(F, \overline{\mathcal{U}}, s) ds$$

for $\mathcal{U} = \mathcal{U}_j, E_u$, with sufficiently large real number s_0. Since $D^*(F, \overline{\mathcal{U}}, s) = 0$ by assuming Proposition 5, we conclude that F is zero i.e.

$$E_{k,\overline{\chi}}^{(2)} = \frac{(-2\pi i)^k \tau_N(\overline{\chi})}{N^k \Gamma(k) L(k, \overline{\chi})} \mathcal{M} E_{k,1,\overline{\chi}}^{\infty}.$$

An explicit calculation of the Fourier coefficients of $E_{k,1,\overline{\chi}}^{\infty}$ gives Theorem 1.

5 Coincidence of the Koecher-Maass series

In this section we prove Proposition 5, the coincidence of the Koecher-Maass series.

It follows from Proposition 3, 4 and (12) that it holds

$$D^*(\tilde{E},\mathcal{U},s) = \frac{2^{k+1}N^{s-k}\pi^{k+1/2-2s}(-i)^k\tau_N(\overline{\chi})}{\Gamma(k)L(k,\overline{\chi})}L(2s-k+1,\chi)$$

$$\times \quad \Gamma(s-1/4+ir/2)\Gamma(s-1/4-ir/2)\sum_{n=1}^{\infty}\frac{e_{\overline{\chi}}^{\infty}(-n)b(-n)n^{3/4}}{n^s}, \tag{20}$$

where $e_{\overline{\chi}}^{\infty}(-n)$ is the Fourier coefficients of $E_{k,1,\overline{\chi}}^{\infty}$ (see (5)).

To get an explicit formula for $D^*(E_{k,\overline{\chi}}^{(2)},\mathcal{U},s)$ we proceed as follows.

Let $F_{k,\overline{\chi}}^{(2)}$ be the twist of $E_{k,\overline{\chi}}^{(2)}$ defined by

$$F_{k,\overline{\chi}}^{(2)}(Z) = E_{k,\overline{\chi}}^{(2)}|_k\omega_N^{(2)}(Z) = N^{-k}\det Z^{-k}E_{k,\overline{\chi}}^{(2)}(-(NZ)^{-1}),$$

where $\omega_N^{(2)}$ is defined by (13). Then we have

$$E_{k,\overline{\chi}}^{(2)} \in M_k(\Gamma_0^{(2)}(N),\chi), \qquad F_{k,\overline{\chi}}^{(2)} \in M_k(\Gamma_0^{(2)}(N),\overline{\chi}).$$

We can get an explicit formula of the Fourier coefficients of $F_{k,\overline{\chi}}^{(2)}$ and from this we have

Theorem 2. *The Koecher-Maass series of $F_{k,\overline{\chi}}^{(2)}$ with a Grossencharacter $\mathcal{U}$ corresponding to the eigenvalue $-\left(\frac{1}{4}+r^2\right)$ of Δ has the form*

$$D^*(F_{k,\overline{\chi}}^{(2)},\mathcal{U},s) = \frac{2^{4-2k}N^{s-k}\pi^{1/2-2s}}{\xi_{2,k}\alpha_k\chi(-1)L(k,\overline{\chi})}$$

$$\times \quad \Gamma(s-1/4+ir/2)\Gamma(s-1/4-ir/2)L(2s-k+1,\overline{\chi})\sum_{n=1}^{\infty}\frac{e_{\chi}^{0}(-n)b(-n)n^{3/4}}{n^s},$$

where $\xi_{2,k}$ and α_k are defined by

$$\alpha_k = \frac{\pi^{k-1/2}}{i^k 2^{k-2}\Gamma(k-1/2)}, \tag{21}$$

$$\xi_{2,k} = (4\pi)^{1/2}(2\pi i)^{-2k}\Gamma(k)\Gamma(k-1/2). \tag{22}$$

respectively.

To regard $D^*(F_{k,\overline{\chi}}^{(2)},\mathcal{U},s)$ as a Rankin-Selberg transform of certain automorphic form and to apply the Rankin-Selberg method, we need the Shimura correspondence for Maass wave forms due to Katok-Sarnak [9] and Duke-Imamoglu

[5], and the Rankin-Selberg method for automorphic forms which are not of rapid decay given in [15], [14].

To state the Shimura correspondence for Maass wave forms, first we introduce Maass wave form of weight 1/2. Let

$$j(\gamma,\tau)=\frac{\theta(\gamma\tau)}{\theta(\tau)},\quad \theta(\tau)=\sum_{n\in\mathbf{Z}}e^{2\pi in^2\tau},\quad \gamma\in\Gamma_0(4)$$

be the well known automorphic factor on $\Gamma_0(4)$. For $r\in\mathbf{C}$ let T_r^+ denote the vector space consisting of all functions g on the upper half-plane H_1 satisfying the following three conditions.

