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GELFAND PAIRS AND BOUNDS FOR VARIOUS FOURIER COEFFICIENTS OF AUTOMORPHIC FUNCTIONS

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ABSTRACT. We explain how the uniqueness of certain invariant functionals on irreducible unitary representations leads to non-trivial spectral identities between various periods of automorphic functions. As an example of an application of these identities, we deduce a non-trivial bounds for the corresponding unipotent and spherical Fourier coefficients of Maass forms.

1. INTRODUCTION

1.1. Rankin-Selberg type identities and Gelfand pairs. The main aim of this note is to present a new method which allows one to obtain non-trivial spectral identities for weighted sums of certain periods of automorphic functions. These identities are modelled on the classical identity of R. Rankin [Ra] and A. Selberg [Se]. We recall that the Rankin-Selberg identity relates weighted sum of Fourier coefficients of a cusp form ϕ to the weighted integral of the inner product of ϕ^2 with the Eisenstein series (see formula (1.2) below).

In this note we explain how to deduce the classical Rankin-Selberg identity and similar new identities from the uniqueness principle in representation theory. The uniqueness principle is a powerful tool in representation theory; it plays an important role in the theory of automorphic functions. We show how one can associate a non-trivial spectral identity to certain *pairs* of different Gelfand *triples* of subgroups inside of the ambient group. Namely, we associate a spectral identity to two triples $\mathcal{F} \subset \mathcal{H}_1 \subset \mathcal{G}$ and $\mathcal{F} \subset \mathcal{H}_2 \subset \mathcal{G}$ of subgroups in a group $\mathcal{G}$ such that pairs $(\mathcal{G}, \mathcal{H}_i)$ and $(\mathcal{H}_i, \mathcal{F})$ for $i = 1, 2$, are strong Gelfand pairs having the same subgroup $\mathcal{F}$ in the intersection. We call such a collection $(\mathcal{G}, \mathcal{H}_1, \mathcal{H}_2, \mathcal{F})$ a strong Gelfand formation.

Rankin-Selberg type identities which are obtained by our method relate two different weighted sums of (generalized) periods of automorphic functions, where periods are taken along closed orbits of various subgroups appearing in the strong Gelfand formation (for the exact representation-theoretic formulation of the setup, see Section 2.1). Our main observation is that for each term in the formation the corresponding automorphic period defines an equivariant functional satisfying the uniqueness principle. These functionals provide two different spectral expansions of the functional given by the period with respect to the smallest subgroup $\mathcal{F}$.

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The weights appearing in Rankin-Selberg type identities lead to a pair of integral transforms which are described in terms of representation theory (i.e., generalized matrix coefficients) without any reference to the automorphic picture. In the simplest case of the classical Rankin-Selberg identity, this pair of transforms consists of the Fourier and the Mellin transforms.

Rankin-Selberg type identities could be used in order to obtain non-trivial bounds for the corresponding periods. In Theorem 1.3 we give such an application by proving non-trivial bound for spherical Fourier coefficients of Maass forms (for the classical unipotent Fourier coefficients the analogous bound, Theorem 1.1, was obtained in [BR1] by a different method). To obtain these bounds, we study analytic properties of the corresponding transforms and in particular establish certain bounds which might be viewed as instances of the "uncertainty principle" for a pair of such transforms. As a corollary, we obtain a subconvexity bound for certain automorphic L-functions.

The novelty of our results mainly lies in the method, as we do not rely on an appropriate unfolding procedure which would give formulas similar to the one proved in Theorem 1.2. Instead, we use the uniqueness of relevant invariant functionals which we explain below in Section 2.1.

We now describe two *analytic applications* of the Rankin-Selberg type spectral identities. We consider two cases: the classical unipotent Fourier coefficients of Maass forms and their spherical analogs.

1.2. **Unipotent Fourier coefficients of Maass forms.** Let $G = PGL_2(\mathbb{R})$ and denote by $K = PO(2)$ the standard maximal compact subgroup of G. Let $\mathbb{H} = G/K$ be the upper half plane endowed with a hyperbolic metric and the corresponding volume element $d\mu_{\mathbb{H}}$.

Let $\Gamma \subset G$ be a non-uniform lattice. We assume for simplicity that, up to equivalence, Γ has a unique cusp which is reduced at ∞. This means that the unique up to conjugation unipotent subgroup $\Gamma_\infty \subset \Gamma$ is generated by $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ (e.g. $\Gamma = PSL_2(\mathbb{Z})$). We denote by $X = \Gamma \setminus G$ the automorphic space and by $Y = X/K = \Gamma \setminus \mathbb{H}$ the corresponding Riemann surface (with possible conic singularities if Γ has elliptic elements). This induces the corresponding Riemannian metric on Y, the volume element $d\mu_Y$ and the Laplace-Beltrami operator Δ. We normalize $d\mu_Y$ to have the total *volume one.*

Let $\phi_\tau \in L^2(Y)$ be a Maass cusp form. In particular, ϕ_τ is an eigenfunction of Δ with the eigenvalue which we write in the form $\mu = \frac{1-\tau^2}{4}$ for some $\tau \in \mathbb{C}$. We will always assume that ϕ_τ is normalized to have L^2*-norm one.* We can view ϕ_τ as a Γ-invariant eigenfunction of the Laplace-Beltrami operator Δ on $\mathbb{H}$. Consider the classical Fourier expansion of ϕ_τ at ∞ given by (see [Iw])

$$\phi_\tau(x+iy) = \sum_{n\neq 0} a_n(\phi_\tau)\mathcal{W}_{\tau,n}(y)e^{2\pi inx} \ . \tag{1.1}$$

Here $\mathcal{W}_{\tau,n}(y)e^{2\pi inx}$ are properly normalized eigenfunctions of Δ on $\mathbb{H}$ with the same eigenvalue μ as that of the function ϕ_τ. The functions $\mathcal{W}_{\tau,n}$ are usually described in terms of the K-Bessel function. It is well-known that the functions $\mathcal{W}_{\tau,n}$ could be described in terms of certain matrix coefficients of unitary representations of G (i.e., Whittaker functionals).

We note that from the group-theoretic point of view, the Fourier expansion (1.1) is a consequence of the decomposition of the function ϕ_τ under the natural action of the group N/Γ_∞ (commuting with Δ). Here N is the standard upper-triangular subgroup and the decomposition is with respect to the characters of the group N/Γ_∞.

The vanishing of the zero Fourier coefficient $a_0(\phi_\tau)$ in (1.1) distinguishes cuspidal Maass forms (for Γ having several inequivalent cusps, the vanishing of the zero Fourier coefficient is required at each cusp).

The coefficients $a_n(\phi_\tau)$ are called the Fourier coefficients of the Maass form ϕ_τ and play a prominent role in analytic number theory.

One of the central problems in the analytic theory of automorphic functions is the following

Problem: Find the best possible constants σ, ρ and C_Γ such that the following bound holds

$$|a_n(\phi_\tau)| \leq C_\Gamma \cdot |n|^\sigma \cdot (1+|\tau|)^\rho .$$

In particular, one asks for constants σ and ρ which are *independent* of ϕ_τ (i.e., depend on Γ only; for a brief discussion of the history of this question, see Remark 1.4.4).

It is easy to obtain a polynomial bound for coefficients $a_n(\phi_\tau)$ using boundness of ϕ_τ on Y. Namely, G. Hardy and E. Hecke essentially proved that the following bound

$$\sum_{|n|\leq T} |a_n(\phi_\tau)|^2 \leq C \cdot \max\{T, 1+|\tau|\} ,$$

holds for any $T \geq 1$, with the constant depending on Γ only (see [Iw]). It would be very interesting to improve this bound for coefficients $a_n(\phi_\tau)$ in the range $|n| \ll |\tau|$.

For a fixed τ, we have the bound $|a_n(\phi_\tau)| \leq C_\tau |n|^{\frac{1}{2}}$. This bound is usually called the standard bound or the Hardy/Hecke bound for the Fourier coefficients of cusp forms (in the n aspect).

The first improvements of the standard bound are due to H. Salié and A. Walfisz using exponential sums. Rankin [Ra] and Selberg [Se] independently discovered the so-called Rankin-Selberg unfolding method (i.e., the formula (1.4) below) which allowed them to show that for any $\varepsilon > 0$, the bound $|a_n(\phi)| \ll |n|^{\frac{3}{10}+\varepsilon}$ holds. Their approach is based on the integral representation for the weighted sum of Fourier coefficients $a_n(\phi)$. To state it, we assume, for simplicity, that the so-called residual spectrum is trivial (i.e., the Eisenstein series $E(s,z)$ are holomorphic for $s \in (0,1)$; e.g, $\Gamma = PGL_2(\mathbb{Z})$). (The reader also should keep in mind that we use the normalization $\mathrm{vol}(Y) = 1$ and $\mathrm{vol}(\Gamma_\infty \setminus N) = 1$.) We have then

$$\sum_n |a_n(\phi)|^2 \hat{\alpha}(n) = \alpha(0) + \frac{1}{2\pi i} \int_{Re(s)=\frac{1}{2}} D(s,\phi,\bar{\phi}) M(\alpha)(s) ds , \tag{1.2}$$

where $\alpha \in C^\infty(\mathbb{R})$ is an appropriate test function with the Fourier transform $\hat{\alpha}$ and the Mellin transform $M(\alpha)(s)$,

$$D(s, \phi, \bar{\phi}) = \Gamma(s, \tau) \cdot < \phi\bar{\phi}, E(s) >_{L^2(Y)} , \tag{1.3}$$

where $E(z, s)$ is an appropriate non-holomorphic Eisenstein series and $\Gamma(s, \tau)$ is given explicitly in terms of the Euler Γ-function (see Remark 1.4.4).

The proof of (1.2), given by Rankin and Selberg, is based on the so-called unfolding trick, which amounts to the following. Let $E(s, z)$ be the Eisenstein series given by $E(s, z) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} y^s(\gamma z)$ for $Re(s) > 1$ (and analytically continued to a meromorphic function for all $s \in \mathbb{C}$). We have the following "unfolding" identity valid for $Re(s) > 1$,

$$< \phi\bar{\phi}, E(z, s) >_{L^2(Y)} = \int_{\Gamma \backslash \mathbb{H}} \phi(z)\bar{\phi}(z) \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} y^s(\gamma z) d\mu_Y = \tag{1.4}$$

$$= \int_{\Gamma_\infty \backslash \mathbb{H}} \phi(z)\bar{\phi}(z) y^s(z) d\mu_{\mathbb{H}} = \int_0^\infty \left(\int_0^1 \phi(x + iy)\bar{\phi}(x + iy)\, dx \right) y^{s-1}\, d^x y \,.$$

This together with the Fourier expansion of cusp forms ϕ, leads to the Rankin-Selberg formula (1.2).

Using the strategy formulated in Section 2.1, in this note we explain how to deduce the Rankin-Selberg formula (1.2) directly from the uniqueness principle in representation theory and hence avoid the use of the unfolding trick (1.4). One of the uniqueness results we are going to use is related to the unipotent subgroup $N \subset G$ such that $\Gamma_\infty \subset N$ (the so-called Γ-cuspidal unipotent subgroup). In fact, the definition of classical Fourier coefficients $a_n(\phi_\tau)$ is implicitly based on the uniqueness of N-equivariant functionals on an irreducible (admissible) representation of G (i.e., on the uniqueness of the so-called Whittaker functional). For this reason, we call the coefficients $a_n(\phi_\tau)$ the *unipotent* Fourier coefficients.

We obtain a somewhat different (a slightly more "geometric") form of the Rankin-Selberg identity (1.2). In particular, we exhibit a connection between analytic properties of the function $D(s, \phi, \bar{\phi})$ and analytic properties of certain invariant functionals on irreducible unitary representations of G. This allows us to deduce subconvexity bounds for Fourier coefficients of Maass forms for a *general* Γ in a more transparent way (here we relay on ideas of A. Good [Go] and on our earlier results [BR1] and [BR3]). Namely, we prove the following bound for the Fourier coefficients $a_n(\phi_\tau)$.

Theorem 1.1. *Let ϕ_τ be a fixed Maass form of L^2-norm one. For any $\varepsilon > 0$, there exists an explicit constant C_ε such that*

$$\sum_{|k-T| \leq T^{\frac{2}{3}}} |a_k(\phi_\tau)|^2 \leq C_\varepsilon \cdot T^{\frac{2}{3}+\varepsilon} \,.$$

In particular, we have $|a_n(\phi_\tau)| \ll |n|^{\frac{1}{3}+\varepsilon}$. This is weaker than the Rankin-Selberg bound, but holds for *general lattices* Γ (i.e., not necessary a congruence subgroup). The bound in the theorem was first claimed in [BR1] and the analogous bound for holomorphic cusp forms was proved by Good [Go] by a different method. Here we give full details of the proof following a slightly different argument.