(M-i) Each $g(\tau)$ is a C^∞ function of $u=\Re\tau$ and $v=\Im\tau$ verifying the transformation formula

$$g(\gamma\tau)=g(\tau)j(\gamma,\tau)|c\tau+d|^{-1/2}$$

for all $\gamma\in\Gamma_0(4)$ and it has a moderate growth at any cusp of $\Gamma_0(4)$.

(M-ii)$g(\tau)$ has a Fourier expansion of the form

$$g(\tau)=\sum_{n\in\mathbf{Z}}B(n,v)e(nu),$$

where the Fourier coefficients $B(n,v)$ for $n\neq 0$ are given by

$$B(n,v)=b(n)W_{sign(n)/4,ir/2}(4\pi|n|v).$$

Here $W_{\alpha,\beta}(v)$ is the usual Whittaker function.

(M-iii)If $n\equiv 2,3 \pmod 4$, then necesarily $B(n,v)=0$.

The following result due to Katok-Sarnak [9] and Duke-Imamoglu [5] gives a Shimura correspondence for Maass wave forms.

Proposition 6. *Let $\mathcal{U}$ be an even Maass wave form i.e. $\mathcal{U}(-\bar{\tau})=\mathcal{U}(\tau)$, and assume that $\Delta\mathcal{U}=-\left(\frac{1}{4}+r^2\right)\mathcal{U}$ with some $r\in\mathbf{C}$. Then there exists $g\in T_r^+$ which satisfies the relation*

$$b(-n)=n^{-3/4}\sum_{\substack{T\in L_2^+/SL_2(\mathbf{Z})\\ \det 2T=n}}\frac{\mathcal{U}(T)}{\epsilon(T)}$$

for any natural number n, where $\epsilon(T)$ is the same as in (10).

For $g\in T_r^+$ which has the Fourier expansion as in (M-ii), we set

$$k_j(\tau)=\sum_{D\equiv -j \pmod 4}(v/4)^{1/4}\overline{B(-D,v/4)}e(Du/4) \tag{23}$$

for $j = 0, 1$.

For the Jacobi Eisenstein series $E^0_{k,1,\chi}$ which has the Fourier expansion as in (4), h_j is defined by

$$h_j(\tau) = \sum_{D>0, D\equiv -j \pmod 4} e^0_\chi(-D) q^{\frac{D}{4}}. \tag{24}$$

We can see that the Koecher-Maass series $D^*(F^{(2)}_{k,\overline{\chi}}, \mathcal{U}, s)$ for the twisted Siegel-Eisenstein series $F^{(2)}_{k,\overline{\chi}}$ is a Rankin-Selberg transformation of certain automorphic form ξ, in other words $D^*(F^{(2)}_{k,\overline{\chi}}, \mathcal{U}, s)$ is the Mellin transformation of the constant term of ξ.

This ξ is define by

$$\xi(\tau) = h_0(\tau)\overline{k_0(\tau)} + h_1(\tau)\overline{k_1(\tau)}, \tag{25}$$

where h_j and k_j are defined by (24) and (23) respectively and we have

Proposition 7. *One has*

$$\xi(\gamma\tau) = \overline{\chi}(d)\frac{(c\tau+d)^k}{|c\tau+d|}\xi(\tau)$$

for any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$.

Define the Rankin-Selberg transform of ξ associated with the cusp $i\infty$ by

$$R_\infty(s) = \int_0^\infty \int_0^1 \xi(\tau) v^{s-2} du dv. \tag{26}$$

Then using the formula

$$\int_0^\infty e^{-\frac{y}{2}} y^{\nu-1} W_{\kappa,\mu}(y) dy = \frac{\Gamma(\nu+1/2-\mu)\Gamma(\nu+1/2+\mu)}{\Gamma(\nu-\kappa+1)} \tag{27}$$

and the Fourier expansion of ξ obtained from (25), (24), (23), we get

Proposition 8. *One has*

$$\begin{aligned} R_\infty(s) &= \frac{2^{-1/2}\pi^{3/4-s}}{\Gamma(s+1/2)} \\ &\times \ \Gamma(s-1/4+ir/2)\Gamma(s-1/4-ir/2)\sum_{n=1}^\infty \frac{e^0_\chi(-n)b(-n)n^{3/4}}{n^s}. \end{aligned}$$

From Theorem 2 and Proposition 8, we see that the Koecher-Maass series $D^*(F_{k,\overline{\chi}}^{(2)}, \mathcal{U}, s)$ is essentially equal to the Rankin-Selberg transform of the automorphic form ξ. Roughly speaking, we can see from Proposition 2 that the Koecher-Maass series $D^*(E_{k,\overline{\chi}}^{(2)}, \mathcal{U}, s)$ is $R_\infty(k-s)$. Hence we want to apply the Rankin-Selberg method for automorphic forms which are not of rapid decay (see [15] and Theorem 2 given in [14]) to get a reasonable Dirichlet series expression for $R_\infty(k-s)$.