Our main goal is different, however. Our main new results deal with another type of Fourier coefficients associated with a Maass form. These Fourier coefficients, which we call spherical, were introduced by H. Petersson and are associated to a compact subgroup of G.

1.3. **Spherical Fourier coefficients.** When dealing with spherical Fourier coefficients we assume, for simplicity, that $\Gamma \subset G$ is a co-compact subgroup and $Y = \Gamma \setminus \mathbb{H}$ is the corresponding compact Riemann surface. Let ϕ_τ be a norm one eigenfunction of the Laplace-Beltrami operator on Y, i.e., a Maass form. We would like to consider a kind of a Taylor series expansion for ϕ_τ at a point on Y. To define this expansion, we view ϕ_τ as a Γ-invariant eigenfunction on $\mathbb{H}$. We fix a point $z_0 \in \mathbb{H}$. Let $z = (r, \theta)$, $r \in \mathbb{R}^+$ and $\theta \in S^1$, be the geodesic polar coordinates centered at z_0 (see [He]). We have the following spherical Fourier expansion of ϕ_τ associated to the point z_0

$$\phi_\tau(z) = \sum_{n\in\mathbb{Z}} b_{n,z_0}(\phi_\tau) P_{\tau,n}(r) e^{in\theta} . \tag{1.5}$$

Here functions $P_{\tau,n}(r)e^{in\theta}$ are properly normalized eigenfunctions of Δ on $\mathbb{H}$ with the same eigenvalue μ as that of the function ϕ_τ. The functions $P_{\tau,n}$ can be described in terms of the classical Gauss hypergeometric function or the Legendre function. It is well-known that one can describe special functions $P_{\tau,n}$ and their normalization in terms of certain matrix coefficients of irreducible unitary representations of G.

We call the coefficients $b_n(\phi_\tau) = b_{n,z_0}(\phi_\tau)$ the spherical (or anisotropic) Fourier coefficients of ϕ_τ (associated to a point z_0). These coefficients were introduced by H. Petersson and played a major role in recent works of Sarnak (e.g., [Sa]). Earlier, it was discovered by J.-L. Waldspurger [Wa] that in certain cases these coefficients are related to special values of L-functions (see Remark 1.4.1).

As in the case of the unipotent expansion (1.1), the spherical expansion (1.5) is the result of an expansion with respect to a group action. Namely, the expansion (1.5) is with respect to characters of the compact subgroup $K_{z_0} = \mathrm{Stab}_{z_0} G$ induced by the natural action of G on $\mathbb{H}$.

The expansion (1.5) exists for *any* eigenfunction of Δ on $\mathbb{H}$. This follows from a simple separation of variables argument applied to the operator Δ on $\mathbb{H}$. For a proof and a discussion of the growth properties of coefficients $b_n(\phi)$ for a general eigenfunction ϕ on $\mathbb{H}$, see [He], [L]. For another approach which is applicable to Maass forms, see [BR2].

Under the normalization we choose, the coefficients $b_n(\phi_\tau)$ are bounded on the average. Namely, one can show that the following bound holds

$$\sum_{|n|\leq T} |b_n(\phi_\tau)|^2 \leq C' \cdot \max\{T, 1 + |\tau|\}$$

for any $T \geq 1$, with the constant C' depending on Γ only (see [R]).

As our approach is based directly on the uniqueness principle, we are able to prove an analog of the Rankin-Selberg formula (1.2) with the group N replaced by a maximal compact subgroup of G. This is the main aim of this note. We obtain an analog of the Rankin-Selberg formula (1.2) for the coefficients $b_n(\phi_\tau)$. Roughly speaking, new formula amounts to the following

Theorem 1.2. *Let $\{\phi_{\lambda_i}\}$ be an orthonormal basis of $L^2(Y)$ consisting of Maass forms. Let ϕ_τ be a fixed Maass form.*

There exists an explicit integral transform $^\sharp : C^\infty(S^1) \to C^\infty(\mathbb{C})$, $u(\theta) \mapsto u^\sharp_\tau(\lambda)$, such that for all $u \in C^\infty(S^1)$, the following relation holds

$$\sum_n |b_n(\phi_\tau)|^2 \hat{u}(n) = u(1) + \sum_{\lambda_i \neq 1} \mathcal{L}_{z_0}(\phi_{\lambda_i}) \cdot u^\sharp_\tau(\lambda_i) \ , \tag{1.6}$$

with some explicit coefficients $\mathcal{L}_{z_0}(\phi_{\lambda_i}) \in \mathbb{C}$ which are independent of u.

Here $\hat{u}(n) = \frac{1}{2\pi} \int_{S^1} u(\theta) e^{-in\theta} d\theta$ and $u(1)$ is the value at $1 \in S^1$.

The definition of the integral transform $^\sharp$ is based on the uniqueness of certain invariant trilinear functionals on irreducible unitary representations of G. These functionals were studied in [BR3] and [BR4]. The main point of the relation (1.6) is that the transform $u^\sharp_\tau(\lambda_i)$ depends only on the parameters λ_i and τ, but not on the choice of Maass forms ϕ_{λ_i} and ϕ_τ. The coefficients $\mathcal{L}_{z_0}(\phi_{\lambda_i})$ are essentially given by the product of the triple product coefficients $< \phi^2_\tau, \phi_{\lambda_i} >_{L^2(Y)}$ and the values of Maass forms ϕ_{λ_i} at the point z_0. In some special cases both types of these coefficients are related to L-functions (see [W], [JN], [Wa] and Remark 1.4.1).

A formula similar to (1.6) holds for a non-uniform lattice Γ as well, and includes the contribution from the Eisenstein series. Also, a similar formula holds for holomorphic forms. We intend to discuss it elsewhere.

The new formula (1.6) allows us to deduce the following bound for the *spherical* Fourier coefficients of Maass forms.

Theorem 1.3. *Let Γ be as above and ϕ_τ a fixed Maass form of L^2-norm one. For any $\varepsilon > 0$, there exists an explicit constant D_ε such that*

$$\sum_{|k-T| \leq T^{\frac{2}{3}}} |b_k(\phi_\tau)|^2 \leq D_\varepsilon \cdot T^{\frac{2}{3}+\varepsilon} \ .$$

In particular, we have $|b_n(\phi_\tau)| \ll |n|^{\frac{1}{3}+\varepsilon}$ for any $\varepsilon > 0$. Analogous bound should hold for the periods of holomorphic forms. We hope to return to this subject elsewhere.

The proof of the bound in the theorem follows from essentially the same argument as in the case of the unipotent Fourier coefficients, once we have the Rankin-Selberg type identity (1.6). In the proof we use bounds for triple products of Maass forms obtained in [BR3], and a well-known bound for the averaged value of eigenfunctions of Δ.

In special cases, the bound in the theorem could be interpreted as a subconvexity bound for some automorphic L-function which we briefly explain now.

1.4. Remarks.

1.4.1. *Special values of L-functions.* One of the reasons one might be interested in bounds for coefficients $b_k(\phi_\tau)$ is their relation to certain automorphic L-functions. It was discovered by J.-L. Waldspurger [Wa] that, in certain cases, the coefficients $b_k(\phi_\tau)$ are related to special values of L-functions. H. Jacquet constructed the appropriate relative trace formula which covers these cases (see [JN]). The simplest case of the formula of Waldspurger is the following. Let $z_0 = i \in SL_2(\mathbb{Z})\setminus\mathbb{H}$ and $E = \mathbb{Q}(i)$. Let π be the automorphic representation which corresponds to ϕ_τ, Π its base change over E and $\chi_n(z) = (z/\bar{z})^{4n}$ the n-th power of the basic Grössencharacter of E. One has then, under appropriate normalization (for details, see [Wa], [JN]), the following beautiful formula

$$|b_n(\phi_\tau)|^2 = \frac{L(\frac{1}{2}, \Pi \otimes \chi_n)}{L(1, Ad\pi)} \quad . \tag{1.7}$$

Using this formula, we can interpret the bound in Theorem 1.3 as a bound on the corresponding L-functions. In particular, we obtain the bound $|L(\frac{1}{2}, \Pi \otimes \chi_n)| \ll |n|^{2/3+\varepsilon}$. This gives a subconvexity bound (with the convexity bound for this L-function being $|L(\frac{1}{2}, \Pi \otimes \chi_n)| \ll |n|^{1+\varepsilon}$).

The subconvexity problem is the classical question in analytic theory of L-functions which received a lot of attention in recent years (we refer to the survey [IS] for the discussion of subconvexity for automorphic L-functions). In fact, Y. Petridis and P. Sarnak [PS] recently considered more general L-functions. Among other things, they have shown that $|L(\frac{1}{2}+it_0, \Pi\otimes\chi_n)| \ll |n|^{\frac{159}{166}+\varepsilon}$ for any fixed $t_0 \in \mathbb{R}$ and any automorphic cuspidal representation Π of $GL_2(E)$ (not necessary a base change). Their method is also spectral in nature although it uses Poincaré series and treats L-functions through (unipotent) Fourier coefficients of cusp forms. We deal directly with periods and the special value of L-functions only appear through the Waldspurger formula. Of course, our interest in Theorem 1.3 lies not so much in the slight improvement of the Petridis-Sarnak bound for these L-functions, but in the fact that we can give a general bound valid for *any* point z_0. (It is clear that for a generic point or a cusp form which is not a Hecke form, coefficients b_n are not related to special values of L-functions.)

Recently, A. Venkatesh [V] announced (among other remarkable results) a slightly weaker subconvexity bound for coefficients $b_n(\phi_\tau)$ for a fixed ϕ_τ. His method seems to be quite different and is based on ergodic theory. In particular, it is not clear how to deduce the identity (1.6) from his considerations. On the other hand, the ergodic method gives a bound for Fourier coefficients for higher rank groups (e.g., on $GL(n)$) while it is not yet clear in what higher-rank cases one can develop Rankin-Selberg type formulas similar to (1.6).

1.4.2. *Fourier expansions along closed geodesics.* There is one more case where we can apply the uniqueness principle to a subgroup of $PGL_2(\mathbb{R})$. Namely, we can consider closed orbits of the diagonal subgroup $A \subset PGL_2(\mathbb{R})$ acting on X. It is well-known that such an orbit corresponds to a closed geodesic on Y (or to a geodesic ray starting and ending at cusps of Y). Such closed geodesics give rise to Rankin-Selberg type formulas similar to ones we considered for closed orbits of subgroups N and K. In special cases the corresponding Fourier coefficients are related to special values of various L-functions (e.g., the standard Hecke L-function of a Hecke-Maass forms which appears for a geodesic connecting cusps of a congruence subgroup of $PSL(2,\mathbb{Z})$). In fact, in the language of representations of adèle groups, which is appropriate for arithmetic Γ, the case of closed geodesics corresponds to real quadratic extensions of $\mathbb{Q}$ (e.g., twisted periods along Heegner cycles) while the anisotropic expansions (at Heegner points) which we considered in Section 1.3 correspond to imaginary quadratic extensions of $\mathbb{Q}$ (e.g., twisted "periods" at Heegner points).

In order to prove an analog of Theorems 1.1 and 1.3 for the Fourier coefficients associated to a closed geodesic, one has to face certain technical complications. Namely, for orbits of the diagonal subgroup A one has to consider contributions from representations of discrete series, while for subgroups N and K this contribution vanishes. It is more cumbersome to compute a contribution from discrete series as these representations do not have nice geometric models. Hence, while the proof of an analog of Theorem 1.2 for closed geodesics is straightforward, one has to study invariant trilinear functionals on discrete series representations more closely in order to deduce bounds for the corresponding coefficients. We hope to return to this subject elsewhere.

1.4.3. *Dependence on the eigenvalue.* From the proof we present it follows that the constants C_ε and D_ε in Theorems 1.2 and 1.3 satisfy the following bound

$$C_\varepsilon,\ D_\varepsilon \leq C(\Gamma)\cdot(1+|\tau|)\cdot|\ln\varepsilon|\ ,$$

for any $0<\varepsilon\leq 0.1$, and some explicit constant $C(\Gamma)$ depending on the lattice Γ only. We will discuss this elsewhere.

1.4.4. *Historical remarks.* The question of the size of Fourier coefficients of cusp forms was posed (in the n aspect) by S. Ramanujan for holomorphic forms (i.e., the celebrated Ramanujan conjecture established in full generality by P. Deligne for the holomorphic Hecke cusp form for congruence subgroups) and extended by H. Petersson to include Maass forms (i.e., the Ramanujan-Petersson conjecture for Maass forms). In recent years the τ aspect of this problem also turned out to be important.