For each cusp κ in the set of representatives of non equivalent cusps of $\Gamma_0(N)$ given by

$$\{i\infty, 0\} \bigcup \{1/\mu; 1 < \mu < N, \mu | N\},$$

we define elements in $SL_2(\mathbf{R})$ by

$$g_\infty = \sigma_\infty, g_0 = \sigma_0 A_1, g_\mu = \sigma_\mu A_\mu, \tag{28}$$

where $A_\mu = \begin{pmatrix} \sqrt{N/\mu} & 0 \\ 0 & \sqrt{\mu/N} \end{pmatrix}$ for $1 \leq \mu < N$ and σ_μ are defined by (7). For each cusp κ, we will also denote g_κ instead of the above g_j by a trivial identification.

Since g_κ are elements in the normalizer of $\Gamma_0(N)$, the conditions

$$g_\kappa(i\infty) = \kappa, \quad \Gamma_0(N) \cap g_\kappa \left\{ \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} : a \in \mathbf{R} \right\} g_\kappa^{-1} = g_\kappa \left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\rangle g_\kappa^{-1}$$

and $\chi(\gamma) = 1$ for all $\gamma \in g_\kappa \left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\rangle g_\kappa^{-1}$ assumed in the section 2.1 of [14] are satisfied.

Using these g_κ defined by (28), we define ξ_κ by

$$\xi_\kappa(\tau) = \frac{|J(g_\kappa, \tau)|}{J(g_\kappa, \tau)^k} \xi(g_\kappa \tau). \tag{29}$$

To apply Theorem 2 in [14], we must check the assumption (b) given there, which is the growth condition for each ξ_κ. This is accomplished by expanding ξ_κ in the Fourier series.

If $\mathcal{U}$ is cuspidal, then $B(0, v/4) = 0$ and if $\mathcal{U}$ is a constant function or non holomorphic Eisenstein series, then $B(0, v/4)$ comes from the constant term of real analytic Cohen's Eisenstein series (see (5.18) of [3] p.228 and Lemma 5 of [5] p.351). Hence we can apply Theorem 2 in [14]. The Rankin-Selberg transform of ξ associated with the cusp κ is defined by

$$R_\kappa(s) = \int_0^\infty \int_0^1 (\xi_\kappa(\tau) - \delta_{0,\kappa}(v/4)^{1/4} B(0, v/4)) v^{s-2} du dv.$$

Using the same notation given in Theorem 2 in [14], we have

$$\varphi_{\infty,\kappa}(s,\chi)$$
$$= \begin{cases} 0, & \text{for } \kappa \neq 0 \\ \dfrac{2^{2-2s}\pi i^k N^{-s}\Gamma(2s-1)L(2s-1,\overline{\chi})}{\Gamma(s-k/2)\Gamma(s+k/2)L(2s,\overline{\chi})}. & \text{for } \kappa = 0 \end{cases}$$

Thus we get

$$\begin{aligned} R_\infty(s) &= \varphi_{\infty,0}(s-\frac{k-1}{2},\chi)R_0(k-s) \\ &= \frac{2^{k+1-2s}\pi i^k N^{-s+k/2-1/2}\Gamma(2s-k)L(2s-k,\overline{\chi})}{\Gamma(s-k+1/2)\Gamma(s+1/2)L(2s-k+1,\overline{\chi})}R_0(k-s). \end{aligned} \tag{30}$$

It follows from Theorem 2, Proposition 8, (30) and the functional equation of the Dirichlet L-function that it holds

$$\begin{aligned} D^*(F^{(2)}_{k,\overline{\chi}},\mathcal{U},s) &= \frac{2^{9/2-2k}N^{k/2-1/2-2s}\pi^{s-k-1/4}\tau_N(\overline{\chi})}{\xi_{2,k}\alpha_k\chi(-1)L(k,\overline{\chi})} \\ &\times\ \Gamma(1/2+k-s)L(1-2s+k,\chi)R_0(k-s), \end{aligned}$$

where $\tau_N(\chi)$ is the Gauss sum $\tau_N(\chi)=\sum_{n=1}^{N}\chi(n)e^{2\pi i n/N}$.