Under the normalization we have chosen, it is expected that the coefficients $a_n(\phi_\tau)$ are at most slowly growing as $n\to\infty$ ([Sa]). Moreover, it is quite possible that the strong uniform bound $|a_n(\phi_\tau)|\ll(|n|(1+|\tau|))^\varepsilon$ holds for any $\varepsilon>0$ (e.g., Ramanujan-Petersson conjecture for Hecke-Maass forms for congruence subgroups of $PSL_2(\mathbb{Z})$). We note, however, that the behavior of Maass forms and holomorphic forms in these questions might be quite different (e.g., high multiplicities of holomorphic forms).

Using the integral representation (1.2) and detailed information about Eisenstein series available only for *congruence subgroups*, Rankin and Selberg showed that for a cusp form ϕ for a congruence subgroup of $PGL(2,\mathbb{Z})$ one has $\sum_{|n|\leq T}|a_n(\phi)|^2 = CT+O(T^{3/5+\varepsilon})$ for any $\varepsilon>0$. In particular, this implies that for any $\varepsilon>0$, $|a_n(\phi)|\ll |n|^{\frac{3}{10}+\varepsilon}$. Since their groundbreaking papers, this bound was improved many times by various methods (with the current record for Hecke-Maass forms being $7/64\approx 0.109...$ due to H. Kim, F. Shahidi and P. Sarnak [KiSa]).

The approach of Rankin and Selberg is based on the integral representation of the Dirichlet series given for $Re(s) > 1$, by the series $D(s, \phi, \bar{\phi}) = \sum_{n>0} \frac{|a_n(\phi)|^2}{n^s}$. The introduction of the so-called Ranking-Selberg L-function $L(s, \phi \otimes \bar{\phi}) = \zeta(2s) D(s, \phi, \bar{\phi})$ played an even more important role in the further development of automorphic forms than the bound for Fourier coefficients which Rankin and Selberg obtained.

Using integral representation (1.3), Rankin and Selberg analytically continued the function $L(s, \phi \otimes \bar{\phi})$ to the whole complex plane and obtained effective bound for the function $L(s, \phi \otimes \bar{\phi})$ on the critical line $s = \frac{1}{2} + it$ for Γ being a congruence subgroup of $SL_2(\mathbb{Z})$. From this, using standard methods in the theory of Dirichlet series, they were able to deduce the first non-trivial bounds for Fourier coefficients of cusp forms. In fact, Rankin and Selberg appealed to the classical Perron formula (in the form given by E. Landau) which relates analytic behavior of a Dirichlet series with non-negative coefficients to partial sums of its coefficients. The necessary analytic properties of $L(s, \phi \otimes \bar{\phi})$ are inferred from properties of the Eisenstein series through the formula (1.3).

A small drawback of the original Rankin-Selberg argument is that their method is applicable to Maass (or holomorphic) forms coming from *congruence subgroups* only. The reason for such a restriction is the absence of methods which would allow one to estimate unitary Eisenstein series for general lattices Γ. Namely, in order to effectively use the Rankin-Selberg formula (1.2) one would have to obtain *polynomial* bounds for the normalized inner product $D(s, \phi, \bar{\phi}) = \Gamma(s, \tau) \cdot < \phi\bar{\phi}, E(s) >_{L^2(Y)}$. This turns out to be notoriously difficult because of the *exponential* growth of the factor $\Gamma(s, \tau) = \frac{2\pi^s \Gamma(s)}{\Gamma^2(s/2)\Gamma(s/2+\tau/2)\Gamma(s/2-\tau/2)}$, for $|s| \to \infty$, $s \in i\mathbb{R}$. For a congruence subgroup, the question could be reduced to known bounds for the Riemann zeta function or for Dirichlet L-functions, as was shown by Rankin and Selberg. The problem of how to treat general Γ was posed by Selberg in his celebrated paper [Se].

The breakthrough in this direction was achieved in works of Good [Go] (for holomorphic forms) and Sarnak [Sa] (in general) who proved non-trivial bounds for Fourier coefficients of cusp forms for a general Γ using spectral methods. The method of Sarnak was finessed in [BR1] by introducing various ideas from the representation theory and further extended in [KS]. The method of our paper is different and avoids the use of analytic continuation which is central for [Sa], [BR1] and [KS]. We also would like to mention that recently R. Bruggeman, M. Jutila and Y. Motohashi (see [Mo] and references therein) developed what they call the inner product method. It is based on the unfolding of an appropriate Poincaré or Petersson type series. The standard unfolding leads to the spectral expansion for the series of the type $\sum_k A_k(\phi) A_{k+h}(\bar{\phi}) W(k)$, where ϕ is a Maass form and A_k are appropriate Fourier coefficients (e.g., unipotent or spherical Fourier coefficients we discussed above). The formulas obtained in such a way are special cases of our Rankin-Selberg type formula for a *special* test vectors v. These vectors are constructed from certain functions on the upper-half plane. As a result, the corresponding weights in the Rankin-Selberg type formulas are reminiscent of *exponentially* small weights considered by Selberg and Rankin. It seems that using our approach one can avoid a difficult task of removing these unwanted weights.

2. Gelfand Formations and Spectral Identities

2.1. **The method.** We explain now a simple representation-theoretic idea which underlies the classical Rankin-Selberg formula and some new similar formulas (e.g., the formulas (1.2) and (1.6) below).

2.1.1. *Gelfand pairs.* In what follows we will need the notion of Gelfand pairs (see [Gr] and references therein). A pair (A, B) of a group A and a subgroup B is called a *strong Gelfand pair* if for any pair of irreducible representations V of A and W of B, the multiplicity one condition $\dim \mathrm{Mor}_B(V, W) \leq 1$ holds.

In this paper we apply the notion of strong Gelfand pair to real Lie groups and to the spaces of *smooth* vectors in irreducible representations of these groups.

We apply the notion of strong Gelfand pairs repeatedly in the following standard situation. Let (A, B) be a strong Gelfand pair. Let $\Gamma_A \subset A$ be a lattice, $X_A = \Gamma_A \setminus A$ an automorphic space of A and $X_B \subset X_A$ a closed B-orbit. We fix some invariant measures on X_A and on X_B. Let (π, L, V) and (σ, M, W) be two abstract unitary irreducible representations of A and B respectively and their subspaces of smooth vectors. Assuming that both representations are automorphic, we fix $\nu_V : V \to L^2(X_A)$ and $\nu_W : W \to L^2(X_B)$ the corresponding isometric imbeddings of the spaces of smooth vectors. We denote the images of these maps by $V^{aut} \subset C^\infty(X_A)$ and $W^{aut} \subset C^\infty(X_B)$ and call these the automorphic realizations of the corresponding representations. Consider the restriction map $r_{X_B} : V^{aut} \to C^\infty(X_B)$. Together with the projection $pr_W : C^\infty(X_B) \to W^{aut}$ and identifications ν_V and ν_W, the map r_{X_B} defines a B-equivariant map $T^{aut}_{X_B} = \nu_W^{-1} \circ pr_W \circ r_{X_B} \circ \nu_V : V \to W$. Assuming that (A, B) is a strong Gelfand pair, the space of such B-equivariant maps is at most one-dimensional.

Usually, the abstract representations (π, L, V) and (σ, M, W) are easy to construct using explicit *models* which are independent of the automorphic realizations (e.g., realizations in the spaces of sections of various vector bundles over appropriate manifolds). Using these explicit models, we construct a *model* B-equivariant map $T^{mod} : V \to W$. Such a map usually could be defined for any representations V and W and not only for the automorphic ones. The uniqueness of such B-equivariant maps then implies that there exists a constant of proportionality a_{X_B,ν_V,ν_W} such that $T^{aut}_{X_B} = a_{X_B,\nu_V,\nu_W} \cdot T^{mod}$. We would like to study these constants. In many cases these constants are related to interesting objects (e.g., Fourier coefficients of cusp forms, special values of L-functions etc.). Of course, these constants depend, among other things, on the choice of model maps. In many cases we hope to find a way to canonically normalize norms of these maps in the adèlic setting (and hence define canonically if not the constants themselves then their absolute values). We hope to discuss these normalizations elsewhere.

We explain now how in certain situations one can obtain spectral identities for the coefficients a_{X_B,ν_V,ν_W}.

2.1.2. *Rankin-Selberg type spectral identities.* Let $\mathcal{G}$ be a (real reductive) group and $\mathcal{F} \subset \mathcal{H}_i \subset \mathcal{G}$, $i = 1, 2$ be a collection of subgroups, which we call a Gelfand *formation*, such that in the following commutative diagram each imbedding is a strong Gelfand pair (i.e., $(\mathcal{G},\mathcal{H}_i)$ and $(\mathcal{H}_i,\mathcal{F})$ are strong Gelfand pairs)

$$\begin{tikzcd} & \mathcal{G} & \\ \mathcal{H}_1 \arrow[ur, hook, "j_1"] & & \mathcal{H}_2 \arrow[ul, hook', "j_2"'] \\ & \mathcal{F} \arrow[ul, hook', "i_1"] \arrow[ur, hook, "i_2"'] & \end{tikzcd} \tag{2.1}$$

Let $\Gamma \subset \mathcal{G}$ be a lattice and denote by $X_{\mathcal{G}} = \Gamma \setminus \mathcal{G}$ the corresponding automorphic space. Let $\mathcal{O}_i \subset X_{\mathcal{G}}$ and $\mathcal{O}_{\mathcal{F}} \subset X_{\mathcal{G}}$ be closed orbits of $\mathcal{H}_i$ and $\mathcal{F}$ respectively, satisfying the following commutative diagram of imbeddings

$$\begin{tikzcd} & X_{\mathcal{G}} & \\ \mathcal{O}_1 \arrow[ur, hook, "j'_1"] & & \mathcal{O}_2 \arrow[ul, hook', "j'_2"'] \\ & \mathcal{O}_{\mathcal{F}} \arrow[ul, hook', "i'_1"] \arrow[ur, hook, "i'_2"'] & \end{tikzcd} \tag{2.2}$$

assumed to be compatible with the diagram (2.1). We endow each orbit (as well as $X_{\mathcal{G}}$) with a measure invariant under the corresponding subgroup (to explain our idea, we assume that all orbits are compact, and hence, these measures could be normalized to have mass one).

Let $\mathcal{V} \subset C^\infty(X_{\mathcal{G}})$ be an automorphic realization of the space of smooth vectors in an irreducible automorphic representation of $\mathcal{G}$. The integration over the orbit $\mathcal{O}_{\mathcal{F}} \subset X_{\mathcal{G}}$ defines an $\mathcal{F}$-invariant functional $I_{\mathcal{O}_{\mathcal{F}}} : \mathcal{V} \to \mathbb{C}$. In general, an $\mathcal{F}$-invariant functional on $\mathcal{V}$ does not satisfy the uniqueness property, as $(\mathcal{G},\mathcal{F})$ is not a Gelfand pair. Instead, we will write two *different* spectral expansions for $I_{\mathcal{O}_{\mathcal{F}}}$ using two intermediate groups $\mathcal{H}_1$ and $\mathcal{H}_2$.

Namely, for any $v \in \mathcal{V}$, we have two different ways to compute the value $I_{\mathcal{O}_{\mathcal{F}}}(v)$: by restricting the function $v \in C^\infty(X_{\mathcal{G}})$ to the orbit $\mathcal{O}_1$ and then integrating over $\mathcal{O}_{\mathcal{F}}$ or, alternatively, by restricting v to $\mathcal{O}_2$ and then integrating over $\mathcal{O}_{\mathcal{F}}$. Hence we have the identity

$$\int_{\mathcal{O}_{\mathcal{F}}} res_{\mathcal{O}_1}(v)d\mu_{\mathcal{O}_{\mathcal{F}}} = I_{\mathcal{O}_{\mathcal{F}}}(v) = \int_{\mathcal{O}_{\mathcal{F}}} res_{\mathcal{O}_2}(v)d\mu_{\mathcal{O}_{\mathcal{F}}} \,.$$

The restriction $res_{\mathcal{O}_1}$ has the spectral expansion $res_{\mathcal{O}_1} = \sum_{W_j \subset L^2(\mathcal{O}_1)} pr_{W_j}(res_{\mathcal{O}_1})$ induced by the decomposition of $L^2(\mathcal{O}_1) = \oplus_j W_j$ into irreducible representations of $\mathcal{H}_1$ (and similarly $res_{\mathcal{O}_2} = \sum_{U_k \subset L^2(\mathcal{O}_2)} pr_{U_k}(res_{\mathcal{O}_2})$ for the group $\mathcal{H}_2$). The integration over the orbit $\mathcal{O}_{\mathcal{F}} \subset \mathcal{O}_1$ defines an $\mathcal{F}$-invariant functional on (the smooth part of) each irreducible representation W_j of $\mathcal{H}_1$ (and correspondingly for U_k). We denote the corresponding $\mathcal{F}$-invariant functional by $I_{\mathcal{O}_{\mathcal{F}},j} : W_j^\infty \to \mathbb{C}$ (and correspondingly an $\mathcal{F}$-invariant functional $J_{\mathcal{O}_{\mathcal{F}},k} : U_k^\infty \to \mathbb{C}$ on irreducible representations U_k of $\mathcal{H}_2$).