This is nothing but $(-1)^k D^*(E^{(2)}_{k,\overline{\chi}},\mathcal{U},k-s)$ by Proposition 2. Hence by replacing $k-s$ by s and using $\chi(-1)=(-1)^k$, we get

$$\begin{aligned} D^*(E^{(2)}_{k,\overline{\chi}},\mathcal{U},s) &= \frac{2^{9/2-2k}N^{2s-3k/2-1/2}\pi^{-s-1/4}\tau_N(\overline{\chi})}{\xi_{2,k}\alpha_k L(k,\overline{\chi})} \\ &\times \Gamma(s+1/2)L(2s-k+1,\chi)R_0(s). \end{aligned} \tag{31}$$

By calculating the Fourier expansion of ξ_0, we can see

$$\begin{aligned} R_0(s) &= \frac{2^{-1/2}\pi^{3/4-s}N^{k/2+1/2-s}}{\Gamma(s+1/2)} \\ &\times\ \Gamma(s-1/4+ir/2)\Gamma(s-1/4-ir/2)\sum_{n=1}^{\infty}\frac{a^0_{\chi}(-N^2n)b(-n)n^{3/4}}{n^s}. \end{aligned}$$

Finally combining the above calculations and Proposition 1, we obtain

$$\begin{aligned} D^*(E^{(2)}_{k,\overline{\chi}},\mathcal{U},s) &= \frac{2^{4-2k}N^{s-k}\pi^{1/2-2s}\tau_N(\overline{\chi})}{\xi_{2,k}\alpha_k L(k,\overline{\chi})} \\ &\times\ \Gamma(s-1/4+ir/2)\Gamma(s-1/4-ir/2)L(2s-k+1,\chi) \\ &\times\ \sum_{n=1}^{\infty}\frac{e^{\infty}_{\overline{\chi}}(-n)b(-n)n^{3/4}}{n^s}. \end{aligned}$$

From the definitions of $\xi_{2,k}$ and α_k given by (22) and (21), we have

$$\xi_{2,k}\alpha_k = 2^{3-3k}\pi^{-k}i^{-3k}\Gamma(k).$$

Comparing the Dirichlet series expressions, we get

$$D^*(\tilde{E},\mathcal{U},s) = D^*(E^{(2)}_{k,\overline{\chi}},\mathcal{U},s)$$

as desired.

References

[1] T. Arakawa, Dirichlet series corresponding to Siegel's modular forms of degree n with level N. Tohoku Math. J. (2) **42** (1990), no. 2, 261–286.

[2] T. Arakawa, Saito-Kurokawa lifting for odd weights. Comment. Math. Univ. St. Paul. **49** (2000), no. 2, 159–176.

[3] T. Arakawa, I. Makino, F. Sato, Converse theorem for not necessarily cuspidal Siegel modular forms of degree 2 and Saito-Kurokawa lifting. Comment. Math. Univ. St. Paul **50**, (2001), no. 2, 197–234.

[4] S. Boecherer, Bemerkungen uber die Dirichletreichen von Koecher und Maass. Math.Gottingensis des Schrift.des SFB. Geometry and Analysis Heft **68** (1986)

[5] W. Duke, O. Imamoglu, A converse theorem and the Saito-Kurokawa lift. Internat. Math. Res. Notices 1996, no. 7, 347–355.

[6] M. Eichler, D. Zagier, The theory of Jacobi forms. Progress in Mathematics, 55. Birkhauser Boston, Inc., Boston, MA, 1985. v+148 pp.

[7] T. Ibukiyama, A survey on the new proof of Saito-Kurokawa lifting after Duke and Imamoglu (in Japanese), in Report of the fifth summer school of number theory "Introduction to Siegel modular forms" (1997), 134-176.

[8] K. Imai, Generalization of Hecke's correspondence to Siegel modular forms. Amer. J. Math. **102** (1980), no. 5, 903–936.

[9] S. Katok, P. Sarnak, Heegner points, cycles and Maass forms, Israel J. Math. **84** (1993), no. 1-2, 193–227.

[10] H. Katsurada, An explicit formula for Siegel series. Amer. J. Math. **121** (1999), no. 2, 415–452.

[11] H. Maass, Siegel's modular forms and Dirichlet series. Dedicated to the last great representative of a passing epoch. Carl Ludwig Siegel on the occasion of his seventy-fifth birthday. Lecture Notes in Mathematics, Vol. 216. Springer-Verlag, Berlin-New York, 1971. v+328 pp

[12] H. Maass, Uber die Fourierkoeffizienten der Eisensteinreihen zweiten Grades. (German) Mat.-Fys. Medd. Danske Vid. Selsk. **38**, no. 14, 13 pp. (1972).

[13] H. Maass, Die Fourierkoeffizienten der Eisensteinreihen zweiten Grades. (German) Mat.-Fys. Medd. Danske Vid. Selsk. **34** 1964 no. 7, 25 pp. (1964).

[14] Y. Mizuno, The Rankin-Selberg convolution for Cohen's Eisenstein series of half integral weight. Abh. Math. Sem. Univ. Hamburg **75** (2005), 1–20.

[15] D. Zagier, The Rankin-Selberg method for automorphic functions which are not of rapid decay, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **28** (1981), 415-437.

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