Hence we obtain two spectral decompositions for the functional $I_{\mathcal{O}_{\mathcal{F}}}$:

$$\sum_{W_j \subset L^2(\mathcal{O}_1)} I_{\mathcal{O}_{\mathcal{F}},j}\left(pr_{W_j}(res_{\mathcal{O}_1}(v))\right) = I_{\mathcal{O}_{\mathcal{F}}}(v) = \sum_{U_k \subset L^2(\mathcal{O}_2)} J_{\mathcal{O}_{\mathcal{F}},k}\left(pr_{U_k}(res_{\mathcal{O}_2}(v))\right) \tag{2.3}$$

for any $v \in \mathcal{V}$. Note that the summation on the left is over the set of irreducible representations of $\mathcal{H}_1$ occurring in $L^2(\mathcal{O}_1)$ and the summation on the right is over the set of irreducible representations of $\mathcal{H}_2$ occurring in $L^2(\mathcal{O}_2)$. Since the groups $\mathcal{H}_1$ and $\mathcal{H}_2$ might be quite different, the identity (2.3) is nontrivial in general.

The identity (2.3) is the origin of our Rankin-Selberg type identities. We show how one can transform it to a more familiar form. To this end we use the standard device of model invariant functionals. Our main observation is that the functionals $I_{\mathcal{O}_{\mathcal{F}},j}$, $J_{\mathcal{O}_{\mathcal{F}},k}$ and the maps $pr_{W_j}(res_{\mathcal{O}_1}) : \mathcal{V} \to W_j$ and $pr_{U_k}(res_{\mathcal{O}_2}) : \mathcal{V} \to U_k$ satisfy the uniqueness property due to the assumption that the pairs $(\mathcal{H}_i, \mathcal{F})$ and $(\mathcal{G}, \mathcal{H}_i)$ are strong Gelfand pairs (in fact, it is enough for $(\mathcal{H}_i, \mathcal{F})$ to be the usual Gelfand pairs).

Hence, by choosing explicit "models" $\mathcal{V}^{mod}$, W_j^{mod}, U_k^{mod} for the corresponding automorphic representations, we can construct model invariant functionals $I_j^{mod} = I_{W_j}^{mod}$, $J_k^{mod} = J_{U_k}^{mod}$ and the model equivariant maps $T_j^{mod} : \mathcal{V}^{mod} \to W_j^{mod}$ and $S_k^{mod} : \mathcal{V}^{mod} \to U_k^{mod}$. The model functionals and maps could be constructed regardless of the automorphic picture and we define them for *any* irreducible representations of $\mathcal{G}$ and $\mathcal{H}_i$. The uniqueness principle then implies the existence of coefficients of proportionality a_j, b_j, c_k, d_k such that

$$I_{\mathcal{O}_{\mathcal{F}},j} = a_j \cdot I_j^{mod} \ , \quad pr_{W_j}(res_{\mathcal{O}_1}) = b_j \cdot T_j^{mod} \quad \text{for any } j,$$

and similarly

$$J_{\mathcal{O}_{\mathcal{F}},k} = c_k \cdot J_k^{mod} \ , \quad pr_{U_k}(res_{\mathcal{O}_2}) = d_k \cdot S_k^{mod} \quad \text{for any } k.$$

This allows us to rewrite the relation (2.3) in the form

$$\sum_{\{W_j\}} \alpha_j \cdot h_j(v) = \sum_{\{U_k\}} \beta_k \cdot g_k(v) \tag{2.4}$$

for any $v \in \mathcal{V}^{mod}$. Where we denoted by $\alpha_j = a_j b_j$, $\beta_k = c_k d_k$, $h_j(v) = I_j^{mod}(T_j^{mod}(v))$ and $g_k(v) = J_k^{mod}(S_k^{mod}(v))$.

This is what we call Rankin-Selberg type spectral identity associated to the diagram (2.2).

Remark. We note that one can associate a non-trivial spectral identity of a kind we described above to a *pair* of different *filtrations* of a group by subgroups forming strong Gelfand pairs. Namely, we associate a spectral identity to two filtrations $\mathcal{F} = G_0 \subset G_1 \subset \cdots \subset G_n = \mathcal{G}$ and $\mathcal{F} = H_0 \subset H_1 \subset \cdots \subset H_m = \mathcal{G}$ of subgroups in the same group $\mathcal{G}$ such that all pairs (G_{i+1}, G_i) and (H_{j+1}, H_j) are strong Gelfand pairs having the same intersection $\mathcal{F}$. One also can "twist" such an identity by a nontrivial character or an irreducible representation of the group $\mathcal{F}$.

2.1.3. *Bounds for coefficients.* The Rankin-Selberg type formulas can be used in order to obtain bounds for coefficients α_j or β_k (e.g, Theorems 1.1 and 1.3). To this end one has to study properties of the integral transforms $h_W = I^{mod}(T_W^{mod}) : \mathcal{V}^{model} \to C(\hat{\mathcal{H}}_1)$, $v \mapsto h_W(v) = I_W^{mod}(T_W^{mod}(v))$ induced by the corresponding *model* functionals and maps (here $\hat{\mathcal{H}}_1$ is the unitary dual of $\mathcal{H}_1$ and $\mathcal{V}^{mod}$ an explicit model of the representation $\mathcal{V}$); similarly for the triple $(\mathcal{G}, \mathcal{H}_2, \mathcal{F})$. This is a problem in harmonic analysis which has nothing to do with the automorphic picture. One can study the corresponding transforms and establish some instance of what might be called an "uncertainty principle" for the pair of such transforms. The idea behind the proof of Theorems 1.1 and 1.3 is quite standard (see [Go]), once we have the appropriate Rankin-Selberg type identity and the necessary information about corresponding integral transforms. Namely, we find a family of test vectors $v_T \in \mathcal{V}$, $T \geq 1$ such that when substituted in the Rankin-Selberg type identity (2.4) it will pick up the (weighted) sum of coefficients α_j for j in certain "short" interval around T (i.e., the transform $h_j(v)$ has essentially small support in $\hat{\mathcal{H}}_1$). We show then that the integral transform $g_k(v)$ of such a vector is a slowly changing function on $\hat{\mathcal{H}}_2$. This allows us to bound the right hand side in (2.4) using Cauchy-Schwartz inequality and the *mean value* (or convexity) bound for the coefficients β_k. The simple way to obtain these mean value bounds was explained by us in [BR3].

We note that in order to obtain bounds for the coefficients in (2.4) one needs to have a kind of positivity which is not always easy to achieve. In our examples we consider representations of the type $\mathcal{V} = V \otimes \bar{V}$ for the group $\mathcal{G} = G \times G$ and V an irreducible representation of G. For such representations the necessary positivity is automatic.

In this paper we implement the above strategy in two cases: for the unipotent subgroup N of $G = PGL_2(\mathbb{R})$ and a compact subgroup $K \subset G$. The first case corresponds to the unipotent Fourier coefficients and the formula we obtain is equivalent to the classical Rankin-Selberg formula. The second case corresponds to the spherical Fourier coefficients which were introduced by H. Peterson long time ago, but the corresponding formula (see Theorem 1.2) has never appeared in print, to the best of our knowledge.

We set $\mathcal{G} = G \times G$, $\mathcal{H}_2 = \Delta G \overset{j_2}{\hookrightarrow} G \times G$ in both cases under consideration and $\mathcal{H}_1 = N \times N$, $\mathcal{F} = \Delta N \overset{i_1}{\hookrightarrow} N \times N \overset{j_1}{\hookrightarrow} G \times G$ for the first case and $\mathcal{H}_1 = K \times K$, $\mathcal{F} = \Delta K \hookrightarrow K \times K \hookrightarrow G \times G$ for the second case. Strictly speaking, the uniqueness principle is only "almost" satisfied for the subgroup N, but the theory of Eisenstein series provides the necessary remedy in the automorphic setting .

Finally, we would like to mention that the method described above also lies behind the proof of the subconvexity for the triple L-function given in [BR4] (but has not been understood at the time). Recently we discovered a variety of other strong Gelfand formations in higher rank groups.

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References

[BR1] J. Bernstein, A. Reznikov, Analytic continuation of representations, Ann. of Math., 150 (1999), 329–352, arXiv: math.RT/9907202.

[BR2] J. Bernstein, A. Reznikov, Sobolev norms of automorphic functionals, IMRN 2002:40 (2002), 2155-2174.

[BR3] J. Bernstein, A. Reznikov, Estimates of automorphic functions, Moscow Math. J. 4 (2004), no. 1, 19–37, arXiv: math.RT/0305351.

[BR4] J. Bernstein, A. Reznikov, Subconvexity of triple L-functions, preprint, (2005).

[Go] A. Good, Cusp forms and eigenfunctions of the Laplacian. Math. Ann. 255 (1981), no. 4, 523–548.

[Gr] B. Gross, Some applications of Gelfand pairs to number theory. Bull. Amer. Math. Soc. (N.S.) 24 (1991), no. 2, 277–301.

[He] S. Helgason, Groups and geometric analysis. Mathematical Surveys and Monographs, 83, AMS, Providence, RI, 2000.

[Iw] H. Iwaniec, Spectral methods of automorphic forms. Graduate Studies in Mathematics, 53. AMS, Providence, RI; Revista Matemática Iberoamericana, Madrid, 2002.

[IS] H. Iwaniec, P. Sarnak, Perspectives on the analytic theory of L-functions. GAFA 2000, Geom. Funct. Anal. 2000, Special Volume, Part II, 705–741.

[JN] H. Jacquet, C. Nan, Positivity of quadratic base change L-functions. Bull. Soc. Math. France 129 (2001), no. 1, 33–90.

[KiSa] H. Kim, P. Sarnak, Appendix to: H. Kim, Functoriality for the exterior square of GL_4 and the symmetric fourth of GL_2. J. Amer. Math. Soc. 16 (2003), no. 1, 139–183.

[KS] B. Krötz, R. Stanton, Holomorphic extensions of representations. I, Ann. of Math. (2) 159 (2004), no. 2, 641–724,

[Ku] T. Kubota, Elementary theory of Eisenstein series. Kodansha Ltd., Tokyo; Halsted Press, 1973.

[L] J. Lewis, Eigenfunctions on symmetric spaces with distribution-valued boundary forms. J. Funct. Anal. 29 (1978), no. 3, 287–307

[M] H. Maass, Über eine neue Art von nichtanalytischen automorphen Funktionen und die Bestimmung Dirichletscher Reihen durch Funktionalgleichungen, Math. Ann. 121, (1949), 141–183.

[Mo] Y. Motohashi, A note on the meanvalue of the zeta and L-functions, preprint, arXiv: math.NT/0401085.

[PS] Y. Petridis, P. Sarnak, Quantum unique ergodicity for $SL_2(\mathcal{O})\backslash\mathbb{H}^3$ and estimates for L-functions, J. Evol. Equ. 1 (2001), no. 3, 277–290.

[R] A. Reznikov, Norms of geodesic restrictions for eigenfunctions on hyperbolic surfaces and representation theory, preprint (2004). arXiv: math.AP/0403437.

[Ra] R. Rankin, Contributions to the theory of Ramanujan's function $\tau(n)$, Proc. Camb. Philos. Soc. 35 (1939), 357–372.

[Sa] P. Sarnak, Integrals of products of eigenfunctions, Internat. Math. Res. Notices, no. 6, (1994), 251–261.

[Se] A. Selberg, On the estimation of Fourier coefficients, in Collected works, Springer-Verlag, New York (1989), 506–520.

[V] A. Venkatesh, Sparse equidistribution problems, period bounds, and subconvexity, preprint (2005). arXiv: math.NT/0506224.

[Wa] J.-L. Waldspurger, Sur les valeurs de certaines fonctions L automorphes en leur centre de symétrie. Compositio Math. 54 (1985), no. 2, 173–242.

[W] T. Watson, Thesis, Princeton, 2001.

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